Linear Logic

Reference:

Lecture Notes by Olivier Laurent (in French)

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Recall LK ?

Groupe logique multiplicatif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \wedge^{\mathrm{mul}}\mathrm{R} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\mathrm{mul}}\mathrm{L}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\mathrm{mul}}\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee^{\mathrm{mul}}\mathrm{L}$$

Groupe logique additif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge^{\mathrm{add}}\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\mathrm{add}}\mathrm{L}_1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{\mathrm{add}}\mathrm{L}_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\mathrm{add}}\mathrm{R}_1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee^{\mathrm{add}}\mathrm{R}_2 \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee^{\mathrm{add}}\mathrm{L}$$

Groupe structurel.

$$\frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \tau(\Delta)} \, ech$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \, ctr\mathrm{L} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \, ctr\mathrm{R}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \, wk\mathrm{L} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \, wk\mathrm{R}$$

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LL Sequent Calculus

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Formulas:

$$A ::= X \mid A \parr A \mid A \,\&\, A \mid ?A \mid X^{\perp} \mid A \otimes A \mid A \oplus A \mid !A$$

Groupe multiplicatif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes\mathrm{R} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes\mathrm{L}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta} \parr\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'} \parr\mathrm{L}$$

Groupe additif.

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta} \&\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&\mathrm{L}_1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&\mathrm{L}_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus\mathrm{R}_1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus\mathrm{R}_2 \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus\mathrm{L}$$

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Groupe identité.

$$\frac{}{A \vdash A} \, ax \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \, cut$$

Groupe négation.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} (.)^{\perp}\mathrm{R} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} (.)^{\perp}\mathrm{L}$$

Groupe exponentiel.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} ?\mathrm{R} \qquad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta} ?\mathrm{L}$$

$$\frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} !\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} !\mathrm{L}$$

$$\frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} \, ctr\mathrm{R} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \, ctr\mathrm{L}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} \, wk\mathrm{R} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \, wk\mathrm{L}$$

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MALL (no structural rules)

$$\frac{}{A \vdash A} \, ax \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \, cut$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \otimes\mathrm{R} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes\mathrm{L}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta} \parr\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'} \parr\mathrm{L}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta} \&\mathrm{R} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&\mathrm{L}_1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&\mathrm{L}_2$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus\mathrm{R}_1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus\mathrm{R}_2 \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus\mathrm{L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} (.)^{\perp}\mathrm{R} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} (.)^{\perp}\mathrm{L}$$

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Montrer que l'on a les distributivités suivantes :

$$A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$$
$$A \parr (B \,\&\, C) \dashv\vdash (A \parr B) \,\&\, (A \parr C)$$

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Define negation:

$$A ::= X \mid A \parr A \mid A \,\&\, A \mid X^{\perp} \mid A \otimes A \mid A \oplus A$$

$$(X^{\perp})^{\perp} = X$$
$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$
$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$
$$(A \,\&\, B)^{\perp} = A^{\perp} \oplus B^{\perp}$$
$$(A \oplus B)^{\perp} = A^{\perp} \,\&\, B^{\perp}$$

ce qui donne $A^{\perp\perp} = A$.

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et on utilise des séquents de la forme $\vdash \Gamma$.

$$\frac{}{\vdash A^{\perp}, A}\,ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus_2$$

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EX
The axiom can be restricted to atomic formulas:
the axiom rule is admissible for every formula

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$$A \multimap B = A^{\perp} \parr B$$

On peut alors dériver les règles de calcul des séquents :

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta, \Delta'}$$

$$\frac{\vdash \Gamma, A^{\perp}, B}{\vdash \Gamma, A \multimap B} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B^{\perp}}{\vdash \Gamma, \Delta, (A \multimap B)^{\perp}}$$

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Élimination des coupures

On va se contenter de donner les cas clefs :

$$\frac{\vdash \Gamma, A \quad \dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash \Gamma, A}\,cut \quad \rightsquigarrow \quad \vdash \Gamma, A$$

$$\frac{\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \quad \dfrac{\vdash \Xi, A^{\perp}, B^{\perp}}{\vdash \Xi, A^{\perp} \parr B^{\perp}}\,\parr}{\vdash \Gamma, \Delta, \Xi}\,cut \quad \rightsquigarrow \quad \frac{\vdash \Gamma, A \quad \dfrac{\vdash \Delta, B \quad \vdash \Xi, A^{\perp}, B^{\perp}}{\vdash \Delta, \Xi, A^{\perp}}\,cut}{\vdash \Gamma, \Delta, \Xi}\,cut$$

$$\frac{\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \quad \dfrac{\vdash \Delta, A^{\perp}}{\vdash \Delta, A^{\perp} \oplus B^{\perp}}\,\oplus_1}{\vdash \Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \quad \dfrac{\vdash \Delta, B^{\perp}}{\vdash \Delta, A^{\perp} \oplus B^{\perp}}\,\oplus_2}{\vdash \Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \frac{\vdash \Gamma, B \quad \vdash \Delta, B^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

EX. 1
Define commutative steps (just give two examples).

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Homework Ex 1

EX. 1
How do we define commutative steps?
Give the following two examples:

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes$$

Cut with a formula inside Γ.
How do we reduce this cut?

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\&$$

Cut with a formula inside Γ.
How do we reduce this cut?

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Re-introducing the Structural Rules

In a controlled way

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$$\begin{array}{rcllllll} A & ::= & X & \mid & A \parr A & \mid & A \,\&\, A & \mid\ ?A \\ & \mid & X^\perp & \mid & A \otimes A & \mid & A \oplus A & \mid\ !A \end{array}$$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?d \qquad \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w \qquad \dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,!$$

La règle $?d$ ("déréliction") permet à toute formule de devenir sujette aux règles structurelles. Les règles $?w$ ("affaiblissement") et $?c$ ("contraction") sont les règles structurelles habituelles mais ne s'appliquent qu'aux formules dont le connecteur principal est $?$. La règle $!$ ("promotion")

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La regle promotion…

est la plus subtile. Elle permet de rendre une formule (et surtout la preuve correspondante) duplicable (ou effaçable), mais ceci nécessite un contexte adapté. On peut comprendre $!A$ comme "A autant de fois que l'on veut". La règle écrite sous la forme :

$$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,!$$

dit que si A est obtenue à partir d'hypothèses utilisables autant de fois que l'on veut alors A elle-même peut être utilisée autant de fois que l'on veut. Une autre manière de comprendre la contrainte de contexte de cette règle est de regarder l'élimination des coupures.

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Cut-elimination steps

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp}\,?d}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash ?\Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash ?\Gamma, \Delta}\,cut$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash \Delta, ?A^\perp}\,?c}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash \Delta, ?A^\perp, ?A^\perp}{\vdash ?\Gamma, \Delta, ?A^\perp}\,cut}{\vdash ?\Gamma, ?\Gamma, \Delta}\,cut}{\vdash ?\Gamma, \Delta}\,?c \text{ (double line)}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?A^\perp}\,?w}{\vdash ?\Gamma, \Delta}\,cut \quad \rightsquigarrow \quad \dfrac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?w \text{ (double line)}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?A^\perp, !B}\,!}{\vdash ?\Gamma, ?\Delta, !B}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \vdash ?\Delta, ?A^\perp, B}{\vdash ?\Gamma, ?\Delta, B}\,cut}{\vdash ?\Gamma, ?\Delta, !B}\,!$$

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Homework Ex 2

Two formulas A and B are (linearly) equivalent, written $A \dashv\vdash B$, if both implications $A \multimap B$ and $B \multimap A$ are provable. Equivalently, $A \dashv\vdash B$ if both $A \vdash B$ and $B \vdash A$ are provable.

Ex. 2 Please pick and prove one equivalence in each of the 3 groups

1. $A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$
   $A \parr (B \,\&\, C) \dashv\vdash (A \parr B) \,\&\, (A \parr C)$

2. $!(A \,\&\, B) \dashv\vdash !A \otimes !B$
   $?(A \oplus B) \dashv\vdash ?A \parr ?B$

3. $!A \otimes !A \dashv\vdash !A$
   $?A \parr ?A \dashv\vdash ?A$

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Neutral Elements

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On définit quatre éléments neutres pour les quatre connecteurs multiplicatifs et additifs : 1 ("un") pour $\otimes$, $\bot$ ("bottom") pour $\parr$, $\top$ ("top") pour & et 0 ("zéro") pour $\oplus$:

$$\begin{array}{rcccccccccccccc} A & ::= & X & \mid & A \parr A & \mid & A \,\&\, A & \mid & \bot & \mid & \top & \mid & ?A \\ & \mid & X^{\bot} & \mid & A \otimes A & \mid & A \oplus A & \mid & 1 & \mid & 0 & \mid & !A \end{array}$$

Les règles sont obtenues comme les cas 0-aires des règles de connecteurs binaires correspondants (en particulier deux règles pour $\oplus$ donc aucune pour 0) :

$$\dfrac{}{\vdash 1}\,1 \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \qquad \dfrac{}{\vdash \Gamma, \top}\,\top$$

Les étapes clefs d'élimination des coupures sont :

$$\dfrac{\dfrac{}{\vdash 1}\,1 \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot}{\vdash \Gamma}\,cut \qquad \rightsquigarrow \qquad \vdash \Gamma$$

puisqu'il n'y a pas de règle pour 0.

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On the Proof Search

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Consequences of Cut-elimination

Theorem 3.4.1 (cut elimination)**.** *For every sequent $\Gamma \vdash \Delta$, there is a proof of $\Gamma \vdash \Delta$ if and only if there is a proof of $\Gamma \vdash \Delta$ that does not use the cut rule.*

Theorem 3.4.3 (subformula property)**.** *A sequent $\Gamma \vdash \Delta$ is provable if and only if it is the conclusion of a proof in which each intermediate conclusion is made of subformulas of the formulas of Γ and Δ.*

Theorem 3.4.4 (consistency)**.** *The empty sequent $\vdash$ is not provable. Subsequently, it is impossible to prove both a formula A and its negation $A^{\bot}$; it is impossible to prove* **0** *or $\bot$.*

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Reversing the Reversible Rules

Définition 22 (Connecteur réversible)
Un connecteur $\circ$ est réversible si lorsque $\vdash \Gamma, A$ est dérivable avec $\circ$ connecteur principal de A, alors $\vdash \Gamma, A$ est dérivable avec comme dernière règle une règle d'introduction de $\circ$.

$$\dfrac{\vdash A \parr B, \Gamma}{\vdash A, B, \Gamma}\,(\parr^{rev}) \qquad \dfrac{\vdash \bot, \Gamma}{\vdash \Gamma}\,(\bot^{rev}) \qquad \dfrac{\vdash A_1 \,\&\, A_2, \Gamma}{\vdash A_i, \Gamma}\,(\&_i^{rev})$$

- $$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\bot}}\,(ax) \qquad \dfrac{}{\vdash B, B^{\bot}}\,(ax)}{\vdash A, B, A^{\bot} \otimes B^{\bot}}\,(\otimes) \qquad \vdash A \parr B, \Gamma}{\vdash A, B, \Gamma}\,(cut)$$

- $$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\bot}}\,(ax)}{\vdash A, A^{\bot} \oplus B^{\bot}}\,(\oplus_1) \qquad \vdash A \,\&\, B, \Gamma}{\vdash A, \Gamma}\,(cut)$$

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Reversible Rules

$$\vdash \Gamma, A \parr B \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A \parr B \qquad \dfrac{\dfrac{}{\vdash A^{\bot}, A}\,ax \qquad \dfrac{}{\vdash B^{\bot}, B}\,ax}{\vdash A^{\bot} \otimes B^{\bot}, A, B}\,\otimes}{\vdash \Gamma, A, B}\,cut}{\vdash \Gamma, A \parr B}\,\parr$$

$$\vdash \Gamma, A \,\&\, B \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Gamma, A \,\&\, B \qquad \dfrac{\dfrac{}{\vdash A^{\bot}, A}\,ax}{\vdash A^{\bot} \oplus B^{\bot}, A}\,\oplus_1}{\vdash \Gamma, A}\,cut \qquad \dfrac{\vdash \Gamma, A \,\&\, B \qquad \dfrac{\dfrac{}{\vdash B^{\bot}, B}\,ax}{\vdash A^{\bot} \oplus B^{\bot}, B}\,\oplus_2}{\vdash \Gamma, B}\,cut}{\vdash \Gamma, A \,\&\, B}\,\&$$

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Reversible Rules

$$\vdash \Gamma, \bot \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma, \bot \qquad \dfrac{}{\vdash 1}\,1}{\vdash \Gamma}\,cut}{\vdash \Gamma, \bot}\,\bot$$

$$\vdash \Gamma, \top \quad \rightsquigarrow \quad \dfrac{}{\vdash \Gamma, \top}\,\top$$

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Positive/negative connective

	Positive		*Negative*	*Class*
α	atom	$A^\perp$	negation	
$A \otimes B$	tensor	$A \parr B$	par	multiplicatives
$\mathbf{1}$	one	$\bot$	bottom	multiplicative units
$A \oplus B$	plus	$A \,\&\, B$	with	additives
$\mathbf{0}$	zero	$\top$	top	additive units
$!A$	of course	$?A$	why not	exponentials

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Focalization

A consequence of this fact is that, **when searching for a proof** of some sequent $\vdash \Gamma$, one can always start by decomposing negative connectives in Γ without losing provability.

For instance:

- $\vdash \Gamma, (A \parr B) \parr (B \,\&\, C)$ is provable
- iff $\vdash \Gamma, A \parr B, B \,\&\, C$ is provable
- iff $\vdash \Gamma, A \parr B, B$ and $\vdash \Gamma, A \parr B, C$ are provable
- iff $\vdash \Gamma, A, B, B$ and $\vdash \Gamma, A, B, C$ are provable

So without loss of generality, we can assume that any proof of $\vdash \Gamma, (A \parr B) \parr (B \,\&\, C)$ ends like

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, A, B, B}{\vdash \Gamma, A \parr B, B}\,(\parr) \qquad \dfrac{\vdash \Gamma, A, B, C}{\vdash \Gamma, A \parr B, C}\,(\parr)}{\vdash \Gamma, A \parr B, B \,\&\, C}\,(\&)}{\vdash \Gamma, (A \parr B) \parr (B \,\&\, C)}\,(\parr)$$

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Any of this two is a focalized proof?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}\,ax \qquad \dfrac{}{\vdash Y, Y^\perp}\,ax}{\vdash X \otimes Y, X^\perp, Y^\perp}\,\otimes \qquad \dfrac{}{\vdash 1}\,1}{\vdash X \otimes Y, X^\perp \otimes 1, Y^\perp}\,\otimes}{\vdash (X \otimes Y) \oplus \bot, X^\perp \otimes 1, Y^\perp}\,\oplus_1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}\,ax \qquad \dfrac{}{\vdash 1}\,1}{\vdash X, X^\perp \otimes 1}\,\otimes \qquad \dfrac{}{\vdash Y, Y^\perp}\,ax}{\vdash X \otimes Y, X^\perp \otimes 1, Y^\perp}\,\otimes}{\vdash (X \otimes Y) \oplus \bot, X^\perp \otimes 1, Y^\perp}\,\oplus_1$$

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Summing-Up
Linear Logic rules

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$$\dfrac{}{\vdash A, A^\perp}\,(identity) \qquad \dfrac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,(cut)$$

Multiplicatives

$$\dfrac{}{\vdash 1}\,(one) \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(false)$$

$$\dfrac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,(times) \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(par)$$

Additives

(no rule for zero) $\qquad \dfrac{}{\vdash \Gamma, \top}\,(true)$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,(left\ plus) \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,(right\ plus) \qquad \dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,(with)$$

Exponentials

$$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,(of\ course) \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\,(weakening)$$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,(dereliction) \qquad \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,(contraction)$$

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Linear negation

$$(X^{\perp})^{\perp} = X$$
$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$
$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$
$$(A \,\&\, B)^{\perp} = A^{\perp} \oplus B^{\perp}$$
$$(A \oplus B)^{\perp} = A^{\perp} \,\&\, B^{\perp}$$

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