MPRI 2–2 TD 2

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1) Let E be a coherence space. We use $\mathsf{Cl}(E)$ for the set of cliques of E.

1.1) Let $X = (|E|, \left\{x \in \mathbb{R}_{\geq 0}^{|E|} \mid \forall u' \in \mathsf{Cl}(E^\perp)\ \sum_{a\in u'} x_a \leq 1\right\})$. Prove that X is a probabilistic coherence space (PCS). We use $\mathsf{p}(E)$ to denote this PCS.

1.2) Let E be a coherence space and let $x \in \mathsf{P}(\mathsf{p}(E))$. Prove that $\|x\|_{\mathsf{p}(E)} = \sup_{u'\in\mathsf{Cl}(E^\perp)} \sum_{a\in u'} x_a$.

1.3) Let $t \in \mathsf{Cl}(E \multimap F)$ (where E and F are coherence spaces). We defined $\mathsf{p}(t) \in \mathbb{R}_{\geq 0}^{|E|\times|F|}$ by

$$\mathsf{p}(t)_{a,b} = \begin{cases} 1 & \text{if } (a,b) \in t \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\mathsf{p}(t) \in \mathbf{Pcoh}(\mathsf{p}(E), \mathsf{p}(F))$.

1.4) Prove that $\mathsf{p}(\_)$ defined in the two previous questions is a functor from the category **Coh** of coherence spaces and linear morphisms to **Pcoh**.

Remember that if $(E_i)_{i\in I}$ is a family of coherence spaces, then the coherence space $\with_{i\in I} E_i$ is defined as follows:

- $|E| = \bigcup_{i\in I} \{i\} \times |E_i|$
- and $(i,a) \coh_E (j,b)$ if $i = j \Rightarrow a \coh_{E_i} b$.

And $F = \bigoplus_{i\in I} E_i$ is defined as follows:

- $|E| = \bigcup_{i\in I} \{i\} \times |E_i|$
- and $(i,a) \frown_F (j,b)$ if $i = j$ and $a \frown_{E_i} b$.

1.5) Let $(E_i)_{i\in I}$ be a family of coherence spaces. Prove that $\mathsf{p}(\with_{i\in I} E_i) = \with_{i\in I} \mathsf{p}(E_i)$ (this property relates the $\with$ of ordinary coherence spaces and the $\with$ of PCSs).

1.6) Let $(E_i)_{i\in I}$ be a family of coherence spaces. Prove that $\mathsf{p}(\bigoplus_{i\in I} E_i) = \bigoplus_{i\in I} \mathsf{p}(E_i)$ (use the characterization of the $\oplus$ of PCSs in terms of the norm).

1.7) Let $\mathcal{S}$ be the least set of coherence spaces which contains 1 (the coherence space whose web is $\{*\}$) and such that

- if $E \in \mathcal{S}$ then $E^\perp \in \mathcal{S}$
- and if $(E_i)_{i\in I}$ is a family of elements of $\mathcal{S}$, then $\with_{i\in I} E_i \in \mathcal{S}$.

Prove that, for any $E \in \mathcal{S}$, one has $\mathsf{p}(E^\perp) = \mathsf{p}(E)^\perp$.

An *embedding* from a coherence space E into a coherence space F is an injective function $f : |E| \to |F|$ such that, for all $a, b \in |E|$, one has $a \coh_E b \Leftrightarrow f(a) \coh_F f(b)$. If there is such an embedding we say that *E embeds in F*.

1.8) If $k \in \mathbb{N}$, let C_k be the coherence space such that $|C_k| = \{1, \dots, k\}$ and where $1 \frown 2$, $2 \frown 3 \dots$, $k \frown 1$ are the only coherent pairs (the cycle of length k). Prove that it is not true that $\mathsf{P}(\mathsf{p}({C_5}^\perp)) = \mathsf{P}(\mathsf{p}(C_5))^\perp$.

1.9) Prove that if C_5 embeds in a coherence space E then it is not true that $\mathsf{P}(\mathsf{p}(E^\perp)) = \mathsf{P}(\mathsf{p}(E))^\perp$.

1.10) Generalize the above to all C_k's with k odd.

2) Let X and Y be PCSs and $f : \mathsf{P}(X) \to \mathsf{P}(Y)$ be a function which is monotone, Scott continuous and linear in the sense that for all $x(1), x(2) \in \mathsf{P}(X)$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$, if $\lambda_1 x(1) + \lambda_2 x(2) \in \mathsf{P}(X)$ then $f(\lambda_1 x(1) + \lambda_2 x(2)) = \lambda_1 f(x(1)) + \lambda_2 f(x(2))$.

2.1) For each $a \in |X|$ let $N(a) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \lambda \mathsf{e}_a \in \mathsf{P}(X)\}$. Prove that $0 < N(a) < \infty$ and that $N(a)\mathsf{e}_a \in \mathsf{P}(X)$.

2.2) We define $s \in \mathbb{R}_{\geq 0}^{|X| \times |Y|}$ by

$$s_{a,b} = \frac{f(N(a)\mathsf{e}_a)_b}{N(a)} \in \mathbb{R}_{\geq 0}\,.$$

Given $x \in \mathsf{P}(X)$ let $\mathsf{supp}(x) = \{a \in |X| \mid x_a \neq 0\}$. Prove that if $\mathsf{supp}(x)$ is a finite set then $f(x) = s \cdot x$.

2.3) Given $x \in \mathsf{P}(X)$ and $I \subseteq |X|$ let $x(I) \in \mathbb{R}_{\geq 0}^{|X|}$ be defined by

$$x(I)_a = \begin{cases} x_a & \text{if } a \in I \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $x(I) \in \mathsf{P}(X)$, that $\{x(I) \mid I \in \mathcal{P}_{\mathrm{fin}}(|X|)\}$ is directed in $\mathsf{P}(X)$ (where $\mathcal{P}_{\mathrm{fin}}(E)$ is the set of all finite subsets of E) and that

$$x = \sup\{x(I) \mid I \in \mathcal{P}_{\mathrm{fin}}(|X|)\}\,.$$

2.4) Prove that $s \in \mathbf{Pcoh}(X, Y)$ and that $\forall x \in \mathsf{P}(X)\ f(x) = s \cdot x$.

3) We define $T_0, T_1 \in \mathbf{Pcoh}(!(!1 \multimap 1) \otimes !1, 1)$, keeping often implicit the monoidality isomorphisms of $\mathbf{Pcoh}$. T_0 is the following composition of morphisms in $\mathbf{Pcoh}$:

$$!(!1 \multimap 1) \otimes !1 \xrightarrow{\mathsf{w} \otimes !1} !1 \xrightarrow{\mathsf{der}} 1$$

and T_1 is the following composition of morphisms in $\mathbf{Pcoh}$:

$$\begin{tikzcd}
!(!1 \multimap 1) \otimes !1 \arrow[r, "\mathsf{c}\otimes\mathsf{c}"] & !(!1 \multimap 1) \otimes !(!1 \multimap 1) \otimes !1 \otimes !1 \arrow[r, "\sim"] & !(!1 \multimap 1) \otimes !1 \otimes !(!1 \multimap 1) \otimes !1 \arrow[d, "\mathsf{der}\otimes !1 \otimes \mathsf{der} \otimes !1"] \\
1 & 1 \otimes 1 \arrow[l, "\sim"'] & (!1 \multimap 1) \otimes !1 \otimes (!1 \multimap 1) \otimes !1 \arrow[l, "\mathsf{ev}\otimes\mathsf{ev}"']
\end{tikzcd}$$

3.1) Let $t \in \mathsf{P}(!1 \multimap 1)$ and $x \in \mathsf{P}(1)$ (that we identify with the unit interval $[0,1]$). Prove that

$$T_0 \cdot (t^{(!)} \otimes x^{(!)}) = x$$
$$T_1 \cdot (t^{(!)} \otimes x^{(!)}) = (t \cdot x^{(!)})^2$$

Let $p \in [0,1]$. We assume that $p \neq 0$.

3.2) Let $S_i = \mathsf{cur}(T_i) \in \mathbf{Pcoh}(!(!1 \multimap 1), !1 \multimap 1) = \mathsf{P}(!(!1 \multimap 1) \multimap (!1 \multimap 1))$ for $i = 1, 2$ and let $S = (1-p)S_0 + pS_1$. Explain why $S_i = \mathsf{cur}(T_i) \in \mathbf{Pcoh}(!(!1 \multimap 1), !1 \multimap 1)$ and show that for any $t \in \mathsf{P}(!1 \multimap 1)$, the morphism $s = S \cdot t^{(!)} \in \mathbf{Pcoh}(!1, 1)$ satisfies

$$\forall x \in \mathsf{P}(1) \quad s \cdot x^{(!)} = (1-p)x + p(t \cdot x^{(!)})^2$$

Remember that the function $F : t \mapsto S \cdot t^{(!)}$ is monotone and Scott continuous, and hence has a least fixed point.

3.3) Let s_0 be the least fixed point of F, and let $f : [0,1] \to [0,1]$ be the associated function (that is $f(x) = s_0 \cdot x^{(!)}$). Prove that we must have

$$\forall x \in [0,1] \quad f(x) = \frac{1 + \alpha(x)\sqrt{1 - 4p(1-p)x}}{2p}$$

where $\forall x \in [0,1]\ \alpha(x) \in \{-1, 1\}$.

3.4) Prove that $f(0) = 0$ and that $\forall x \in]0,1]\ \alpha(x) = -1$.

3.5) Plot the function f for $p = \frac{1}{4}$, $p = \frac{1}{2}$, $p = \frac{3}{4}$ and $p = 1$.

3.6) Since $s_0 \in \mathsf{P}(!1 \multimap 1)$, we can consider s_0 as an element of $\mathbb{R}_{\geq 0}^{\mathbb{N}}$. Using the Taylor expansion of $\sqrt{1-u}$ compute the value of $(s_0)_n$ for each $n \in \mathbb{N}$.

4) 4.1) Let X and Y be PCSs, let $\varphi : |X| \to |Y|$ be a bijection and let $v : |X| \to \mathbb{R}_{\geq 0}$ be such that $\forall a \in |X|\ v(a) \neq 0$. Let $s = \mathsf{mat}(\varphi, v) \in \mathbb{R}_{\geq 0}^{|X| \times |Y|}$ be given by

$$\mathsf{mat}(\varphi, v)_{a,b} = \begin{cases} v(a) & \text{if } \varphi(a) = b \\ 0 & \text{otherwise.} \end{cases}$$

We assume that

$$\forall u \in \mathbb{R}_{\geq 0}^{|X|} \quad u \in \mathsf{P}(X) \Leftrightarrow \mathsf{mat}(\varphi, v) \cdot u \in \mathsf{P}(Y)\,.$$

Prove that $\mathsf{mat}(\varphi, v)$ is an iso in the category **Pcoh**, with inverse $\mathsf{mat}(\varphi^{-1}, v')$ where $v'(b) = 1/v(\varphi^{-1}(b))$. An iso of shape $\mathsf{mat}(\varphi, v)$ will be called quasi-strong.

We want to prove that any iso of PCS is quasi-strong. So let $t \in \mathbf{Pcoh}(X, Y)$ be an iso and $t^{-1} \in \mathbf{Pcoh}(Y, X)$ be its inverse.
Let $a \in |X|$ and let $\alpha = \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \lambda \mathsf{e}_a \in \mathsf{P}(X)\}$. Remember that $\alpha > 0$.

4.2) Prove that $\alpha \mathsf{e}_a \in \mathsf{P}(X)$ and that $t \cdot \alpha \mathsf{e}_a \neq 0$.

Let $b, b' \in |Y|$ be such that $(t \cdot \alpha \mathsf{e}_a)_b \neq 0$ and $(t \cdot \alpha \mathsf{e}_a)_{b'} \neq 0$. Let $\beta = (t \cdot \alpha \mathsf{e}_a)_b$ and $\beta' = (t \cdot \alpha \mathsf{e}_a)_{b'}$.

4.3) Observe that, for the standard order relation on PCSs ($x(1) \leq x(2)$ if $\forall a' \in |X|\ x(1)_{a'} \leq x(2)_{a'}$) one has $\beta \mathsf{e}_b \leq t \cdot \alpha \mathsf{e}_a$ and deduce that there must exist $\gamma \leq \alpha$ such that $t^{-1} \cdot \beta \mathsf{e}_b = \gamma \mathsf{e}_a$. Prove that $\gamma \neq 0$.

Similarly we have $\gamma' > 0$ such that $t^{-1} \cdot \beta' \mathsf{e}_{b'} = \gamma' \mathsf{e}_a$. Without loss of generality we can assume that $\gamma' \leq \gamma$.

4.4) Prove that $\frac{\gamma' \beta}{\gamma} \mathsf{e}_b = \beta' \mathsf{e}_{b'}$. Deduce that $b = b'$. As a consequence $\beta' = (t \cdot \alpha \mathsf{e}_a)_b = \beta$.

4.5) Prove that $t \cdot \alpha \mathsf{e}_a = \beta \mathsf{e}_b$.

4.6) Deduce from the above that there is a function $\varphi : |X| \to |Y|$ and a function $v : |X| \to \mathbb{R}_{\geq 0} \setminus \{0\}$ such that $\forall a \in |X|\ t_{a,b} \neq 0 \Rightarrow b = \varphi(a)$ and $t_{a,\varphi(a)} = v(a)$.

4.7) Prove that t is quasi-strong.