

MPRI SEMPL

TD1: Probabilistic Coherence Spaces

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Warning: here, we call ω -cpo any partially ordered set D where any increasing sequence $(x_n)_{n \in \mathbb{N}}$ of elements of D has a least upper bound (lub) $\sup_{n \in \mathbb{N}} x_n$. When dealing with probabilities, this notion of completeness is more relevant than the standard completeness used in domain theory which is based on arbitrary directed sets.

Accordingly, if D and E are ω -cpo's, an increasing function $f : D \rightarrow E$ is ω -Scott continuous if f commutes with the lubs of increasing sequences.

Notice that $[0, 1] \subseteq \mathbb{R}$, ordered with the standard order relation on the real numbers, is an ω -cpo.

1. Let I be a finite set and let $(u_n \in \mathbb{R}_{\geq 0}^I)_{n \in \mathbb{N}}$ be an increasing sequence (for the product order).
 - (a) Assume first that for some $i \in I$, the sequence $(u_{n,i})_{n \in \mathbb{N}}$ is unbounded. Prove that $\sum_{i \in I} \sup_{n \in \mathbb{N}} u_{n,i} = \sup_{n \in \mathbb{N}} \sum_{i \in I} u_{n,i} = \infty$.

Assume now that for each $i \in I$, the sequence $(u_{n,i})_{n \in \mathbb{N}}$ is bounded and let $u_i \in \mathbb{R}_{\geq 0}$ be its least upper bound.

- (b) Prove that $\sum_{i \in I} u_i \geq \sup_{n \in \mathbb{N}} \sum_{i \in I} u_{n,i}$
- (c) Let $\varepsilon > 0$. Prove that there is $N \in \mathbb{N}$ such that $\forall i \in I \forall n \in \mathbb{N} n \geq N \Rightarrow (u_i - u_{n,i}) \leq \varepsilon / (\#I + 1)$.
- (d) Prove that, for $n \geq N$, one has $\sum_{i \in I} u_{n,i} \geq (\sum_{i \in I} u_i) - \varepsilon$, and conclude that $\sum_{i \in I} \sup_{n \in \mathbb{N}} u_{n,i} = \sup_{n \in \mathbb{N}} \sum_{i \in I} u_{n,i}$.

We do not assume that I is finite anymore.

- (e) Prove that $\sum_{i \in I} \sup_{n \in \mathbb{N}} u_{n,i} = \sup_{n \in \mathbb{N}} \sum_{i \in I} u_{n,i}$.
 - (f) Give an example showing that this equation does not necessarily hold if we don't assume $(u_n)_{n \in \mathbb{N}}$ to be increasing.
 - (g) Let X be a probabilistic coherence space (PCS for short). Let $(u_n \in \mathcal{P}(X))_{n \in \mathbb{N}}$ be an increasing sequence (for the pointwise order). Prove that $u = \sup_{n \in \mathbb{N}} u_n$ (given by $u_a = \sup_{n \in \mathbb{N}} u_{n,a}$ for each $a \in |X|$) satisfies $u \in \mathcal{P}(X)$.
2. Let X be a probabilistic coherence space (PCS for short) and let $\mathcal{A} \subseteq \mathbb{R}_{\geq 0}^{|X|}$ be such that $\mathcal{P}(X) = \mathcal{A}^{\perp\perp}$.
 - (a) Let Y be a PCS and let $t \in \mathbb{R}_{\geq 0}^{|X| \times |Y|}$. Prove that $t \in \mathcal{P}(X \multimap Y)$ iff $\forall u \in \mathcal{A} \ t \cdot u \in \mathcal{P}(Y)$.
 - (b) Let $u' \in \mathcal{P}(X^\perp)$, prove that $\|u'\|_{X^\perp} = \sup\{\langle u, u' \rangle \mid u \in \mathcal{A}\}$. Remember that, by definition, $\|u'\|_{X^\perp} = \sup\{\langle u, u' \rangle \mid u \in \mathcal{P}(X)\}$.
 - (c) Give examples of a PCS X , of $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $\mathcal{P}(X) = \mathcal{A}^{\perp\perp}$, of a PCS Y and of $s, t \in \mathcal{P}(X \multimap Y)$ with $s \neq t$ such that $\forall u \in \mathcal{A} \ t \cdot u = s \cdot u$.