

MPRI SEMPL

TD2: PCSs and cones

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Remember that if $u \in \mathbb{R}_{\geq 0}^I$ and $v \in \mathbb{R}_{\geq 0}^J$, then $u \otimes v \in \mathbb{R}_{\geq 0}^{I \times J}$ is defined by $(u \otimes v)_{i,j} = u_i v_j$. Remember also that if $u \in \mathbf{P}(X)$ then we set

$$\|u\|_X = \sup\{\langle u, u' \rangle \mid u' \in \mathbf{P}(X^\perp)\} \leq 1.$$

We also recall that the “technical condition”, satisfied by a PCS X , means that for all $a \in |X|$:

- there is $u \in \mathbf{P}(X)$ such that $u_a > 0$, in other words, there is $\alpha \in \mathbb{R}_{\geq 0}$ such that $\alpha > 0$ and $\alpha e_a \in \mathbf{P}(X)$
 - and there is $\lambda \in \mathbb{R}_{\geq 0}$ such that for all $u \in \mathbf{P}(X)$ one has $u_a \leq \lambda$.
1. (a) Let X and Y be PCS. Prove that $X \multimap Y$ satisfies the “technical condition”.
 2. (a) Prove that for all $u \in \mathbf{P}(X)$ and $u' \in \mathbf{P}(X^\perp)$, one has $\langle u, u' \rangle \leq \|u\|_X \|u'\|_{X^\perp}$.
 (b) Let $t \in \mathbf{Pcoh}(X, Y)$. Prove that $\|t\|_{X \multimap Y} = \sup_{u \in \mathbf{P}(X)} \|t \cdot u\|_Y$. Prove also that $\forall u \in \mathbf{P}(X) \quad \|t \cdot u\|_Y \leq \|t\|_{X \multimap Y} \|u\|_X$.
 (c) Given $u' \in \mathbf{P}(X^\perp)$ and $v \in \mathbf{P}(Y)$, prove that $u' \otimes v \in \mathbf{P}(X \multimap Y)$.
 (d) Let $u \in \mathbf{P}(X)$ and $v \in \mathbf{P}(Y)$, prove that $\|u \otimes v\|_{X \otimes Y} = \|u\|_X \|v\|_Y$.
 3. We define a *pre-cone* as a set C equipped with
 - a commutative monoid structure (neutral element $0 \in C$ and operation $+$: $C^2 \rightarrow C$ called *addition*) which is *cancellative*, that is $u + v = u' + v \Rightarrow u = u'$.
 - a scalar multiplication $\mathbb{R}_{\geq 0} \times C \rightarrow C$ mapping (λ, u) to λu satisfying the usual module laws:
 - $\lambda(u + v) = \lambda u + \lambda v$ (which implies $\lambda 0 = 0$ by cancellativity upon taking $u = v = 0$)
 - $(\lambda + \mu)u = \lambda u + \mu u$ (which implies $0u = 0$ by cancellativity upon taking $\lambda = \mu = 0$)
 - $1u = u$ and $(\lambda\mu)u = \lambda(\mu u)$.
 - and a norm $\|_ \| : C \rightarrow \mathbb{R}_{\geq 0}$ subject to:
 - $\|u\| = 0 \Rightarrow u = 0$
 - $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \|\lambda u\| = \lambda \|u\|$ (which implies $\|0\| = 0$ by taking $u = 0$ and $\lambda = 0$)
 - $\|u + v\| \leq \|u\| + \|v\|$
 - $\|u\| \leq \|u + v\|$
 for all $u, v \in C$.
 - (a) Using the norm, prove that a pre-cone is *positive*, that is $u + v = 0 \Rightarrow u = 0$.
 - (b) We equip each pre-cone C with a binary relation $\leq$ defined by $u \leq v$ if $\exists w \in C \quad u + w = v$. Prove that $\leq$ is an order relation on C . Notice that by cancellativity, if there is w such that $u + w = v$, this w is unique, so it makes sense to introduce a notation for it: we set $w = v - u$, but be careful that $v - u$ makes sense only if we already know that $u \leq v$!

(c) Prove some basic properties of this order relation, in particular:

- if $\lambda_1 \leq \lambda_2$ in $\mathbb{R}_{\geq 0}$ and $u \in P(X)$ then $\lambda_1 u \leq \lambda_2 u$ and $\lambda_2 u - \lambda_1 u = (\lambda_2 - \lambda_1)u$
- $u_1 \leq u_2 \Rightarrow u_1 + v \leq u_2 + v$, and then $(u_2 + v) - (u_1 + v) = u_2 - u_1$
- $u_1 \leq u_2 \Rightarrow \lambda u_1 \leq \lambda u_2$, and then $\lambda u_2 - \lambda u_1 = \lambda(u_2 - u_1)$
- $v \leq u_1 \leq u_2 \Rightarrow u_1 - v \leq u_2 - v$, and then $(u_2 - v) - (u_1 - v) = u_2 - u_1$
- $v_1 \leq v_2 \leq u \Rightarrow u - v_2 \leq u - v_1$, and then $(u - v_1) - (u - v_2) = v_2 - v_1$
- $u_1 \leq u_2 \Rightarrow \|u_1\| \leq \|u_2\|$.

4. In this exercise, we complete the definition of cones, by introducing an order-theoretic completeness condition. We say that a subset A of a cone C is bounded if there is a $\lambda \in \mathbb{R}_{\geq 0}$ such that $\forall u \in A \ \|u\| \leq \lambda$.

- (a) Let C be a pre-cone and let $A \subseteq C$. Assume that A has a least upper bound (lub) u_0 in C . Given $v \in C$ and $\lambda \in \mathbb{R}_{\geq 0}$, prove that the sets $A + v = \{u + v \mid u \in A\}$ and $\lambda A = \{\lambda u \mid u \in A\}$ have a lub in C , which are respectively $u + v$ (under the proviso that $A \neq \emptyset$) and λu .

A cone is a pre-cone C which is (ω -)complete in the following sense: each increasing sequence $(u_n)_{n \in \mathbb{N}}$ of elements of C such that $\forall n \in \mathbb{N} \ \|u_n\| \leq 1$ has a least upper bound u , and u satisfies $\|u\| \leq 1$. In the next questions we assume that C is a cone.

- (b) Let $\lambda \in \mathbb{R}_{\geq 0}$ and $(u_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements of C such that $\forall n \in \mathbb{N} \ \|u_n\| \leq \lambda$. Prove that $(u_n)_{n \in \mathbb{N}}$ has a least upper bound u in C and that $\|u\| \leq \lambda$.
- (c) Given $v \in C$ and an increasing sequence $(u_n)_{n \in \mathbb{N}}$ which is bounded (remember that this means that the $\|u_n\|$ s are bounded in $\mathbb{R}_{\geq 0}$), prove that $\sup_{n \in \mathbb{N}}(u_n + v) = (\sup_{n \in \mathbb{N}} u_n) + v$. Given $\lambda \in \mathbb{R}_{\geq 0}$, we know that $\sup_{n \in \mathbb{N}} \lambda u_n = \lambda \sup_{n \in \mathbb{N}} u_n$ by one of the questions above.
- (d) Let $(u_n)_{n \in \mathbb{N}}$ be an increasing sequence and let $u \in C$ be such that $\forall n \in \mathbb{N} \ u_n \leq u$ and $\lim_{n \rightarrow \infty} \|u - u_n\| = 0$. Prove that $u = \sup_{n \in \mathbb{N}} u_n$.
- (e) Let $u \in C$ and let λ_n be an increasing sequence in $\mathbb{R}_{\geq 0}$ such that $\sup_{n \in \mathbb{N}} \lambda_n = 1$. Prove that $\sup_{n \in \mathbb{N}} \lambda_n u = u$. More generally let $(\lambda_n)_{n \in \mathbb{N}}$ be an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$. Prove that $\sup_{n \in \mathbb{N}} \lambda_n u = (\sup_{n \in \mathbb{N}} \lambda_n) u$.
- (f) Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be two increasing and bounded sequences in C , and assume that $(u_n + v_n)_{n \in \mathbb{N}}$ is also increasing. Prove that $\sup_{n \in \mathbb{N}}(u_n + v_n) = \sup_{n \in \mathbb{N}} u_n + \sup_{n \in \mathbb{N}} v_n$.
- (g) Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be two increasing sequences in C such that $(v_n)_{n \in \mathbb{N}}$ is bounded, $\forall n \in \mathbb{N} \ u_n \leq v_n$ and $(v_n - u_n)_{n \in \mathbb{N}}$ is increasing. Prove that $\sup_{n \in \mathbb{N}}(v_n - u_n) = \sup_{n \in \mathbb{N}} v_n - \sup_{n \in \mathbb{N}} u_n$.

5. Let X be a PCS, we set $\bar{P}(X) = \{u \in \mathbb{R}_{\geq 0}^{|X|} \mid \exists \lambda \in \mathbb{R}_{\geq 0} \ u \in \lambda P(X)\} = \bigcup_{\lambda \in \mathbb{R}_{\geq 0}} \lambda P(X) \subseteq \mathbb{R}_{\geq 0}^{|X|}$.

- (a) Prove that, equipped with the map $\|_ \| : \bar{P}(X) \rightarrow \mathbb{R}_{\geq 0}$ defined by $\|u\| = \inf\{\lambda \in \mathbb{R}_{\geq 0} \mid u \in \lambda P(X)\}$ that you will prove to be a norm, $\bar{P}(X)$ is a cone (algebraic operations being defined in the obvious pointwise way). Prove also that, in this cone, $u \leq v$ holds iff $\forall a \in |X| \ u_a \leq v_a$. You can use, without giving the proof, some results on PCSs which have been presented during the lectures.
- (b) Consider the PCS $B = \bigotimes_{n \in \mathbb{N}} 1$, that is, up to trivial iso: $|B| = \mathbb{N}$ and $P(B) = \{u \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \forall n \in \mathbb{N} \ u_n \leq 1\}$. Prove that $\bar{P}(B)$ is the set of all $u \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ such that $(u_i)_{i \in \mathbb{N}}$ is bounded and that $\|u\|_B = \sup_{i \in \mathbb{N}} u_i$.
- (c) Exhibit an increasing sequence $(u_n \in \bar{P}(B))_{n \in \mathbb{N}}$ and an element $u \in \bar{P}(B)$ such that $\sup_{n \in \mathbb{N}} u_n = u$ and $\forall n \in \mathbb{N} \ \|u - u_n\| = 1$. Comment.