Proof Nets

A graph syntax for proofs

Reference:
Notes on proof-nets by Olivier Laurent

(Note: most slides are taken from the notes of Olivier Laurent)

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MLL

$$\dfrac{}{\vdash A^\perp, A}\,ax \qquad \dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut$$

$$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

Recall that linear negation is defined :

$$(X^\perp)^\perp = X$$
$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$
$$(A \parr B)^\perp = A^\perp \otimes B^\perp$$

2

Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes) \quad \dfrac{}{\vdash C, C^\perp}(ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr) \quad \dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(cut)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr)$$

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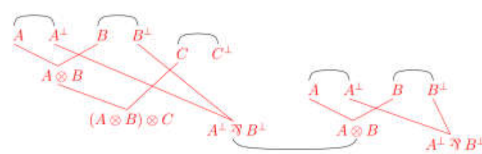
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes) \quad \dfrac{}{\vdash C, C^\perp}(ax)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr) \quad \dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \quad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A \otimes B, A^\perp, B^\perp}(\otimes)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}(cut)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}(\parr)$$



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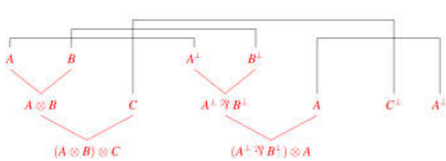
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}ax\ (1) \quad \dfrac{}{\vdash B, B^\perp}ax\ (2)}{\vdash A \otimes B, A^\perp, B^\perp}\otimes\ (3) \quad \dfrac{}{\vdash C, C^\perp}ax\ (4)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}\otimes\ (5)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}\parr\ (6) \quad \dfrac{}{\vdash A, A^\perp}ax\ (7)}{\vdash (A \otimes B) \otimes C, (A^\perp \parr B^\perp) \otimes A, C^\perp, A^\perp}\otimes\ (8)$$



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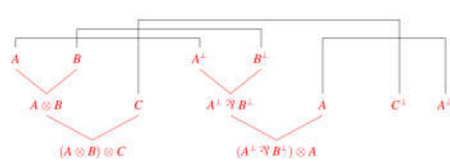
Forgetting Sequential Structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}ax\ (1) \quad \dfrac{}{\vdash B, B^\perp}ax\ (2)}{\vdash A \otimes B, A^\perp, B^\perp}\otimes\ (3) \quad \dfrac{}{\vdash C, C^\perp}ax\ (4)}{\vdash (A \otimes B) \otimes C, A^\perp, B^\perp, C^\perp}\otimes\ (5)}{\vdash (A \otimes B) \otimes C, A^\perp \parr B^\perp, C^\perp}\parr\ (6) \quad \dfrac{}{\vdash A, A^\perp}ax\ (7)}{\vdash (A \otimes B) \otimes C, (A^\perp \parr B^\perp) \otimes A, C^\perp, A^\perp}\otimes\ (8)$$



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Proof Nets

A graph syntax for proofs

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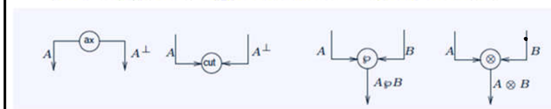
Proof structures

A ***proof structure*** $\mathcal{M}$ *is* ***a labelled directed acyclic graph (DAG)*** *with possibly pending edges (i.e. some edges may have no source and/or no target) built over the alphabet of nodes which is represented below.*
(Note: in figures, the edges orientation is always top-bottom.)



- *The* ***nodes*** *are labelled by* ax, cut, $\otimes$, $\parr$
- *The* ***edges*** *are labelled by* MLL *formulas.*

For each node/link: ***premisses*** *= entering edges,* ***conclusions*** *= exiting edges*

The ***conclusions*** *of* $\mathcal{M}$ *is the set of pending edges of* $\mathcal{M}$.

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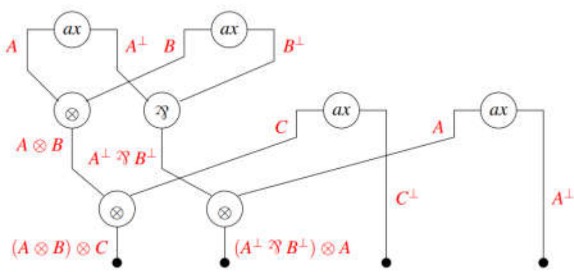
In the graphical representation of a proof structure, we do not mention explicitly the direction of edges, but we draw them in such a way that direction in represented in a top-down way:



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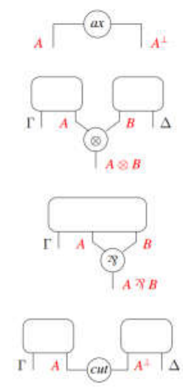
From proofs to proofs structures

$$\frac{}{\vdash A, A^\perp}\,ax \quad \mapsto$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \quad \mapsto$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \mapsto$$

$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,cut \quad \mapsto$$



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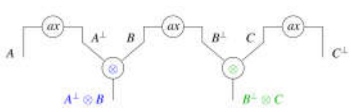
example

Translate each of these sequent calculus proofs. Start from axioms....

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax \quad \dfrac{}{\vdash B, B^\perp}\,ax}{\vdash A, A^\perp \otimes B, B^\perp}\,\otimes \quad \dfrac{}{\vdash C, C^\perp}\,ax}{\vdash A, A^\perp \otimes B, B^\perp \otimes C, C^\perp}\,\otimes \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,ax \quad \dfrac{\dfrac{}{\vdash B, B^\perp}\,ax \quad \dfrac{}{\vdash C, C^\perp}\,ax}{\vdash B, B^\perp \otimes C, C^\perp}\,\otimes}{\vdash A, A^\perp \otimes B, B^\perp \otimes C, C^\perp}\,\otimes$$



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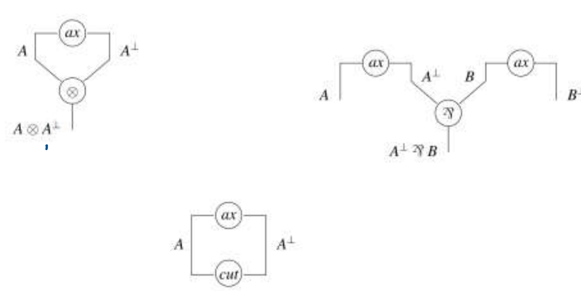
Is every structure the image of an MLL proof?



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Proof Nets

A PROOF NET is a proof structure which is the image of an MLL proof

Internal condition!

Purely geometrical conditions (correction) characterize the proof structures which are proof nets

Theorem 1. *A proof structure is **correct** iff it is a **proof net**.*

Correctness criteria:

- (LT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...

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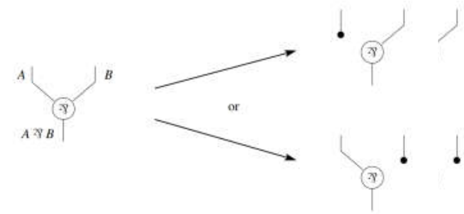
Danos-Regnier Criterion

Correctness Criterion

Switching Graphs



Correctness

- Switching graphs are acyclic.
- Switching graphs are connected.

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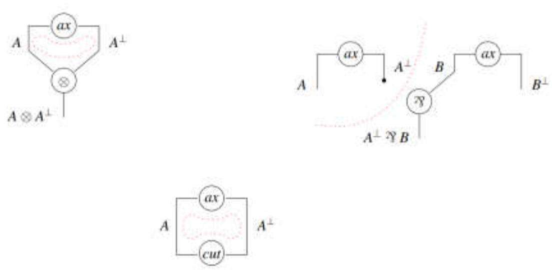

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Definition 2 (Correctness criterion AC (Danos-Regnier)). *Let R be a proof structure.*

*A **switching** s is a function on the nodes of R, which chooses, for each $\parr$-link, either the left or the right premise.*

*A proof structure R is **correct** if **for each switching**, the **unoriented graph** obtained by erasing for each $\parr$-link of R the edges not chosen by s. is:*

connected and acyclic

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Acyclicity. A multiplicative proof structure is *acyclic* if its switching graphs do not contain any undirected cycle.

A proof structure with p $\parr$ nodes induces 2^p switchings and thus 2^p switching graphs. A switching graph is not a proof structure in general since its $\parr$ nodes have only one premisse.

A *connected component* of a switching graph is a connected component of its underlying (undirected) multigraph.

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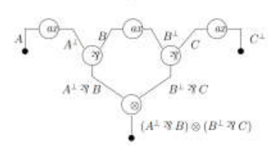
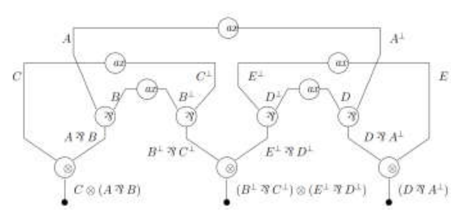
Is this correct?

PN1:



PN2:



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- Correctness guarantees:
  - ✓ Graph is image of a proof (sequentialization)
  - ✓ Normalization progresses (no deadlocks)
  - ✓ Normalization terminates (no infinite cycles)

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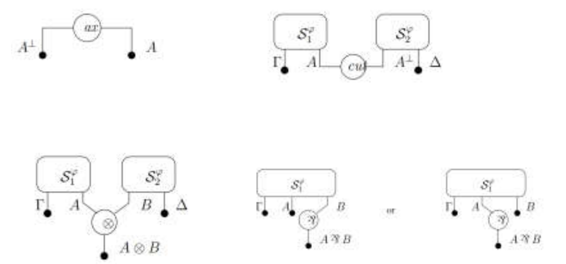
Soundness

Proposition 4.1.1 (Soundness of Correctness). *The translation of a sequent calculus proof of MLL is a connected multiplicative proof net.*



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Sequentialization

Theorem 4.1.1 (Sequentialization). *Any connected multiplicative proof net is the translation of a sequent calculus proof of MLL.*

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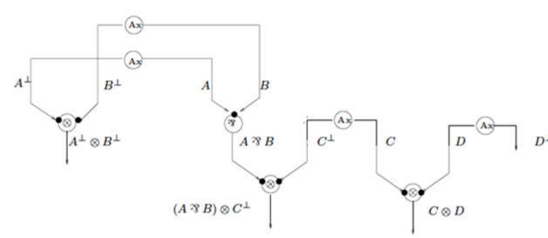
Sequentialization answers the question:

We have a proof net. The problem: it is the image of a sequent calculus proof? And which?



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The beauty of proof nets is normalization

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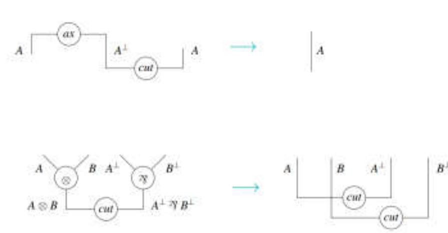
Normalization (local graph reductions!)



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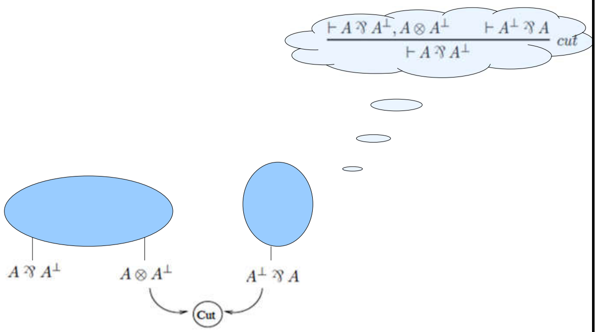
Let us try out!



Write a proof net with this conclusion... and normalize it

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How we write a proof net of these conclusions?

$A \parr A^\perp$ must type an edge conclusion of a par link, with premisses ....

$A \otimes A^\perp$ must type an edge conclusion of a tensor link, with premisses ....

Then we have to choose the axiom links!

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Properties of normalization

Lemma (preservation of correctness)

If the proof structure R is correct and reduces to R', then R' is correct.

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Properties of normalization

1. Confluence?
2. Is normalization weakly/strongly normalizing?
3. Would you be able to define a normalizing strategy?
4. Would you be able to define a normalizing strategy which reaches normal form in a **minimal** number of steps?

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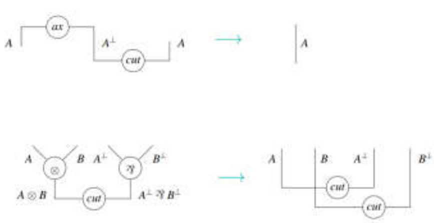
Normalization of MLL proof-nets:



- Strongly normalizing
- Confluent
- Cut elimination:

a proof-net in normal form contains no cuts

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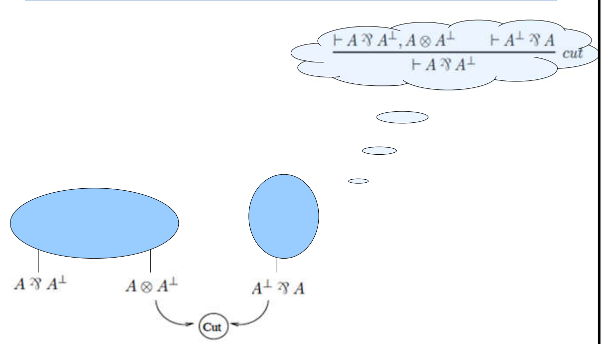
Let us try out!



Write a proof net with this conclusion... and normalize it

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Slide 27

CF1 Claudia F, 3/8/2021

- Let us try out another example

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Let us try one more. First, write a proof net with this conclusion...

$(X \otimes X) \multimap (X \otimes X)$ =

$(X \otimes X)^{\perp} \parr (X \otimes X)$ = $(X^{\perp} \parr X^{\perp}) \parr (X \otimes X)$

TIP: How we write a proof net? As before, all proof nets with the same conclusion, start with the same nodes (the formula tree!)
What distinguishes different proofs are the axiom links

To distinguish the different occurrences of atoms, let us write indices:

$(X_1^{\perp} \parr X_2^{\perp}) \parr (X_3 \otimes X_4)$

In this case, we have two possible proofs, corresponding to two possible way to write axioms:
1,3 and 2,4
OR
1,4 and 2,3

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In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

$$\dfrac{\dfrac{\dfrac{\overline{\vdash X_3^{\perp}, X_1} \quad \overline{\vdash X_4^{\perp}, X_2}}{\vdash X_1^{\perp}, X_2^{\perp}, X_3 \otimes X_4}\otimes}{\vdash X_1^{\perp} \parr X_2^{\perp}, X_3 \otimes X_4}\parr}{\vdash (X_1 \otimes X_3)^{\perp} \parr (X_2 \otimes X_4)}\parr \qquad \dfrac{\dfrac{\dfrac{\overline{\vdash X_3^{\perp}, X_2} \quad \overline{\vdash X_4^{\perp}, X_1}}{\vdash X_1^{\perp}, X_2^{\perp}, X_3 \otimes X_4}\otimes}{\vdash X_1^{\perp} \parr X_2^{\perp}, X_3 \otimes X_4}\parr}{\vdash (X_1 \otimes X_3)^{\perp} \parr (X_2 \otimes X_4)}\parr$$

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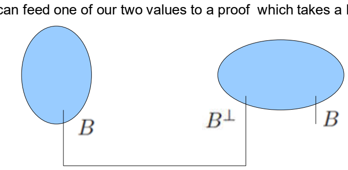
When we have a formula whose normal proofs are exactly two, we have a good candidate to code BOOLEANS :)

Let us indicate the formula $(X^{\perp} \parr X^{\perp}) \parr (X \otimes X)$ with B (for boolean).
We call one proof ***true***, and the other ***false***...

We can feed one of our two values to a proof which takes a boolean, and return a boolean.



We know that the normal form (i.e the result of computation) will be of type B... Hence one of our two values.

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Try to normalize one of the proofs of $(X_1^{\perp} \parr X_2^{\perp}) \parr (X_3 \otimes X_4)$

with the proof net which has conclusions

$(X_1 \otimes X_2) \otimes (X_3^{\perp} \parr X_4^{\perp}) \qquad (X_5^{\perp} \parr X_6^{\perp}) \parr (X_7 \otimes X_8)$

and axiom links: (1,6) (2,5) (3,7) (4,8)

What is the function coded by this proof net?

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Correctness criterion, think again

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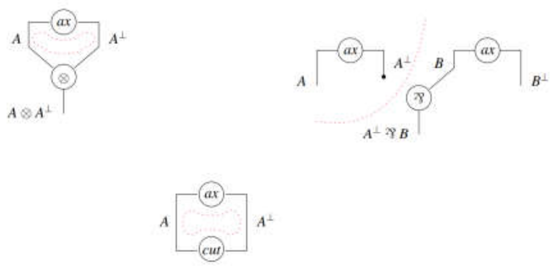

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Correctness: if we focus on acylicity, Danos-Regnier criterion can be reformulated (in equivalent way)

Let R be a proof structure; a *switching path* of R is a path which does not use any two edges entering on the same ⅋ link (such edges are called *switching edges*); a *switching cycle* is a switching path which is a cycle.

Definition 3 (Correctness criterion). *A proof structure is correct if it does not contain any switching cycle.*

$$\frac{}{\vdash A,\ A^\perp}(Ax) \qquad \frac{\vdash \Gamma,\ A \quad \vdash \Delta,\ A^\perp}{\vdash \Gamma,\ \Delta}(Cut)$$

$$\frac{\vdash \Gamma,\ A \quad \vdash \Delta,\ B}{\vdash \Gamma,\ A \otimes B,\ \Delta}(\otimes) \qquad \frac{\vdash \Gamma,\ A,\ B}{\vdash \Gamma,\ A ⅋ B}(⅋)$$

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}(Mix)$$

we can throw away MIX later
By requiring **connectness**

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Exponentials

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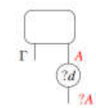
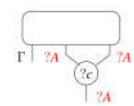
From proofs to proofs structures

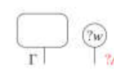
$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \quad \mapsto$$



$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \quad \mapsto$$



$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \quad \mapsto$$



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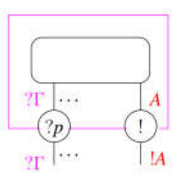
From proofs to proofs structures

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \quad \mapsto$$



- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

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Cut-elimination steps

$$\frac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp}\,?d}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \frac{\vdash ?\Gamma, A \quad \vdash \Delta, A^\perp}{\vdash ?\Gamma, \Delta}\,cut$$

$$\frac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \dfrac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash \Delta, ?A^\perp}\,?c}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \vdash \Delta, ?A^\perp, ?A^\perp}{\vdash ?\Gamma, \Delta, ?A^\perp}\,cut}{\vdash ?\Gamma, ?\Gamma, \Delta}\,cut}{\vdash ?\Gamma, \Delta}\,?c$$

$$\frac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \dfrac{\vdash \Delta}{\vdash \Delta, ?A^\perp}\,?w}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \frac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?w$$

$$\frac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \dfrac{\vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?A^\perp, !B}\,!}{\vdash ?\Gamma, ?\Delta, !B}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \quad \vdash ?\Delta, ?A^\perp, B}{\vdash ?\Gamma, ?\Delta, B}\,cut}{\vdash ?\Gamma, ?\Delta, !B}\,!$$

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What is the associated proof-net?

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax}{\vdash A \parr A^{\perp}}\,\parr}{\vdash A \parr A^{\perp}, ?B}\,?w \qquad \mapsto$$

$$!A \otimes !A \dashv\vdash !A$$
$$?A \parr ?A \dashv\vdash ?A$$

Can you write the proof so that all axioms are atomic (ie on atomic formulas)?

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Reduction steps

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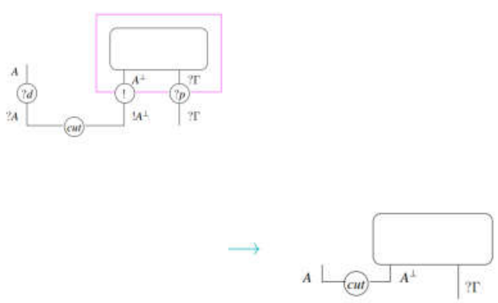
Reduction Steps: $?d$



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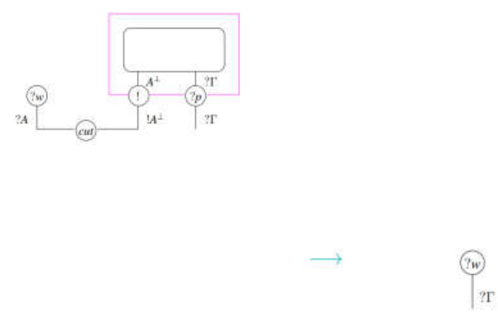
Reduction Steps: $?w$



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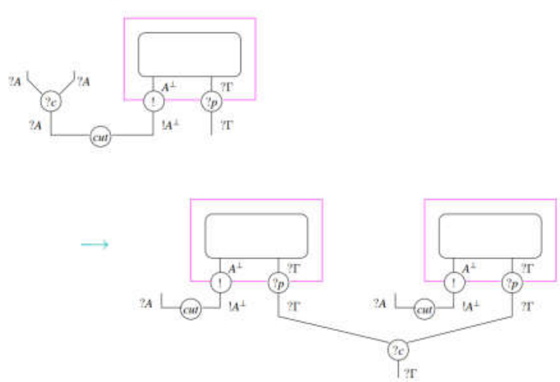
Reduction Steps: $?c$



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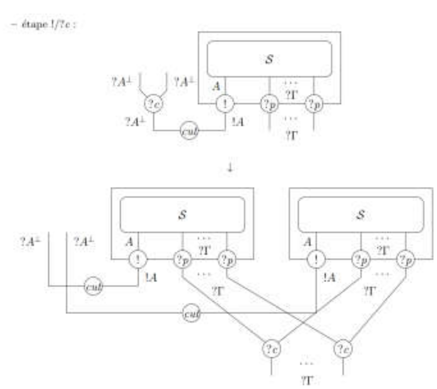

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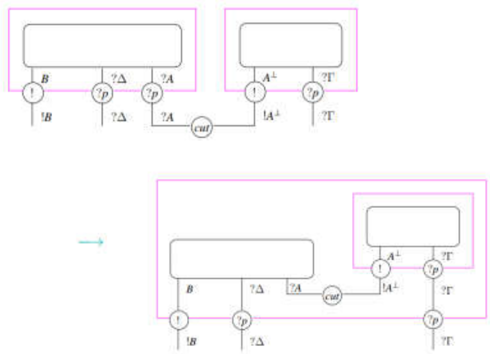
Reduction Steps: ?p



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Confluence?
Weak Normalization?
Strong Normalization?

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