

MPRI SEMPL

TD3: PCSs and cones

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1. Let A be a set which has at least two elements. We use $A^{<\omega}$ for the set of all finite sequences of elements of A and A^ω for the set of all infinite sequences. We use $\langle a_1, \dots, a_n \rangle$ for the element of $A^{<\omega}$ given explicitly as a sequence of n elements of A , the empty sequence is $\langle \rangle$. We use simple juxtaposition for the concatenation of sequences. If $s, t \in A^{<\omega}$ then $s \leq t$ means that s is a prefix of t , that is, there is $s' \in A^{<\omega}$ such that $ss' = t$. Similarly if $s \in A^{<\omega}$ and $\alpha \in A^\omega$, then $s < \alpha$ means that s is a prefix of α . If $s \in A^{<\omega}$ we set $\uparrow s = \{t \in A^{<\omega} \mid s \leq t\}$ and $\uparrow^\omega s = \{\alpha \in A^\omega \mid s < \alpha\}$. If σ is a finite or infinite sequence and $i \in \mathbb{N}$, we use $\sigma(i)$ for its i th element (when defined). Notice that the first element of a sequence is indexed by 1 (not by 0). If C is a subset of $A^{<\omega}$ or A^ω , and if $s \in A^{<\omega}$, we set $sC = \{s\sigma \mid \sigma \in C\}$.

An *antichain* is a subset B of $A^{<\omega}$ such that $\forall s, t \in B \ s \leq t \Rightarrow s = t$.

A *bar* is a subset B of $A^{<\omega}$ such that $\forall \alpha \in A^\omega \exists s \in B \ s < \alpha$.

A *tree* is a non-empty subset T of $A^{<\omega}$ such that $\forall s, t \in A^{<\omega} \ (s \leq t \text{ and } t \in T) \Rightarrow s \in T$.

An *infinite branche* of a tree T is an $\alpha \in A^\omega$ such that $\forall s \in A^{<\omega} \ s < \alpha \Rightarrow s \in T$.

A tree T is *finitely branching* if for all $s \in A^{<\omega}$, the set $\{a \in A \mid s\langle a \rangle \in T\}$ is finite.

- (a) Prove König's lemma: if T is infinite (as a set) and finitely branching, then T has an infinite branche.
- (b) Let $s, t \in A^{<\omega}$. Prove that $\uparrow^\omega s \subseteq \uparrow^\omega t \Leftrightarrow t \leq s$.
- (c) We say that a subset U of A^ω is open if for all $\alpha \in U$ there is $s \in A^{<\omega}$ such that $s < \alpha$ and $\uparrow^\omega s \subseteq U$. Prove that we have defined a topology: open subsets of A^ω are closed under arbitrary unions and finite intersections. We use $\mathcal{O}(A^\omega)$ for the set of all open sets of A^ω .
- (d) One say that $F \subseteq A^\omega$ is closed if $A^\omega \setminus F$ is open. Prove that for each $s \in A^{<\omega}$, the set $\uparrow^\omega s$ is open and closed. Prove that for each $\alpha \in A^\omega$, the set $\{\alpha\}$ is closed but not open.
- (e) Let $U \subseteq A^\omega$ be open. Prove that there is an antichain B of $A^{<\omega}$ such that $U = \bigcup_{s \in B} \uparrow^\omega s$. Exhibit such antichains for the open set $A^\omega \setminus \{\alpha\}$ (for some $\alpha \in A^\omega$), $\uparrow^\omega s$ and $A^\omega \setminus \uparrow^\omega s$ (for some $s \in A^{<\omega}$).

We set $\uparrow^\omega B = \bigcup_{s \in B} \uparrow^\omega s$.

From now on, we assume that A is finite.

- (f) Prove that if $B \subseteq A^{<\omega}$ is an antichain and a bar then B is finite.

If I and J are sets and $J \subseteq I$, we use ε_J for the element of $\mathbb{R}_{\geq 0}^I$ such that $(\varepsilon_J)_j = 1$ if $j \in J$ and $(\varepsilon_J)_j = 0$ otherwise. One defines a PCS S by setting $|S| = A^{<\omega}$ and $P(S) = \{\varepsilon_B \mid B \subseteq |S| \text{ and } B \text{ is an antichain}\}^\perp$ (the technical condition trivially holds, you are not asked to check this fact). In other words, $u \in \mathbb{R}_{\geq 0}^{|S|}$ is in $P(S)$ iff for all antichain B of $A^{<\omega}$, one has $\sum_{s \in B} u_s \leq 1$.

We define $\theta \in \mathbb{R}_{\geq 0}^{|S| \times |S|}$ by

$$\theta_{s,s'} = \begin{cases} 1 & \text{if } s = s'a \text{ for some } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

(g) Prove that $\theta \in \mathbf{Pcoh}(S, S)$.

Let $M = \{u \in P(S) \mid \theta \cdot u = u\}$. In other words, $u \in M$ if $u \in P(S)$ and, for all $s \in |S|$, one has $u_s = \sum_{a \in A} u_{s\langle a \rangle}$.

(h) Let $u \in M$ and let B be an antichain which is a bar. Prove that, for all $s \in A^{<\omega}$, one has $\sum_{t \in B} u_{st} = u_s$. [*Hint*: Use the fact that B is finite and prove the result by induction on $\#B$.]

Let $B, C \subseteq |S|$ be antichains such that $\uparrow^\omega B = \uparrow^\omega C$.

- (i) Given $s \in B$ let $B^+(s) = \{s' \in A^{<\omega} \mid ss' \in C \text{ and } s' \neq \langle \rangle\}$. Given $t \in C$, define similarly $C^+(t) = \{t' \in A^{<\omega} \mid tt' \in B \text{ and } t' \neq \langle \rangle\}$. Prove that if $B^+(s) \neq \emptyset$ then $B^+(s)$ is a bar.
- (j) Prove that if $s \notin B \cap C$ and $B^+(s) = \emptyset$, then there is $t \in C$ such that $s \in tC^+(t)$.

Deduce from this fact that B is the disjoint unions for the following sets:

- $B \cap C$;
- the set of all $s \in B$ such that $B^+(s) \neq \emptyset$;
- and, for each $t \in C$ such that $C^+(t) \neq \emptyset$, the set $tC^+(t)$.

Explain why these sets are indeed disjoint.

(k) Let $u \in M$. Prove that $\sum_{s \in B} u_s = \sum_{t \in C} u_t$.

So, given $u \in M$, we can define a map $\mu_u : \mathcal{O}(A^\omega) \rightarrow [0, 1]$ by $\mu_u(U) = \sum_{s \in B} u_s$ where B is an antichain such that $\uparrow^\omega B = U$: we have proven that this value does not depend on the choice of B .

Continuing on this line it is possible to prove that M can be identified with the set of all subprobability measures on the topological space A^ω , equipped with its Borel σ -algebra.

- (l) Prove that if $u \in M$ then $\|u\|_S = u_{\langle \rangle}$. [*Hint*: Given a non-empty and finite antichain B , define the weight $w(B)$ of B as the sum of the lengths of its elements, which is in $\mathbb{N}$. Prove by induction on $w(B)$ that $\sum_{s \in B} u_s \leq u_{\langle \rangle}$. For this purpose observe that if $w(B) > 0$ then B has an element s whose length is maximal and > 0 so that $s = t\langle a \rangle$ for some $t \in A^{<\omega}$ and $a \in A$. Prove that $B' = (B \setminus \{t\langle b \rangle \mid b \in A\}) \cup \{t\}$ is an antichain (which is obviously finite and non-empty) with $w(B') < w(B)$ and that $\sum_{s' \in B} u_{s'} \leq \sum_{s' \in B'} u_{s'}$.]

Let C be a cone (see TD2). We say that $p \in C$ is an atom if $p \neq 0$ and, whenever $u_1, u_2 \in C$ satisfy $p \leq u_1 + u_2$, there is $\varepsilon \in \mathbb{R}_{>0}$ such that $\varepsilon p \leq u_i$ for $i = 1$ or $i = 2$.

- (m) Prove that if p is an atom of C and $\lambda \in \mathbb{R}_{>0}$, then λp is an atom.
- (n) Let X be a PCS; prove that in the cone $\bar{P}(X)$, if $u \in \bar{P}(X)$ and $u \neq 0$, there is an atom p such that $p \leq u$.

Let $\bar{M} = \{u \in \bar{P}(S) \mid \theta \cdot u = u\} = \{\lambda u \mid u \in M\}$, which is obviously a cone (for the same cone operations as those of $\bar{P}(S)$, in particular the norm of M is defined exactly as in $\bar{P}(S)$).

- (o) Given $\alpha \in A^\omega$, we define $\delta(\alpha) \in \mathbb{R}_{\geq 0}^{|S|}$ by $\delta(\alpha)_s = 1$ if $s < \alpha$ and $\delta(\alpha)_s = 0$ otherwise. Prove that $\delta(\alpha) \in M$.

- (p) Prove that if $p \in \overline{M}$ is an atom, then $p = \lambda\delta(\alpha)$ for some $\lambda \in \mathbb{R}_{>0}$ and $\alpha \in A^\omega$. [*Hint:* Let $\lambda = p_{\langle \rangle}$. Prove first that the set $D = \{s \in A^{<\omega} \mid p_s = \lambda\}$ is totally ordered by the prefix order and $\forall s, t \in |S| \ s \leq t \in D \Rightarrow s \in D$. If D is finite, derive a contradiction.].
- (q) Let $u \in \mathbb{R}_{\geq 0}^{|S|}$ be defined by $u_s = k^{-\text{len}(s)}$ where $k = \#A$. Prove that $u \in M$ and that there is not atom p in $\overline{M}$ such that $p \leq u$. Comment!