MPRI 2-02 2021-2022 — Semantics
Part II

Denotational semantics of functional languages and linear logic

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https://www.irif.fr/~ehrhard/pub/mpri-2020-2021.pdf

What is Denotational Semantics about?

Denotational semantics (initially *mathematical semantics*) has been invented by Christopher Strachey and Dana Scott in 1969.

The goal: provide a mathematical interpretation of programs. Strachey was promoting such an interpretation since the beginning of the 1960's. What do programs do?

Strachey's idea: programs as functions

- A functional program maps "values" to "values",
- a program (with side effects) maps states (of the machine) to states.

Problem

What kind of functions acting on what kind of spaces?

D. Scott: the invention of DS

Dana Scott (a logician, student of Alonzo Church), probably inspired by the Rice Shapiro theorem (1959), found an answer.

Scott was also looking for models of the pure λ-calculus, that is a "universe" where we can have a non-trivial object $\mathcal{X}$ such that

$$(\mathcal{X} \Rightarrow \mathcal{X}) \subseteq \mathcal{X}$$

(impossible in **Set**, the category of sets and functions, for cardinality reasons).

Partial recursive functions

A partial recursive function is a partial function $\mathbb{N} \to \mathbb{N}$ which can be computed by a program. Partiality: for some values of the argument, the program may loop.

The partial rec. fun. φ_n: partial recursive functions are defined by programs which are finite sequences of symbols, and so there are only countably many programs, we can enumerate them $p_0, p_1, \ldots$ Then φ_n is the partial recursive function computed by program p_n.

$F \subseteq \mathbb{N}$ recursively enumerable: there is an integer n such that

$$k \in F \Leftrightarrow \varphi_n(k) \text{ is defined.}$$

Finite function: a partial function $\theta : \mathbb{N} \to \mathbb{N}$ is finite if the set of n's such that $\theta(n)$ is defined is finite. Any such finite θ is partial recursive.

Theorem (Rice Shapiro)

Let F be a set of partial recursive functions $\mathbb{N} \to \mathbb{N}$ such that $\{n \mid \varphi_n \in F\}$ is recursively enumerable.

Then $\psi \in F$ if and only if there is a finite function $\theta \subseteq \psi$ such that $\theta \in F$.

The hypothesis means $\exists k \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \quad \varphi_n \in F \Leftrightarrow \varphi_k(n) \text{ defined.}$$

More intuitively but less accurately: let $\mathcal{N}$ be the set of partial recursive functions $\mathbb{N} \to \mathbb{N}$ and $F : \mathcal{N} \to \mathbb{N}$ be partial "computable". Then

- if $f, g \in \mathcal{N}$ with $f \subseteq g$ (as graphs) and $F(f) = n$ then $F(g) = n$ (F is monotone)
- and if $F(f) = n$ there is a finite function $f_0 \subseteq f$ such that $F(f_0) = n$.

Intuition

A computation takes a finite amount of time and hence, to produce a finite information (here n), a program (here F) can explore only a finite part (here f_0) of its argument (here f).

Dana Scott's great idea: recursion theory without computability!

Forget computability, keep only the order theoretic aspects of Rice Shapiro and generalize it to all types.

Replace $\mathcal{N}$ by the set of all partial functions $\mathbb{N} \to \mathbb{N}$, not only the computable ones, ordered by inclusion of graphs.

Then consider partial $F : \mathcal{N} \to \mathbb{N}$ such that

- if $f, g \in \mathcal{N}$ with $f \subseteq g$ (as graphs) and $F(f) = n$ then $F(g) = n$ (F is monotone)
- and if $F(f) = n$ there is a finite function $f_0 \subseteq f$ such that $F(f_0) = n$.

This property is exactly Scott continuity!

It can be extended to much more general objects than $\mathcal{N}$: *domains* (partially ordered sets with some order-completeness properties).

Scott and Strachey: use Scott continuous functions to interpret programs.

Scott: it also works for the pure λ-calculus.

This is the basic idea of Denotational Semantics.

Cartesian closed categories (CCC)

In the 1980's, one understands that categories are a useful tool for describing such denotational models, *especially when morphisms are not functions*.

The notion of cartesian closed categories (CCC) is the right setting for describing denotational models of the λ-calculus and of PCF.

Beyond Scott's initial idea. . .

Several refinements of Scott continuity:

- sequential functions (many people: Jean Vuillemin, Vladimir Sazonov, Robin Milner) but does not give a CCC
- stable functions (Gérard Berry, rediscovered by Jean-Yves Girard) CCC
- sequential algorithms (Berry and Pierre-Louis Curien) CCC
- strong stability (Antonio Bucciarelli and E.), a CCC whose "type 1" morphisms are sequential
- various game models
- combinations, refinements etc of the above.

Jean-Yves Girard and DS

- In the early 1980's, Girard develops a model of System F (a second-order typed λ-calculus he discovered 15 years earlier) using *stable functions* on *qualitative domains*.
- He understands that, in this model, (1) standard implication can be decomposed using this new implication: $(X \Rightarrow Y) = (!X \multimap Y)$ where $X \multimap Y$ is a space of *linear* stable maps,
- that (2) only very special qd's arise as interpretations of types: *coherence spaces*,
- and that (3) *linear stable maps* between coherence spaces lead to *involutive* linear negation $X \mapsto (X \multimap \bot)$.

This is at the same time

- the origin of Linear Logic
- and a new approach to denotational semantics based on a notion of linear morphisms.

Major influence on the development of game models in the 1990's.

For this reason, the linear logical structure of models is central in this lecture.

LL is everywhere!

Many models appear to have a linear logical structure or have been directly defined as models of LL:

- Scott semantics itself (Scott continuous functions on prime-algebraic complete lattices) though neither Scott nor Girard did notice
- Hypercoherence semantics (strongly stable functions on hypercoherence spaces, accounting for sequentiality)
- the relational model (objects are sets, morphisms are relations)
- models based on linear algebra: Köthe spaces, finiteness spaces, probabilistic coherence spaces
- and a large number of refinements or combinations of these models.

Another great idea of Dana Scott: introduce a simple, Turing complete, functional programming language for defining denotational interpretations. This is PCF.

Allows to study very cleanly the connection between *operational semantics* (execution of programs etc) and DS.

The language PCF

Lecture notes, Sections 1.1, 1.2 and 1.3

Syntax of PCF

PCF = Programming Computable Functions

Published for the first time in a paper by Gordon Plotkin in 1977.

Our version of PCF

A simply typed λ-calculus with one ground data-type (integers) and a fixpoint operator to implement general recursion.

$$A, B, \ldots := \iota \mid A \Rightarrow B$$

$$\begin{aligned} M, N, P, \ldots := {} & x \mid \underline{n} \mid \mathrm{succ}(M) \mid \mathrm{if}(M, N, x \cdot P) \\ & \mid \lambda x^A\, M \mid (M)\, N \mid \mathrm{fix}(M) \end{aligned}$$

For each $n \in \mathbb{N}$ there is a constant $\underline{n}$ in the language.

Typing rules for PCF

Typing context: $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$, x_i's pairwise distinct variables (finite partial function from variables to types).

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A\, M : A \Rightarrow B} \qquad \frac{\Gamma \vdash M : A \Rightarrow A}{\Gamma \vdash \mathrm{fix}(M) : A}$$

$$\frac{}{\Gamma \vdash \underline{n} : \iota} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \mathsf{succ}(M) : \iota}$$

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : A \qquad \Gamma, z : \iota \vdash Q : A}{\Gamma \vdash \mathsf{if}(M, P, z \cdot Q) : A}$$

Intuition: pattern matching

Main difference wrt. Plotkin's PCF: conditional is a case analysis on M, of type ι:

- (zero) $\quad \mathsf{if}(\underline{0}, N, x \cdot P) \rightsquigarrow N$
- (successor) $\quad \mathsf{if}(\underline{n+1}, N, x \cdot P) \rightsquigarrow P\,[\underline{n}/x]$.

Similar to a pattern matching in Ocaml.

Intuition: fixpoint operator

Purpose: define recursive functions.

In Ocaml (or similar functional languages) on can write

$$\text{let rec } f = M$$

where f can occur free in M, to define f of type A.

For this to make sense we need $f : A \vdash M : A$.

With a fixpoint operator like ours, this would be written

$$\text{let } f = \text{fix}(\lambda f^A\, M)$$

Substitution

Substitution $M\,[N/x]$ is defined as usual, terms are considered up to α-converion to avoid meaningless variable bindings as in:

$$(\lambda x^A\, y)\,[x/y] = \lambda x^A\, x$$

Replace first the substituted term $\lambda x^A\, y$ with the α-equivalent $\lambda z^A\, y$ and then apply the substitution:

$$(\lambda z^A\, y)\,[x/y] = \lambda z^A\, x\,.$$

Operational semantics

How do we compute with this language?

We will provide

- a set of general reduction rules β that turns the language into a rewriting system
- and a rewriting subsystem β_{wh} which is a deterministic strategy, turning PCF into a programming language: *weak head-reduction*.

This strategy can be implemented by means of an abstract machine.

Rewriting rules β

They are presented as a *deduction system* which allows to prove statements of shape $M \; \beta \; M'$ expressing that M reduces to M' in PCF.

Red underlined terms are called *redexes*.

Axioms. Standard β-reduction:

$$\frac{}{\underline{(\lambda x^A \, M) \, N} \; \beta \; M \, [N/x]}$$

Fixpoint unfolding:

$$\frac{}{\underline{\mathrm{fix}(M)} \; \beta \; (M) \, \mathrm{fix}(M)}$$

Case analysis:

$$\frac{}{\mathrm{if}(\underline{0}, P, z \cdot Q) \;\beta\; P} \qquad \frac{}{\mathrm{if}(\underline{n+1}, P, z \cdot Q) \;\beta\; Q\,[\underline{n}/z]}$$

Successor:

$$\frac{}{\mathrm{succ}(\underline{n}) \;\beta\; \underline{n+1}}$$

Important

To reduce $\mathrm{if}(M, P, z \cdot Q)$ we require M to be an integer constant $\underline{n}$, for instance the reduction

$$\mathrm{if}(\mathrm{succ}(N), P, z \cdot Q) \;\beta\; Q\,[N/z]$$

is *not valid*. The integers are dealt with in Call by Value style.

The deduction rules express that reduction can be performed in any context.

$$\frac{M \ \beta \ M'}{\lambda x^A \ M \ \beta \ \lambda x^A \ M'}$$

$$\frac{M \ \beta \ M'}{(M)\,N \ \beta \ (M')\,N} \qquad \frac{N \ \beta \ N'}{(M)\,N \ \beta \ (M)\,N'}$$

$$\frac{M \ \beta \ M'}{\mathrm{fix}(M) \ \beta \ \mathrm{fix}(M')}$$

$$\frac{M \mathbin{\beta} M'}{\mathsf{if}(M, P, z \cdot Q) \mathbin{\beta} \mathsf{if}(M', P, z \cdot Q)}$$

$$\frac{P \mathbin{\beta} P'}{\mathsf{if}(M, P, z \cdot Q) \mathbin{\beta} \mathsf{if}(M, P', z \cdot Q)}$$

$$\frac{Q \mathbin{\beta} Q'}{\mathsf{if}(M, P, z \cdot Q) \mathbin{\beta} \mathsf{if}(M, P, z \cdot Q')}$$

$$\frac{M \mathbin{\beta} M'}{\mathsf{succ}(M) \mathbin{\beta} \mathsf{succ}(M')}$$

β preserves types

Lemma (Substitution)

Let $P, Q \in \mathrm{PCF}$. If $\Gamma, x : A \vdash P : B$ and if $\Gamma \vdash Q : A$, then $\Gamma \vdash P\,[Q/x] : B$.

The proof is a simple induction on P.

Theorem (Subject reduction)

Let $M \in \mathrm{PCF}$. If $\Gamma \vdash M : A$ et $M\ \beta\ M'$, then $\Gamma \vdash M' : A$.

The proof is a simple induction on the derivation that $M\ \beta\ M'$. One uses the Substitution Lemma when $M = (\lambda x^A\, P)\, Q$ and $M' = P\,[Q/x]$.

Abstract rewriting systems

An ARS is a pair (T, θ) where T is a set (the "terms") and θ is a binary relation on a set T (that is $\theta \subseteq T \times T$), a "rewriting relation".

- We use θ^* for the least binary relation φ on T such that φ is transitive and reflexive and $\theta \subseteq \varphi$. Given $t, t' \in T$, one has $t\ \theta^*\ t'$ if and only if there are $t_1, \ldots, t_n \in T$ with $n \geq 1$ such that $t = t_1$, $t' = t_n$ and $t_i\ \theta\ t_{i+1}$ for $i = 1, \ldots, n-1$. It is called the *reflexive transitive closure* of θ.
- We define similarly θ^- as the *reflexive closure* of θ: $t\ \theta^-\ t'$ if $t = t'$ or $t\ \theta\ t'$.
- $t \in T$ is *θ-normal* if there is no $t' \in T$ such that $t\ \theta\ t'$.

The Church Rosser property

- We say that θ has the Diamond Property (DP) if

$$\forall t, t_1, t_2 \in T \; t \; \theta \; t_1 \text{ and } t \; \theta \; t_2 \Rightarrow \exists t' \in T \;\; t_1 \; \theta \; t' \text{ and } t_2 \; \theta \; t'$$

$$\begin{tikzcd}
& t \arrow[dl, "\theta"'] \arrow[dr, "\theta"] & \\
t_1 \arrow[dr, dashed, "\theta"'] & & t_2 \arrow[dl, dashed, "\theta"] \\
& t' &
\end{tikzcd}$$

- and that θ has the Church Rosser Property (CR) if θ^* has the Diamond property.

Theorem (PCF is Church Rosser)

The relation β has the Church Rosser property.

We outline a very general and efficient method to this kind of result: the Tait Martin-Löf method of *parallel reductions*.

Good to know it because it can be used is many different settings.

Why isn't PCF trivially CR?

Theorem (Easy)

If θ^- has the Diamond Property (DP) then θ has the Church Rosser property.

Idea of the proof

However β^- has not the DP: let $I = \lambda x^\iota\, x$ and $I' = \lambda g^{\iota \Rightarrow \iota}\, g$ so that $\vdash I : \iota \Rightarrow \iota$, $\vdash I' : (\iota \Rightarrow \iota) \Rightarrow \iota \Rightarrow \iota$ and $\vdash (I')\, I : \iota \Rightarrow \iota$.

$$
\begin{tikzcd}[column sep=small]
 & M = (\lambda f^{\iota \Rightarrow \iota}\, (f)\, (f)\, \underline{0})\, (I')\, I \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\
((I')\, I)\, ((I')\, I)\, \underline{0} \arrow[dr, "\beta^2"'] & & (\lambda f^{\iota \Rightarrow \iota}\, (f)\, (f)\, \underline{0})\, I \arrow[dl, "\beta"] \\
 & (I)\, (I)\, \underline{0} &
\end{tikzcd}
$$

Impossible to close this diagram in one step on both sides: on the left we have to reduce 2 copies of the redex $(I')\, I$, on the right only one.

Tait Martin-Löf proof idea

Crucial observation

We can close the diagram reducing only redexes which were present in the original term, namely:

- $\underline{M}$ itself
- and $\underline{(I')\, I}$

but we need to be allowed to reduce several of them.

We never need to reduce the new redex $(I)\,(I)\,\underline{0}$ which has been created during the reduction.

Sketch of the proof

Strategy of the proof

- Define a *parallel reduction* relation ρ which performs an arbitrary number of reduction of redexes *present in the initial term* (such as the red and the blue ones), so that $\beta \subseteq \rho \subseteq \beta^*$ and hence $\rho^* = \beta^*$.
- Prove that ρ has the diamond property.

The parallel reduction ρ

As usual we present it as a deduction system.

$$\frac{}{\underline{n}\ \rho\ \underline{n}} \qquad \frac{}{x\ \rho\ x}$$

$$\frac{M\ \rho\ M' \qquad N\ \rho\ N'}{(\lambda x^A\, M)\, N\ \rho\ M'\,[N'/x]} \qquad \frac{M\ \rho\ M'}{\lambda x^A\, M\ \rho\ \lambda x^A\, M'}$$

$$\frac{M\ \rho\ M' \qquad M\ \rho\ M''}{\mathrm{fix}(M)\ \rho\ (M')\,\mathrm{fix}(M'')}$$

$$\frac{P\ \rho\ P'}{\mathrm{if}(\underline{0}, P, z \cdot Q)\ \rho\ P'} \qquad \frac{Q\ \rho\ Q'}{\mathrm{if}(\underline{n+1}, P, z \cdot Q)\ \rho\ Q'\,[\underline{n}/z]}$$

$$\frac{M\ \rho\ M'}{\mathrm{succ}(M)\ \rho\ \mathrm{succ}(M')} \qquad \frac{}{\mathrm{succ}(\underline{n})\ \rho\ \underline{n+1}}$$

$$\frac{M \mathrel{\rho} M' \qquad N \mathrel{\rho} N'}{(M)\,N \mathrel{\rho} (M')\,N'}$$

$$\frac{M \mathrel{\rho} M'}{\mathrm{fix}(M) \mathrel{\rho} \mathrm{fix}(M')}$$

$$\frac{M \mathrel{\rho} M' \qquad P \mathrel{\rho} P' \qquad Q \mathrel{\rho} Q'}{\mathrm{if}(M, P, z \cdot Q) \mathrel{\rho} \mathrm{if}(M', P', z \cdot Q')}$$

Relation between ρ and β

In all these statements we assume that $\Gamma \vdash M : A$.

Lemma

If $M \; \beta \; M'$ then $M \; \rho \; M'$.

Easy induction on the derivation that $M \; \beta \; M'$. We also use the following easy property:

Lemma

$M \; \rho \; M$.

Proof by induction on M (or on its typing derivation).

Lemma

Assume that $\Gamma, x : A \vdash N : B$ *and* $N \,\beta\, N'$ *then* $N\,[M/x]\,\beta\, N'\,[M/x]$*. Hence* $N\,\beta^*\, N' \Rightarrow N\,[M/x]\,\beta^*\, N'\,[M/x]$*.*

Easy induction on the derivation that $N\,\beta\, N'$. Assume for instance that $N = (\lambda y^C\, P)\, Q$ and $N' = P\,[Q/y]$ so that the derivation consists of an axiom.

Then $N'\,[M/x] = \underline{P\,[Q/y]\,[M/x] = P\,[M/x]\,[Q\,[M/x]/y]}$

because we can assume that y does not occur free in M. And

$$
\begin{aligned}
N\,[M/x] = ((\lambda y^C\, P)\, Q)\,[M/x] &= (\lambda x^C\, P\,[M/x])\, Q\,[M/x] \\
&\beta\; P\,[M/x]\,[Q\,[M/x]/y]
\end{aligned}
$$

Lemma

Assume that $\Gamma, x : A \vdash N : B$ *and* $M \,\beta\, M'$ *then* $N\,[M/x]\,\beta^*\; N\,[M'/x]$.

Easy induction in the derivation that $\Gamma, x : A \vdash N : B$.

Assume for instance that $N = (P)\,Q$ with $\Gamma, x : A \vdash P : C \Rightarrow B$ and $\Gamma, x : A \vdash Q : C$.

By inductive hypothesis we have $P\,[M/x]\,\beta^*\; P\,[M'/x]$ and $Q\,[M/x]\,\beta^*\; Q\,[M'/x]$. Therefore since $((P)\,Q)\,[M/x] = (P\,[M/x])\,Q\,[M/x]$ we have

$$N\,[M/x] \xrightarrow{\beta^*} (P\,[M'/x])\,Q\,[M/x] \xrightarrow{\beta^*} (P\,[M'/x])\,Q\,[M'/x]$$

Combining these two results:

Lemma

Assume that $\Gamma, x : A \vdash N : B$, $N \,\beta^*\, N'$ *and* $M \,\beta^*\, M'$ *then* $N\,[M/x]\,\beta^*\, N'\,[M'/x]$.

Using this lemma, one proves

Lemma

If $\Gamma \vdash M : B$ *and* $M \,\rho\, M'$ *then* $M \,\beta^*\, M'$*. As a consequence* $\Gamma \vdash M' : B$*.*

By induction on the derivation that $M \,\rho\, M'$. Assume for instance that $M = (\lambda x^A\, P)\, Q$ and $M' = P'\,[Q'/x]$ with $P \,\rho\, P'$ and $Q \,\rho\, Q'$.

By inductive hypothesis $P \,\beta^*\, P'$ and $Q \,\beta^*\, Q'$ and hence by the lemma above $P\,[Q/x]\,\beta^*\, P'\,[Q'/x]$, hence

$$M = (\lambda x^A\, P)\, Q \xrightarrow{\beta} P\,[Q/x] \xrightarrow{\beta^*} P'\,[Q'/x] = M'$$

Main properties of ρ

The crucial property of ρ is:

Theorem

Assume that $\Gamma, x : A \vdash M : B$ *and* $\Gamma \vdash N : A$ *and assume that* $M \,\rho\, M'$ *and* $N \,\rho\, N'$*. Then*

$$M\,[N/x]\;\rho\;M'\,[N'/x]\ .$$

The proof is by induction on the derivation that $M \,\rho\, M'$.

One step in the proof

Assume that $M = (\lambda y^C\, P)\, Q$ with

- $\Gamma, x : A, y : C \vdash P : B$ and $\Gamma, x : A \vdash Q : C$,
- $P\ \rho\ P'$ and $Q\ \rho\ Q'$,
- and $M' = P'\,[Q'/y]$.

We have $M\,[N/x] = (\lambda y^C\, P\,[N/x])\, Q\,[N/x]$ (we assume that y does not occur free in N).

The inductive hypothesis (IH) tells us that $P\,[N/x]\ \rho\ P'\,[N'/x]$ and $Q\,[N/x]\ \rho\ Q'\,[N'/x]$.

By definition of ρ we have
$M\,[N/x]\ \rho\ P'\,[N'/x]\,[Q'\,[N'/x]/y] = M'\,[N'/x]$ because we can also assume that y is not free in N'.

Theorem

The relation ρ has the Diamond property: assume that $\Gamma \vdash M : A$ and that $M \; \rho \; M_i$ for $i = 1, 2$. Then there is a term R such that $M_i \; \rho \; R$ for $i = 1, 2$.

By induction on the structure of M, considering all possible last rules in the deduction that $M \; \rho \; M_1$ and $M \; \rho \; M_2$ and applying the above lemma.

Assume for instance that $M = (\lambda y^B \, P) \, Q$ and:

- $M_1 = (\lambda y^B \, P_1) \, Q_1$ with $P \; \rho \; P_1$ and $Q \; \rho \; Q_1$
- $M_2 = P_2 \, [Q_2/y]$ with $P \; \rho \; P_2$ and $Q \; \rho \; Q_2$.

By IH there are terms P_0 and Q_0 such that $P_i \; \rho \; P_0$ and $Q_i \; \rho \; Q_0$ for $i = 1, 2$. By definition of ρ we have
$M_1 = (\lambda y^B \, P_1) \, Q_1 \; \rho \; P_0 \, [Q_0/y]$ and by the lemma we have
$M_2 = P_2 \, [Q_2/y] \; \rho \; P_0 \, [Q_0/y] = R$.

Consequences of Church Rosser

Given $\theta \subseteq T \times T$ let $\sim_\theta$ be the symmetric, reflexive and transitive closure of θ.

Theorem

If θ is Church Rosser then

$$\forall t_1, t_2 \in T \quad t_1 \sim_\theta t_2 \Leftrightarrow \exists t' \in T \;\; t_1 \; \theta^* \; t' \textit{ and } t_2 \; \theta^* \; t'$$

Idea of the proof

Uniqueness of value

Another crucial consequence of Church Rosser:

Theorem

Assume that $\vdash M : \iota$. If there exists $n \in \mathbb{N}$ such that $M\ \beta^*\ \underline{n}$, there is only one such n. If M has a value, it has exactly one value!

If $M\ \beta^*\ \underline{n'}$ for another $n' \in \mathbb{N}$ then by Church Rosser there is M' such that $\underline{n}\ \beta^*\ M'$ and $\underline{n'}\ \beta^*\ M'$. This implies $n = n'$.

General recursive functions in PCF

So if $\vdash M : \iota \Rightarrow \iota$ we can define a partial function $f_M : \mathbb{N} \rightarrow \mathbb{N}$ by

$$f_M(n) = \begin{cases} k & \text{if } (M)\,\underline{n}\,\beta^*\,\underline{k} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Theorem (Turing completeness of PCF)

The class of partial functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that there exists $\vdash M : \iota \Rightarrow \iota$ such that $f = f_M$ is exactly the class of all partial recursive functions.

Weak head reduction

Problem

How do we execute PCF terms in a machine?

In $\vdash M : \iota$ there may be a lot of redexes, which one should we choose to reduce?

Worse: some sequences of reductions could be infinite whereas there is $n \in \mathbb{N}$ such that $M \,\beta^*\, \underline{n}$.

Example

$$M = (\lambda x^\iota\, \underline{0})\, \mathrm{fix}(\lambda z^\iota\, z)\,.$$

We have $M \,\beta\, \underline{0}$ and $M \,\beta\, (\lambda x^\iota\, \underline{0})\, (\lambda z^\iota\, z)\, \mathrm{fix}(\lambda z^\iota\, z) \,\beta\, M$.

Definition of β_{wh}

We define a sub-relation β_{wh} of β. The axioms are the same as for β:

$$\frac{}{(\lambda x^A\, M)\, N\ \beta_{\mathrm{wh}}\ M\,[N/x]}$$

$$\frac{}{\mathrm{fix}(M)\ \beta_{\mathrm{wh}}\ (M)\,\mathrm{fix}(M)}$$

$$\frac{}{\mathrm{if}(\underline{0}, P, z\cdot Q)\ \beta_{\mathrm{wh}}\ P} \qquad \frac{}{\mathrm{if}(\underline{n+1}, P, z\cdot Q)\ \beta_{\mathrm{wh}}\ Q\,[\underline{n}/z]}$$

$$\frac{}{\mathrm{succ}(\underline{n})\ \beta_{\mathrm{wh}}\ \underline{n+1}}$$

But there are much less *deduction rules*, in other words there are less contexts where redexes can be reduced.

$$\frac{M \;\beta_{\mathrm{wh}}\; M'}{(M)\,N \;\beta_{\mathrm{wh}}\; (M')\,N}$$

$$\frac{M \;\beta_{\mathrm{wh}}\; M'}{\mathrm{if}(M, P, z \cdot Q) \;\beta_{\mathrm{wh}}\; \mathrm{if}(M', P, z \cdot Q)}$$

$$\frac{M \;\beta_{\mathrm{wh}}\; M'}{\mathrm{succ}(M) \;\beta_{\mathrm{wh}}\; \mathrm{succ}(M')}$$

We have

$$(\lambda x^{\iota}\,\underline{0})\,\mathrm{fix}(\lambda z^{\iota}\,z) \;\beta_{\mathrm{wh}}\; \underline{0}$$

$$M = (\lambda x^{\iota}\,\underline{0})\,\mathrm{fix}(\lambda z^{\iota}\,z) \;\not\beta^{*}_{\mathrm{wh}}\; M$$

Notice that β_{wh} is a "deterministic strategy" in the sense that for any term M there is at most one redex which can be reduced by a β_{wh} reduction.

Notation

$$(M)\, M_1 \cdots M_n = (\cdots (M)\, M_1 \cdots)\, M_n$$

Lemma

To have $M \; \beta_{\mathrm{wh}} \; M'$, M must be of shape

$$M = (H)\, M_1 \cdots M_n$$

with $n \geq 0$ and

- *either H is a redex with $H \; \beta_{\mathrm{wh}} \; H'$ and then* $M' = (H')\, M_1 \cdots M_n$
- *or $H = \mathrm{if}(K, P, z \cdot Q)$, $K \; \beta_{\mathrm{wh}} \; K'$ and* $M' = (\mathrm{if}(K', P, z \cdot Q))\, M_1 \cdots M_n$.

β_{wh}-normal closed terms of type ι

Fact

If $\vdash M : \iota$ and M is β_{wh}-normal (no β_{wh}-reduction from M) then $M = \underline{k}$ for some $k \in \mathbb{N}$.

By induction on M.
We can write $M = (M_0)\, M_1 \cdots M_n$ where M_0 is not of shape $(P)\, Q$.

- If $M_0 = \lambda x^A\, P$ we must have $n \geq 1$ because $\vdash M : \iota$ and $(M_0)\, M_1\ \beta_{\mathrm{wh}}\ P\,[M_1/x]$ and hence $M\ \beta_{\mathrm{wh}}\ (P\,[M_1/x])\, M_2 \cdots M_n$ hence M is not β_{wh}-normal. So this case is impossible.
- If $M_0 = \mathrm{if}(K, P, x \cdot Q)$ then we must have $\vdash K : \iota$ and K must be β_{wh}-normal, which by induction implies $K = \underline{k}$ for some $k \in \mathbb{N}$ but then M_0 is not β_{wh} normal and neither is M.

- $M_0 = \mathrm{fix}(P)$ is impossible because $\mathrm{fix}(P)\ \beta_{\mathrm{wh}}\ (P)\,\mathrm{fix}(P)$.
- If $M_0 = \mathrm{succ}(P)$ then we must have $\vdash P : \iota$ and P must be β_{wh}-normal (by typing we must have $n = 0$ and if $P\ \beta_{\mathrm{wh}}\ P'$ then $M\ \beta_{\mathrm{wh}}\ \mathrm{succ}(P')$, contradiction). By inductive hypothesis $P = \underline{k}$ for some $k \in \mathbb{N}$. Then $M\ \beta_{\mathrm{wh}}\ \underline{k+1}$, contradiction.
- The only left possibility is that $M_0 = \underline{k}$ for some $k \in \mathbb{N}$ which implies $n = 0$ by typing.

β_{wh} is complete

Let $\vdash M : \iota$. Of course if $M \; \beta^*_{\mathrm{wh}} \; \underline{n}$ then $M \; \beta^* \; \underline{n}$. In a few weeks we shall be able to prove

Theorem

If $M \; \beta^* \; \underline{n}$ then $M \; \beta^*_{\mathrm{wh}} \; \underline{n}$.

Examples of PCF programs

Addition:

$$\mathsf{add} = \lambda x^{\iota}\,\mathsf{fix}(\lambda a^{\iota\Rightarrow\iota}\,\lambda y^{\iota}\,\mathsf{if}(y, x, z \cdot \mathsf{succ}((a)\,z)))$$
$$\text{with} \quad \vdash \mathsf{add} : \iota \Rightarrow (\iota \Rightarrow \iota)$$

Comparison:

$$\mathsf{cmp} = \mathsf{fix}(\lambda c^{\iota\Rightarrow(\iota\Rightarrow\iota)}\,\lambda x^{\iota}\,\lambda y^{\iota}\,\mathsf{if}(x, \underline{0}, z \cdot \mathsf{if}(y, \underline{1}, z' \cdot (c)\,z\,z')))$$
$$\text{et on a} \quad \vdash \mathsf{cmp} : \iota \Rightarrow (\iota \Rightarrow \iota)$$

Search:

$$\lambda f^{\iota\Rightarrow\iota}\,(\mathsf{fix}(\lambda g^{\iota\Rightarrow\iota}\,\lambda x^{\iota}\,\mathsf{if}((f)\,x, x, z \cdot (g)\,\mathsf{succ}(x))))\,\underline{0}$$

Morris equivalence

We have a notion of equivalence $\sim_\beta$ on terms, but it is very weak. For instance the two terms

$$M_1 = \lambda x_1^\iota\, \lambda x_2^\iota\, \mathrm{if}(x_1, \mathrm{if}(x_2, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$
$$M_2 = \lambda x_2^\iota\, \lambda x_1^\iota\, \mathrm{if}(x_1, \mathrm{if}(x_2, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$

obviously do the same thing (not in the same order). But it is not true that $M_1 \sim_\beta M_2$.

They are Morris (or observationally) equivalent if they can be used indifferently in any context.

Definition

Let M_1 and M_2 be such that $\vdash M_i : A$ for $i = 1, 2$. We say that M_1 and M_2 are observationally equivalent (written $M_1 \sim M_2$) if for any term C such that $\vdash C : A \Rightarrow \iota$ one has

$$(C)\, M_1 \; \beta^*_{\mathrm{wh}} \; \underline{0} \Leftrightarrow (C)\, M_2 \; \beta^*_{\mathrm{wh}} \; \underline{0}\,.$$

The idea behind this definition is that the only type whose values can be observed (by a human, that is, a finite being) is $\mathbb{N}$.

- This is an equivalence relation (on closed terms of type A).
- The choice of convergence to $\underline{0}$ as a criterion is irrelevant, we would define *exactly the same equivalence relation* if we define $M_1 \sim M_2$ by

$$(\exists n \in \mathbb{N}\ (C)\, M_1\ \beta^*_{\mathrm{wh}}\ \underline{n}) \Leftrightarrow (\exists n \in \mathbb{N}\ (C)\, M_2\ \beta^*_{\mathrm{wh}}\ \underline{n})$$

Theorem

Let $\vdash M_1, M_2 : A$*. If* $M_1 \sim_\beta M_2$ *then* $M_1 \sim M_2$*.*

Assume $M_1 \sim_\beta M_2$.

Let C with $\vdash C : A \Rightarrow \iota$ and assume $(C)\,M_1\ \beta^*_{\mathrm{wh}}\ \underline{0}$, which implies $(C)\,M_1\ \beta^*\ \underline{0}$.

Since $M_1 \sim_\beta M_2$ we have $(C)\,M_1 \sim_\beta (C)\,M_2$ and hence $(C)\,M_2\ \beta^*\ \underline{0}$ by Church Rosser.

Hence $(C)\,M_2\ \beta^*_{\mathrm{wh}}\ \underline{0}$ by completeness of β_{wh}.

If

$$M_1 = \lambda x_1^\iota \, \lambda x_2^\iota \, \mathrm{if}(x_1, \mathrm{if}(x_2, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$
$$M_2 = \lambda x_1^\iota \, \lambda x_2^\iota \, \mathrm{if}(x_2, \mathrm{if}(x_1, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$

then

$$M_1 \sim M_2$$

Not easy to prove because of the $\forall C$ in the definition of $\sim$.

Fact

Easy to prove using denotational semantics: it suffices to prove that M_1 and M_2 have the same interpretation.

We'll see that this implies $M_1 \sim M_2$.

The relational model

Lecture notes, Section 6.7 Relational semantics

What is a categorical model of LL?

A tuple $(\underline{\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \gamma}, \underline{\perp}, \underline{!\_, \mathrm{der}, \mathrm{dig}, \mathrm{m}^0, \mathrm{m}^2})$ consisting of:

- a symmetric monoidal closed category (SMCC) which is cartesian
- together with an object $\perp$ of $\mathcal{L}$ which turns this SMCC into a $*$-autonomous category
- and a symmetric monoidal comonad on $\mathcal{L}$.

What we do now

We explain what this means, giving **Rel** as an example.

Linear Logic: short reminder

Formulas: A, B, $A_1 \ldots$

	positive	negative
mutiplicative	1, $A \otimes B$	$\bot$, $A \parr B$
additive	0, $A \oplus B$	$\top$, $A \& B$
exponential	$!A$	$?A$

Linear Negation

Defined by induction on formulas

$$
\begin{aligned}
1^\perp &= \perp & (A \otimes B)^\perp &= A^\perp \parr B^\perp \\
\perp^\perp &= 1 & (A \parr B)^\perp &= A^\perp \otimes B^\perp \\
0^\perp &= \top & A \oplus B^\perp &= A^\perp \,\&\, B^\perp \\
\top^\perp &= 0 & (A \,\&\, B)^\perp &= A^\perp \oplus B^\perp \\
(!A)^\perp &= ?A^\perp & (?A)^\perp &= !A^\perp
\end{aligned}
$$

Fact

$A^{\perp\perp} = A$

Sequents $\vdash A_1, \ldots, A_n$

There is a logical system which allows to build trees π which are proofs of sequents

$$\begin{array}{c} \vdots\ \pi \\ \vdash \Gamma \end{array}$$

And a cut-elimination rewriting system on proofs of the same sequent $\pi \to \pi'$.

Categorical semantics

A category $\mathcal{L}$

A correspondence

$$A \leadsto [\![A]\!] \quad \text{object of } \mathcal{L}$$
$$\Gamma \leadsto [\![\Gamma]\!] \quad \text{object of } \mathcal{L}$$
$$\pi \leadsto [\![\pi]\!] \quad \text{morphism of } \mathcal{L}$$

In such a way that

$$\pi \to \pi' \quad \Rightarrow \quad [\![\pi]\!] = [\![\pi']\!]$$

Main feature: modularity

With each linear connective is associated a functor, for instance

$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$$

With each logical rule, an operation on morphisms. If π is the proof tree

$$\dfrac{\begin{array}{c}\vdots\,\lambda \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\rho \\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}$$

then $[\![\pi]\!] = T([\![\lambda]\!], [\![\rho]\!])$ where T is a well defined operation on morphisms.

Methodology

We do not define directly the interpretation on LL.

Rather, we define a general notion of category where the interpretation is possible and satisfies these requirements (modularity, invariance by cut-elim).

This is much better because the categorical language is extremely precise and explicit. Though not always very convenient logically.

It took several years after the discovery of LL, to find the right categorical setting.

To check that something is a model of LL, it suffices to check these categorical axioms, without coming back to LL iself.

The category **Rel**

It is probably the simplest denotational model of LL.

Very roughly: coherence spaces... without coherence.

It is also a model of PCF.

Rel as a category

Objects of **Rel**: all sets.

$\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$

Identity at E: $\mathrm{Id}_E = \{(a, a) \mid a \in E\}$

Composition: if $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, G)$ then $t\,s \in \mathbf{Rel}(E, G)$ is

$$t\,s = \{(a, c) \in E \times G \mid \exists b \in F \ (a, b) \in s \text{ and } (b, c) \in t\}$$

Fact

Rel *is a category.*

Isomorphisms in **Rel**

Remember:

Definition

$t \in \mathcal{L}(X, Y)$ is an iso if there is $t' \in \mathcal{L}(Y, X)$ such that $t'\, t = \mathsf{Id}_X$ and $t\, t' = \mathsf{Id}_Y$. Then we know that there is a unique such t', it is denoted as t^{-1}.

Fact

$t \in \mathbf{Rel}(E, F)$ is an iso iff t is (the graph of) a bijection $E \to F$ and then t^{-1} is the inverse of this bijection.

Symmetric monoidal category (SMC)

Imporant

An SMC is not a category, it is a category equipped with a monoidal structure, just as a monoid is not a set, but a set equipped with a structure of monoid.

An SMC is a tuple

$$(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \gamma)$$

where

- $\mathcal{L}$ is a category
- $1 \in \mathrm{Obj}(\mathcal{L})$ and $\otimes$ is a functor $\mathcal{L}^2 \to \mathcal{L}$
- and λ, ρ, α and γ are natural isomorphisms.

Monoidality isomorphisms in $\mathcal{L}$

$$
\begin{aligned}
\lambda_X &: 1 \otimes X \to X \\
\rho_X &: X \otimes 1 \to X \\
\alpha_{X_1,X_2,X_3} &: (X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes (X_2 \otimes X_3) \\
\gamma_{X_1,X_2} &: X_1 \otimes X_2 \to X_2 \otimes X_1
\end{aligned}
$$

Satisfying *coherence diagrams*.

Idea: if we consider the isos of the monoidal structure as rewriting rules, there are "critical pairs", for instance

$$\begin{tikzcd}
(X_1 \otimes X_2) \otimes X_3 \arrow[r, "\alpha_{X_1,X_2,X_3}"] \arrow[d, "\gamma_{X_1,X_2} \otimes X_3"'] & X_1 \otimes (X_2 \otimes X_3) \\
(X_2 \otimes X_1) \otimes X_3 &
\end{tikzcd}$$

then the coherence diagrams explain how to solve these conflicts.

Examples of coherence diagram

$$
\begin{tikzcd}
(X_1 \otimes X_2) \otimes X_3 \arrow[r, "\alpha_{X_1,X_2,X_3}"] \arrow[d, "\gamma_{X_1,X_2} \otimes X_3"'] & X_1 \otimes (X_2 \otimes X_3) \arrow[d, "\gamma_{X_1,X_2 \otimes X_3}"] \\
(X_2 \otimes X_1) \otimes X_3 \arrow[d, "\alpha_{X_2,X_1,X_3}"'] & (X_2 \otimes X_3) \otimes X_1 \arrow[d, "\alpha_{X_2,X_3,X_1}"] \\
X_2 \otimes (X_1 \otimes X_3) \arrow[r, "X_2 \otimes \gamma_{X_1,X_3}"] & X_2 \otimes (X_3 \otimes X_1)
\end{tikzcd}
$$

McLane's Pentagon

$$
\begin{tikzcd}
((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \arrow[rr, "\alpha_{X_1 \otimes X_2, X_3, X_4}"] \arrow[d, "\alpha_{X_1, X_2, X_3} \otimes X_4"'] & & (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \arrow[d, "\alpha_{X_1, X_2, X_3 \otimes X_4}"] \\
(X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 \arrow[d, "\alpha_{X_1, X_2 \otimes X_3, X_4}"'] & & X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \\
X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) \arrow[urr, "X_1 \otimes \alpha_{X_2, X_3, X_4}"'] & &
\end{tikzcd}
$$

McLane's theorem on monoidal categories

One major effect of these coherence diagrams is that in a (symmetric) monoidal category $\mathcal{L}$, if $X_1, \ldots, X_n$ are objects, if X and X' are two ways of putting parenthesis in $X_1 \otimes \cdots \otimes X_n$, there is a unique canonical iso from X to X'.

Example

$n = 5$, $X = X_1 \otimes ((X_2 \otimes X_3) \otimes (X_4 \otimes X_5))$,
$X' = (((X_1 \otimes X_2) \otimes X_3) \otimes X_4) \otimes X_5$.

Using α, we can define several isos from X to X'. McLane's Theorem tells us that they are all equal.

Consequence

We can write $X_1 \otimes \cdots \otimes X_n$ without parentheses.

The other commutations are similar (see the lecture notes).

One special commutation, which holds in **Rel**, is not exactly of that kind and corresponds to the adjective "symmetric":

$$\begin{tikzcd}
X_1 \otimes X_2 \arrow[r, "\gamma_{X_1,X_2}"] \arrow[dr, "\mathrm{Id}_{X_1 \otimes X_2}"'] & X_1 \otimes X_2 \arrow[d, "\gamma_{X_2,X_1}"] \\
& X_1 \otimes X_2
\end{tikzcd}$$

There are other, weaker, possibilities for γ. One of them corresponds to *braided monoidal categories*.

Monoidal structure of **Rel**

We set $E_1 \otimes E_2 = E_1 \times E_2$.

If $s_i \in \mathbf{Rel}(E_i, F_i)$ for $i = 1, 2$, we set

$$\begin{aligned} s_1 \otimes s_2 &= \{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in s_i \text{ for } i = 1, 2\} \\ &\in \mathbf{Rel}(E_1 \otimes E_2, F_1 \otimes F_2) \end{aligned}$$

Fact

$\otimes$ *is a functor* $\mathbf{Rel}^2 \to \mathbf{Rel}$.

One has to prove that $\mathrm{Id}_{E_1} \otimes \mathrm{Id}_{E_2} = \mathrm{Id}_{E_1 \otimes E_2}$ and if $s_i \in \mathbf{Rel}(E_i, F_i)$ and $t_i \in \mathbf{Rel}(F_i, G_i)$ then

$$(t_1 \otimes t_2)\,(s_1 \otimes s_2) = (t_1\, s_1) \otimes (t_2\, s_2)$$

All proofs are easy!

$1 = \{*\}$

We have (trivial) natural isomorphisms

$$
\begin{aligned}
\lambda_E &: 1 \otimes E \to E \\
\rho_E &: E \otimes 1 \to E \\
\alpha_{E_1, E_2, E_3} &: (E_1 \otimes E_2) \otimes E_3 \to E_1 \otimes (E_2 \otimes E_3) \\
\gamma_{E_1, E_2} &: E_1 \otimes E_2 \to E_2 \otimes E_1
\end{aligned}
$$

For instance

$$
\begin{aligned}
\lambda_E &= \{((*, a), a) \mid a \in E\} \\
\alpha_{E_1, E_2, E_3} &= \{(((a_1, a_2), a_3), (a_1, (a_2, a_3))) \mid a_i \in E_i \text{ for } i = 1, 2, 3\}
\end{aligned}
$$

and similarly for the others.

Remember that the naturality of γ (for instance) means that if $s_i \in \mathbf{Rel}(E_i, F_i)$ for $i = 1, 2$ then the following diagram commutes in **Rel**:

$$\begin{tikzcd}
E_1 \otimes E_2 \arrow[r, "\gamma_{E_1,E_2}"] \arrow[d, "s_1 \otimes s_2"'] & E_2 \otimes E_1 \arrow[d, "s_2 \otimes s_1"] \\
F_1 \otimes F_2 \arrow[r, "\gamma_{F_1,F_2}"] & F_2 \otimes F_1
\end{tikzcd}$$

To prove such a commutation:

- take $(a_1, a_2) \in E_1 \otimes E_2$ and $(b_2, b_1) \in F_2 \otimes F_1$
- prove that

$$((a_1, a_2), (b_2, b_1)) \in (s_2 \otimes s_1)\ \gamma_{E_1,E_2}$$
$$\Rightarrow ((a_1, a_2), (b_2, b_1)) \in \gamma_{F_1,F_2}\ (s_1 \otimes s_2)$$

- and the converse implication.

In this case, the proof is trivial.

Fact

$(\mathbf{Rel}, 1, \otimes, \lambda, \rho, \alpha, \gamma)$ *is a symmetric monoidal category (SMC).*

Points

In an SMC $\mathcal{L}$, a *point* of an object X is a morphism $x \in \mathcal{L}(1, X)$, $\mathrm{Pt}_{\mathcal{L}}(X) = \mathcal{L}(1, X)$.

Can be seen as a functor: $\mathrm{Pt}_{\mathcal{L}} : \mathcal{L} \to \mathbf{Set}$

If $s \in \mathcal{L}(X, Y)$ then

$$\begin{aligned} \mathrm{Pt}_{\mathcal{L}}(s) : \mathrm{Pt}_{\mathcal{L}}(X) &\to \mathrm{Pt}_{\mathcal{L}}(Y) \\ x &\mapsto s\, x \end{aligned}$$

Points in **Rel** (up to trivial iso)

$\mathrm{Pt}_{\mathbf{Rel}}(E) = \mathcal{P}(E)$ and if $s \in \mathbf{Rel}(E, F)$ then

$$\begin{aligned} \mathrm{Pt}_{\mathbf{Rel}}(s) : \mathcal{P}(E) &\to \mathcal{P}(F) \\ u &\mapsto s \cdot u = \{b \in F \mid \exists a \in u\ (a, b) \in s\} \end{aligned}$$

Monoidal closedness

$(\mathcal{L}, \dots)$ an SMC.

A *linear hom object* from X to Y (objects of $\mathcal{L}$) is a pair $(X \multimap Y, \mathrm{ev})$ which means that

- $X \multimap Y$ is an object of $\mathcal{L}$
- $\mathrm{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$
- such that for any $s \in \mathcal{L}(Z \otimes X, Y)$ there is exactly one morphism $\mathrm{cur}(s) \in \mathcal{L}(Z, X \multimap Y)$ such that

$$\begin{tikzcd}
Z \otimes X \arrow[r, "\mathrm{cur}(s) \otimes X"] \arrow[dr, "s"'] & (X \multimap Y) \otimes X \arrow[d, "\mathrm{ev}"] \\
& Y
\end{tikzcd}$$

Equational characterization

It is useful to know that the linear hom object is characterized by the following equations:

- $\mathrm{ev}\ (\mathrm{cur}(s) \otimes X) = s$ for $s \in \mathbf{Rel}(Z \otimes X, Y)$, this is just the last commutation
- $\mathrm{cur}(s)\, t = \mathrm{cur}(s\ (t \otimes X))$ for $s \in \mathbf{Rel}(Z \otimes X, Y)$ and $t \in \mathbf{Rel}(T, Z)$
- and $\mathrm{cur}(\mathrm{ev}) = \mathrm{Id}_{X \multimap Y}$.

Definition

The SMC $(\mathcal{L}, \dots)$ is closed if any $X, Y \in \mathrm{Obj}(\mathcal{L})$ have a linear hom object $(X \multimap Y, \mathrm{ev})$.

Since linear hom objects are defined by a universal property, being closed is a *property* of an SMC, not an additional structure.

Equivalent definition

An SMC $\mathcal{L}$ is closed if for any object Z of $\mathcal{L}$, the functor $Z \otimes \_ : \mathcal{L} \to \mathcal{L}$ has a right adjoint.

Rel is an SMCC

Concretely

$$
\begin{aligned}
E \multimap F &= E \times F \\
\mathrm{ev} &= \{(((a, b), a), b) \mid a \in E \text{ and } b \in F\} \\
&\in \mathbf{Rel}((E \multimap F) \otimes E, F) \\
\mathrm{cur}(s) &= \{(c, (a, b)) \mid ((c, a), b) \in s\} \\
&\in \mathbf{Rel}(G, E \multimap F)
\end{aligned}
$$

for $s \in \mathbf{Rel}(G \otimes E, F)$.

Linear hom object as a functor

Fact

If $\mathcal{L}$ is an SMCC then $\_ \multimap \_$ is a functor $\mathcal{L}^{\mathrm{op}} \times \mathcal{L} \to \mathcal{L}$.
Explicitly, if $s \in \mathcal{L}(X', X)$ and $t \in \mathcal{L}(Y, Y')$, then $s \multimap t = \mathrm{cur}(u) \in \mathcal{L}(X \multimap Y, X' \multimap Y')$ where u is the following morphism:

$$(X \multimap Y) \otimes X' \xrightarrow{(X \multimap Y) \otimes s} (X \multimap Y) \otimes X \xrightarrow{\mathrm{ev}} Y \xrightarrow{t} Y'$$

$*$-autonomy

Definition

An SMCC $\mathcal{L}$ is $*$-autonomous if it is equipped with an objet $\perp$ of $\mathcal{L}$ such that the natural morphism

$$\eta_X = \mathrm{cur}(s) \in \mathcal{L}(X, (X \multimap \perp) \multimap \perp)$$

is an isomorphism, where s is the following morphism

$$X \otimes (X \multimap \perp) \xrightarrow{\gamma} (X \multimap \perp) \otimes X \xrightarrow{\mathrm{ev}} \perp$$

Then the functor $(\_)^{\perp} = \_ \multimap \perp : \mathcal{L}^{\mathrm{op}} \to \mathcal{L}$ is “involutive up to iso”.

Fact

With $\perp = 1 = \{*\}$, **Rel** *is* $*$*-autonomous. Indeed*

$$\eta_E = \{(a, ((a, *), *)) \mid a \in E\}$$

is trivially an iso.

Linear negation

We can identify the functor

$$\_ \multimap \perp : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$$

with the functor $\_^{\perp}$ defined by

- $E^{\perp} = E$
- and if $s \in \mathbf{Rel}(E, F)$ then

$$s^{\perp} = \{(b, a) \mid (a, b) \in s\} \in \mathbf{Rel}(F, E)$$

which is strictly involutive. If we see s as a $E \times F$-matrix then $s^{\perp}$ is its transpose.

Cotensor or *par* bifunctor

In a $*$-autonomous category $\mathcal{L}$ (using $\_^{\perp}$ for the involutive dualizing contravariant functor $\_ \multimap \perp$) we have a binary functor

$$\parr : \mathcal{L}^2 \to \mathcal{L}$$

- On objects: $X \parr Y = (X^{\perp} \otimes Y^{\perp})^{\perp}$
- and similarly for morphisms.

With $\perp$ as unit and suitable natural isos λ', ρ', α' and γ', this is another SMC structure on $\mathcal{L}$.

Fact

In **Rel** *this symmetric monoidal structure coincides with* $(1, \otimes, \lambda, \rho, \alpha, \gamma)$. *In particular* $E \parr F = E \otimes F = E \times F$.

This is due to the fact that the objects of **Rel** have no structure, they are just sets.

In coherence spaces (for instance), 1 and $\perp$ are the same object but $\otimes$ and $\parr$ are distinct functors.

Products and coproducts

We also require $\mathcal{L}$ to be cartesian, that is, any finite family $(X_i)_{i\in I}$ has a cartesian product $(X,(\pi_i)_{i\in I})$, this means the following.

- X is an objet of $\mathcal{L}$ and $\pi_i \in \mathcal{L}(X, X_i)$
- and the following universal property holds: for any object Y of $\mathcal{L}$ and any family $(s_i)_{i\in I}$ with $s_i \in \mathcal{L}(Y, X_i)$, there is exactly one $s \in \mathcal{L}(Y, X)$ such that $\forall i \in I\ \pi_i\, s = s_i$.

Remark

As usual for objects characterized by a universal property: if $(X', (\pi'_i)_{i \in I})$ is another cartesian product of the X_i's, there is exactly one morphism $t \in \mathcal{L}(X, X')$ such that $\forall i \in I \ \pi'_i t = \pi_i$. Moreover, this morphism t is an iso.

$(X, (\pi_i)_{i \in I})$ and $(X', (\pi'_i)_{i \in I})$ are identical in the strongest categorical sense.

Notations:

- $X = \&_{i \in I} X_i$ and in the binary case $X = X_1 \& X_2$.
- if $s_i \in \mathcal{L}(Y, X_i)$ for each $i \in I$, we use $\langle s_i \rangle_{i \in I}$ for the unique element of $\mathcal{L}(Y, \&_{i \in I} X_i)$ such that $\forall i \in I\ \pi_i\ \langle s_j \rangle_{j \in I} = s_i$. In the binary case: $\langle s_1, s_2 \rangle : Y \to X_1 \& X_2$.
- If $I = \emptyset$ then X is the *terminal object* denoted as $\top$, characterized by: for any object Y of $\mathcal{L}$, the set $\mathcal{L}(Y, \top)$ is a singleton $\{\mathrm{t}_Y\}$.

Equational characterization

The following properties characterize the cartesian product:

- for any family $(s_i)_{i \in I}$ with $s_i \in \mathcal{L}(Y, X_i)$ for each $i \in I$ one has $\forall i \in I \; \pi_i \langle s_j \rangle_{j \in I} = s_i$
- moreover, if $t \in \mathcal{L}(Z, Y)$, one has $\langle s_i \rangle_{i \in I} \, t = \langle s_i \, t \rangle_{i \in I}$
- and last $\langle \pi_i \rangle_{i \in I} = \mathrm{Id}_{\&_{i \in I} X_i}$.

Remark

Most often models of linear logic have cartesian products of all countable families of objects, not only of finite families.

Cart. prod. as an SM functor

Given $s_i \in \mathcal{L}(X_i, Y_i)$ for $i = 1, 2$, we have $s_i\,\pi_i \in \mathcal{L}(X_1 \,\&\, X_2, Y_i)$ for $i = 1, 2$ and hence we have exactly one morphism

$$s_1 \,\&\, s_2 = \langle s_1\,\pi_1\,, s_2\,\pi_2\rangle \in \mathcal{L}(X_1 \,\&\, X_2, Y_1 \,\&\, Y_2)$$

such that

$$\begin{array}{ccc} X_1 \,\&\, X_2 & \xrightarrow{s_1 \& s_2} & Y_1 \,\&\, Y_2 \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ X_i & \xrightarrow{\quad s_i \quad} & Y_i \end{array} \qquad \text{for } i = 1, 2\,.$$

In this way we have defined a functor $\mathcal{L}^2 \to \mathcal{L}$.

- $\pi_2 \in \mathcal{L}(\top \mathbin{\&} X, X)$ is an iso (inverse $\langle \mathsf{t}_X, \mathsf{Id}_X \rangle$).
- $\pi_1 \in \mathcal{L}(X \mathbin{\&} \top, X)$ is an iso (inverse $\langle \mathsf{Id}_X, \mathsf{t}_X \rangle$).
- $\langle \pi_1\, \pi_1, \langle \pi_2\, \pi_1, \pi_2 \rangle \rangle \in \mathcal{L}((X_1 \mathbin{\&} X_2) \mathbin{\&} X_3, X_1 \mathbin{\&} (X_2 \mathbin{\&} X_3))$ is an iso (inverse $\langle \langle \pi_1, \pi_1\, \pi_2 \rangle, \pi_2\, \pi_2 \rangle$).
- $\langle \pi_2, \pi_1 \rangle \in \mathcal{L}(X_1 \mathbin{\&} X_2, X_2 \mathbin{\&} X_1)$ is an iso (inverse $\langle \pi_2, \pi_1 \rangle$).

These isos define another SM structure on $\mathcal{L}$.

Coproduct

We define $\oplus_{i\in I} X_i = (\&_{i\in I} X_i^{\perp})^{\perp}$ and

$$\overline{\pi}_i = \pi_i^{\perp} \in \mathcal{L}(X_i, \bigoplus_{j\in I} X_j)$$

then $(\oplus_{i\in I} X_i, (\overline{\pi}_i)_{i\in I})$ is the coproduct of the X_i's in $\mathcal{L}$ that is, we have the following universal property:

for any family of morphisms $(s_i)_{i\in I}$ with $s_i \in \mathcal{L}(X_i, Y)$, there is exactly one morphism $s \in \mathcal{L}(\oplus_{i\in I} X_i, Y)$ such that $s\,\overline{\pi}_i = s_i$ for each $i \in I$.

The cartesian product in **Rel**

Given a family $(E_i)_{i\in I}$ of sets, we define

$$\mathop{\&}_{i\in I} E_i = \bigcup_{i\in I} \{i\} \times E_i$$

$$\pi_j = \left\{((j,a),a) \mid a \in E_j\right\} \in \mathbf{Rel}(\mathop{\&}_{i\in I} E_i, E_j) \quad \text{for each } j \in I$$

Fact

$(\&_{i\in I} E_i, (\pi_i)_{i\in I})$ is the cartesian product of the E_i's in **Rel**.

Given $s_i \in \mathbf{Rel}(F, E_i)$ for each $i \in I$ then

$$\begin{aligned}\langle s_i \rangle_{i\in I} &= \{(b,(i,a)) \mid \forall i \in I \ (b,a) \in s_i\} \\ &\in \mathbf{Rel}(F, \mathop{\&}_{i\in I} E_i)\end{aligned}$$

Coproduct

$$\bigoplus_{i\in I} E_i = \left(\mathop{\&}_{i\in I} E_i^{\perp} \right)^{\perp} = \bigoplus_{i\in I} E_i = \bigcup_{i\in I} \{i\} \times E_i$$

$$\begin{aligned} \overline{\pi}_j &= \pi_j^{\perp} \in \mathbf{Rel}(E_j, \bigoplus_{i\in I} E_i) \\ &= \left\{ (a, (j, a)) \mid a \in E_j \right\} \end{aligned}$$

Exponential

Let $(\mathcal{L}, \dots)$ be a $*$-autonomous category which is cartesian (that is, has all finite cartesian products).

An exponential on $(\mathcal{L}, \dots)$ is a tuple $(!\_, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2_{,})$ where

- $(!\_, \mathsf{der}, \mathsf{dig})$ is a comonad on $\mathcal{L}$
- and $(\mathsf{m}^0, \mathsf{m}^2_{,})$ is a symmetric monoidal structure on this comonad: the *Seely isomorphisms*.

Let's explain. . .

Comonad

- $!\_ : \mathcal{L} \to \mathcal{L}$ is a functor
- and $\mathrm{der}_X \in \mathcal{L}(!X, X)$ and $\mathrm{dig}_X \in \mathcal{L}(!X, !!X)$ are natural in X

and moreover:

$$\begin{tikzcd} !X \arrow[r, "\mathrm{dig}_X"] \arrow[dr, "\mathrm{Id}"'] & !!X \arrow[d, "\mathrm{der}_{!X}"] \\ & !X \end{tikzcd} \qquad \begin{tikzcd} !X \arrow[r, "\mathrm{dig}_X"] \arrow[dr, "\mathrm{Id}"'] & !!X \arrow[d, "!\mathrm{der}_X"] \\ & !X \end{tikzcd}$$

$$\begin{tikzcd} !X \arrow[r, "\mathrm{dig}_X"] \arrow[d, "\mathrm{dig}_X"'] & !!X \arrow[d, "!\mathrm{dig}_X"] \\ !!X \arrow[r, "\mathrm{dig}_{!X}"] & !!!X \end{tikzcd}$$

Seely isomorphisms

$$\mathrm{m}^0 : 1 \to !\top$$

$$\mathrm{m}^2_{X_1,X_2} : !X_1 \otimes !X_2 \to !(X_1 \mathbin{\&} X_2)$$

are isos in $\mathcal{L}$, and $\mathrm{m}^2_{X_1,X_2}$ is natural in X_1 and X_2. Moreover some symmetric monoidality commutations hold such as

$$\begin{tikzcd}
(!X_1 \otimes !X_2) \otimes !X_3 \arrow[r, "\alpha_{!X_1,!X_2,!X_3}"] \arrow[d, "\mathrm{m}^2_{X_1,X_2} \otimes !X_3"'] & !X_1 \otimes (!X_2 \otimes !X_3) \arrow[d, "!X_1 \otimes \mathrm{m}^2_{X_2,X_3}"] \\
!(X_1 \mathbin{\&} X_2) \otimes !X_3 \arrow[d, "\mathrm{m}^2_{X_1 \& X_2, X_3}"'] & !X_1 \otimes !(X_2 \mathbin{\&} X_3) \arrow[d, "\mathrm{m}^2_{X_1, X_2 \& X_3}"] \\
!((X_1 \mathbin{\&} X_2) \mathbin{\&} X_3) \arrow[r, "\langle \pi_1\, \pi_1, \langle \pi_2\, \pi_1, \pi_2 \rangle\rangle"] & !(X_1 \mathbin{\&} (X_2 \mathbin{\&} X_3))
\end{tikzcd}$$

or

$$\begin{tikzcd}
!X_1 \otimes !X_2 \arrow[r, "\gamma_{!X_1,!X_2}"] \arrow[d, "\mathrm{m}^2_{X_1,X_2}"'] & !X_2 \otimes !X_1 \arrow[d, "\mathrm{m}^2_{X_2,X_1}"] \\
!(X_1 \mathbin{\&} X_2) \arrow[r, "!\langle \pi_2,\pi_1 \rangle"] & !(X_2 \mathbin{\&} X_1)
\end{tikzcd}$$

Plus an additional diagram (compatibility with dig)

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "\mathrm{m}^2_{X,Y}"] \arrow[dd, "\mathrm{dig}_X \otimes \mathrm{dig}_Y"'] & !(X \mathbin{\&} Y) \arrow[d, "\mathrm{dig}_{X\&Y}"] \\
 & !!(X \mathbin{\&} Y) \arrow[d, "!\langle !\pi_1,!\pi_2 \rangle"] \\
!!X \otimes !!Y \arrow[r, "\mathrm{m}^2_{!X,!Y}"] & !(!X \mathbin{\&} !Y)
\end{tikzcd}$$

As usual this allows to define canonically

$$\mathrm{m}^n_{X_1,\dots,X_n} \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, !(X_1 \mathbin{\&} \cdots \mathbin{\&} X_n))$$

This is obtained by combining instances of m^2 and associativity isos of $\otimes$ and $\&$; the specific combination chosen is irrelevant thanks to the monoidality commutations.

Remark

We use the fact that $!X_1 \otimes \cdots \otimes !X_n$ *without parenthesis* makes sense because $(\mathcal{L}, \otimes)$ is monoidal, and similarly for $X_1 \mathbin{\&} \cdots \mathbin{\&} X_n$ because $(\mathcal{L}, \&)$ is monoidal.

The comonad in **Rel**

The canonical choice is to take

$$!E = \mathcal{M}_{\mathrm{fin}}(E) = \{\text{finite multisets of elements of } E\}$$

Definition

An element of $\mathcal{M}_{\mathrm{fin}}(E)$ is a function $m : E \to \mathbb{N}$ such that

$$\mathrm{supp}(m) = \{a \in E \mid m(a) \neq 0\}$$

is finite.

Notations on multisets

- $[]$ the empty multiset
- $m_1 + m_2$, $\sum_{i=1}^{k} m_i$ defined pointwise
- if $a_1, \ldots, a_k \in E$ then $m = [a_1, \ldots, a_k]$ defined by

$$m(a) = \#\{i \in \{1, \ldots, k\} \mid a_i = a\}$$

- if moreover P is a predicate on $\{1, \ldots, k\}$, then $m = [a_i \mid P(i)] \in \mathcal{M}_{\mathrm{fin}}(E)$ defined by

$$m(a) = \#\{i \in \{1, \ldots, k\} \mid a_i = a \text{ and } P(i)\}$$

- $!E = \mathcal{M}_{\text{fin}}(E)$
- if $s \in \mathbf{Rel}(E, F)$ then

$$\begin{aligned} !s = \{([a_1, \ldots, a_k], [b_1, \ldots, b_k]) \mid \\ k \in \mathbb{N} \text{ and } \forall i \in \{1, \ldots k\}\ (a_i, b_i) \in s\} \end{aligned}$$

In other words, $(m, p) \in !s$ iff we can write $m = [a_1, \ldots, a_k]$ and $p = [b_1, \ldots, b_k]$ with $\forall i \in \{1, \ldots k\}\ (a_i, b_i) \in s$.

Example

$E = \{1, 2\}$, $F = \{1, 2, 3\}$, $s = \{(1, 2), (2, 2), (1, 3)\}$.

$$\begin{array}{ll} ([1, 1], [2, 3]) \in !s & ([1, 2], [2, 2]) \in !s \\ ([1, 2], [2, 3]) \in !s & ([1], [2, 3]) \notin !s \end{array}$$

Functoriality of $!\_$

$s \in \mathbf{Rel}(E, F)$, $t \in \mathbf{Rel}(F, G)$, we must prove that

$$\begin{tikzcd} !E \arrow[r, "!s"] \arrow[dr, "!(t\,s)"'] & !F \arrow[d, "!t"] \\ & !G \end{tikzcd}$$

How to prove such a commutation in **Rel**

Take $(m, q) \in\ !E \times\ !G$ and prove that

$$(m, q) \in\ !(t\,s) \Leftrightarrow (m, q) \in\ !t\ !s$$

Assume first $(m, q) \in !(t\, s)$. We can write

$$m = [a_1, \ldots, a_k] \text{ and } q = [c_1, \ldots, c_k]$$

with

$$\forall i \in \{1, \ldots, k\} \quad (a_i, c_i) \in t\, s$$

so for each $i \in \{1, \ldots, k\}$ there is $b_i \in F$ such that

$$\forall i \in \{1, \ldots, k\} \quad (a_i, b_i) \in s \text{ and } (b_i, c_i) \in t\,.$$

We set $p = [b_1, \ldots, b_k]$.

Then we have $(m, p) \in !s$ and $(p, q) \in !t$ and hence $(m, q) \in !t\, !s$.

Conversely assume $(m, q) \in {!t}\,{!s}$.

So let $p \in {!F}$ be such that $(m, p) \in {!s}$ and $(p, q) \in {!t}$.

We can write $m = [a_1, \dots, a_k]$ and $p = [b_1, \dots, b_k]$ such that

$$\forall i \in \{1, \dots, k\} \quad (a_i, b_i) \in s$$

and we can write $q = [c_1, \dots, c_k]$ with

$$\forall i \in \{1, \dots, k\} \quad (b_i, c_i) \in t$$

and it follows that $\forall i \in \{1, \dots, k\}\ (a_i, c_i) \in t\, s$ and hence $(m, q) \in {!(t\, s)}$.

One proves in the same way that $!\mathrm{Id}_E = \mathrm{Id}_{!E}$.

The comonad structure in **Rel**

$$
\begin{aligned}
\mathsf{der}_E &= \{([a], a) \mid a \in E\} \in \mathbf{Rel}(!E, E) \\
\mathsf{dig}_E &= \{(m_1 + \cdots + m_k, [m_1, \ldots, m_k]) \\
&\qquad\qquad \mid k \in \mathbb{N} \text{ and } m_1, \ldots, m_k \in !E\} \in \mathbf{Rel}(!E, !!E)
\end{aligned}
$$

These morphisms are natural in E.

Notation

If $M = [m_1, \ldots, m_k] \in \,!!E$ then $\Sigma M = \sum_{i=1}^{k} m_i \in \,!E$.

With this notation

$$\mathrm{dig}_E = \{(\Sigma M, M) \mid M \in \,!!E)\} \ .$$

A simple lemma

Lemma

Let $(m, p) \in\, !s$ *for some* $s \in \mathbf{Rel}(E, F)$.
Let $P = [p_1, \ldots, p_k] \in\, !!F$ *be such that* $\sum_{i=1}^{k} p_i = p$.
Then there are $m_1, \ldots, m_k \in\, !E$ *such that*

- $\forall i \in \{1, \ldots, k\}\ \ (m_i, p_i) \in\, !s$
- $m = \sum_{i=1}^{k} m_i$

In other words: if $m \in\, !E$ *and* $P \in\, !!F$ *satisfy* $(m, \Sigma P) \in\, !s$ *then there exists* $M \in\, !!E$ *such that* $m = \Sigma M$ *and* $(M, P) \in\, !!s$.

Proof of the lemma

Write $p = [b_1, \ldots, b_n]$.

Since $p = \sum_{i=1}^{k} p_i$ we can find $I_1, \ldots, I_k \subseteq \{1, \ldots, n\}$ *pairwise disjoint* such that $\bigcup_{i=1}^{k} I_i = \{1, \ldots, n\}$ and $p_i = [b_j \mid j \in I_i]$ for $i = 1, \ldots, k$.

Since $(m, p) \in !s$ we can write $m = [a_1, \ldots, a_n]$ with $(a_j, b_j) \in s$ for $j = 1, \ldots, n$.

For $i = 1, \ldots, k$ let $m_i = [a_j \mid j \in I_i]$, we have $\sum_{i=1}^{k} m_i = m$ and $\forall i \in \{1, \ldots, k\}\ \ (m_i, p_i) \in !s$.

Naturality of dig_E

Take $s \in \mathbf{Rel}(E, F)$ and prove that

$$\begin{array}{ccc} !E & \xrightarrow{!s} & !F \\ {\scriptstyle \mathsf{dig}_E}\downarrow & & \downarrow{\scriptstyle \mathsf{dig}_F} \\ !!E & \xrightarrow{!!s} & !!F \end{array}$$

Take $(m, P) \in !E \times !!F$ and prove that

$$(m, P) \in !!s\ \mathsf{dig}_E \Leftrightarrow (m, P) \in \mathsf{dig}_F\ !s\,.$$

Assume first $(m, P) \in\ !!s\ \mathrm{dig}_E$. So let $M \in\ !!E$ be such that

$$(m, M) \in \mathrm{dig}_E \text{ and } (M, P) \in\ !!s$$

This means that we can write $M = [m_1, \ldots, m_k]$ and $P = [p_1, \ldots, p_k]$ with

$$m = \sum_{i=1}^{k} m_i \quad \text{and} \quad \forall i \in \{1, \ldots, k\}\ (m_i, p_i) \in\ !s\,.$$

By the second property we have $(\sum_{i=1}^{k} m_i, \sum_{i=1}^{k} p_i) \in\ !s$.

Let $p = \sum_{i=1}^{k} p_i$, we have $(p, P) \in \mathrm{dig}_F$, $(m, p) \in\ !s$ and hence $(m, P) \in \mathrm{dig}_F\ !s$.

Conversely assume $(m, P) \in \mathrm{dig}_F\ !s$. So let $p \in !F$ be such that

$$(m, p) \in !s \quad \text{and} \quad (p, P) \in \mathrm{dig}_F$$

Let us write $P = [p_1, \dots, p_k]$ so that $\sum_{i=1}^{k} p_i = p$. By the Lemma we can find $m_1, \dots, m_k \in !E$ such that $\sum_{i=1}^{k} m_i = m$ and $\forall i \in \{1, \dots, k\}\ \ (m_i, p_i) \in !s$.

Let $M = [m_1, \dots, m_k]$. We have $(M, P) \in !!s$ and $(m, M) \in \mathrm{dig}_E$ hence $(m, P) \in !!s\ \mathrm{dig}_E$.

Comonadicity in **Rel**

Remember: one has to prove the following commutations

$$
\begin{tikzcd}
!E \arrow[r, "\mathrm{dig}_E"] \arrow[dr, "\mathrm{Id}"'] & !!E \arrow[d, "\mathrm{der}_{!E}"] \\
& !E
\end{tikzcd}
\qquad
\begin{tikzcd}
!E \arrow[r, "\mathrm{dig}_E"] \arrow[dr, "\mathrm{Id}"'] & !!E \arrow[d, "!\mathrm{der}_E"] \\
& !E
\end{tikzcd}
$$

$$
\begin{tikzcd}
!E \arrow[r, "\mathrm{dig}_E"] \arrow[d, "\mathrm{dig}_E"'] & !!E \arrow[d, "!\mathrm{dig}_E"] \\
!!E \arrow[r, "\mathrm{dig}_{!E}"] & !!!E
\end{tikzcd}
$$

Let us prove the last commutation: take $(m, \mathcal{M}) \in {!E} \times {!!!E}$.

Assume first $(m, \mathcal{M}) \in \mathrm{dig}_{!E}\ \mathrm{dig}_E$, we prove $(m, \mathcal{M}) \in {!\mathrm{dig}_E}\ \mathrm{dig}_E$.

Let $M \in {!!E}$ with $(m, M) \in \mathrm{dig}_E$ and $(M, \mathcal{M}) \in \mathrm{dig}_{!E}$ that is

$$\Sigma M = m \quad \text{and} \quad \Sigma\mathcal{M} = M\,.$$

We write $\mathcal{M} = [M_1, \ldots, M_k]$ and set $M' = [\Sigma M_1, \ldots, \Sigma M_k] \in {!!E}$.

Then

$$\Sigma M' = \sum_{i=1}^{k} \Sigma M_i = \Sigma(\sum_{i=1}^{k} M_i) = \Sigma\Sigma\mathcal{M} = \Sigma M = m$$

that is $(m, M') \in \mathrm{dig}_E$.

For $i = 1, \ldots, k$ we have $(\Sigma M_i, M_i) \in \mathrm{dig}_E$ and hence $(M', \mathcal{M}) \in {!\mathrm{dig}_E}$. So $(m, \mathcal{M}) \in {!\mathrm{dig}_E}\ \mathrm{dig}_E$.

Assume conversely that $(m, \mathcal{M}) \in !\mathrm{dig}_E\ \mathrm{dig}_E$.

Let $M \in !E$ with $(m, M) \in \mathrm{dig}_E$ and $(M, \mathcal{M}) \in !\mathrm{dig}_E$.

We can write $\mathcal{M} = [M_1, \dots, M_k]$ and $M = [m_1, \dots, m_k]$ with $(m_i, M_i) \in \mathrm{dig}_E$, that is $\Sigma M_i = m_i$, for $i = 1, \dots, k$.

Let $M' = \Sigma\mathcal{M}$ so that $(M', \mathcal{M}) \in \mathrm{dig}_{!E}$. We have

$$\Sigma M' = \Sigma\Sigma\mathcal{M} = \Sigma(\sum_{i=1}^{k} M_i) = \sum_{i=1}^{k} \Sigma M_i = \sum_{i=1}^{k} m_i = m$$

since $(m, M) \in \mathrm{dig}_E$ that is $\Sigma M = m$.

This shows that $(m, M') \in \mathrm{dig}_E$ and hence $(m, \mathcal{M}) \in \mathrm{dig}_{!E}\ \mathrm{dig}_E$.

The Seely isomorphisms in **Rel**

$$\mathrm{m}^0 : 1 \to\, !\top$$
$$\mathrm{m}^2_{E_1,E_2} : !E_1 \otimes !E_2 \to\, !(E_1 \,\&\, E_2)$$

are the isos defined by

$$\mathrm{m}^0 = \{(*, [\,])\}$$
$$\mathrm{m}^2_{E_1,E_2} = \{((m_1, m_2), 1 \cdot m_1 + 2 \cdot m_2) \mid m_i \in\, !E_i \text{ for } i = 1, 2\}$$

where $l \cdot [a_1, \ldots, a_k] = [(l, a_1), \ldots, (l, a_k)]$. The inverse of $\mathrm{m}^2_{E_1,E_2}$ is

$$\{([(1, a_1), \ldots, (1, a_k), (2, b_1), \ldots, (2, b_n)], ([a_1, \ldots, a_k], [b_1, \ldots, b_n])) \mid a_1, \ldots, a_k \in E_1 \text{ and } b_1, \ldots, b_n \in E_2\}$$

One has to check the Seely commutations.

Derived structures in a model of LL, with illustration in **Rel**

Structural morphisms

In any model of LL $(\mathcal{L}, \dots)$ as described, we have

$$\mathsf{w}_X \in \mathcal{L}(!X, 1) \quad \text{weakening}$$
$$\mathsf{c}_X \in \mathcal{L}(!X, !X \otimes !X) \quad \text{contraction}$$

defined by (remember that $\mathcal{L}(X, \top) = \{\mathsf{t}_X\}$)

$$!X \xrightarrow{!\mathsf{t}_X} !\top \xrightarrow{(\mathsf{m}^0)^{-1}} 1$$

$$!X \xrightarrow{!\langle \mathsf{Id}_X, \mathsf{Id}_X \rangle} !(X \mathbin{\&} X) \xrightarrow{(\mathsf{m}^2_{X,X})^{-1}} !X \otimes !X$$

Intuition

The elements of $!X$ are discardable and duplicable.

Then $(!X, \mathsf{w}_X, \mathsf{c}_X)$ is a commutative comonoid in $\mathcal{L}$, meaning that the following diagrams commute.

Associativity:

$$\begin{tikzcd}
!X \arrow[r, "\mathsf{c}_X"] \arrow[d, "\mathsf{c}_X"'] & !X \otimes !X \arrow[r, "\mathsf{c}_X \otimes !X"] & (!X \otimes !X) \otimes !X \arrow[d, "\alpha_{!X,!X,!X}"] \\
!X \otimes !X \arrow[rr, "!X \otimes \mathsf{c}_X"] & & !X \otimes (!X \otimes !X)
\end{tikzcd}$$

Comes from the monoidality of m^2.

Left neutrality

$$\begin{tikzcd}
!X \arrow[r, "\mathsf{c}_X"] \arrow[drr, "\mathsf{Id}_{!X}"'] & !X \otimes !X \arrow[r, "\mathsf{w}_X \otimes !X"] & 1 \otimes !X \arrow[d, "\lambda_{!X}"] \\
& & !X
\end{tikzcd}$$

Commutativity

$$\begin{tikzcd}
!X \arrow[r, "\mathsf{c}_X"] \arrow[dr, "\mathsf{c}_X"'] & !X \otimes !X \arrow[d, "\gamma_{!X,!X}"] \\
& !X \otimes !X
\end{tikzcd}$$

Promotion

This is sometimes called the *lifting* of the comonad: given $s \in \mathcal{L}(!X, Y)$, one defined $s^! \in \mathcal{L}(!X, !Y)$ as

$$!X \xrightarrow{\mathrm{dig}_X} !!X \xrightarrow{!s} !Y$$

In **Rel**, given $s \in \mathbf{Rel}(!E, F)$ and $s^! \in \mathbf{Rel}(!E, !F)$ is

$$s^! = \{(m_1 + \cdots + m_k, [b_1, \ldots, b_k]) \mid (m_i, b_i) \in s \text{ for } i = 1, \ldots, k\}$$

Comonoid structure of $!E$ in **Rel**

We have

$$\begin{aligned} \mathsf{w}_E &= \{([\,], *)\} \in \mathbf{Rel}(!E, 1) \\ \mathsf{c}_E &= \{(m_1 + m_2, (m_1, m_2)) \mid m_1, m_2 \in !E\} \in \mathbf{Rel}(!E, !E \otimes !E) \end{aligned}$$

Lax symmetric monoidal structure of $!\_$

Remember

The Seely morphisms m^0 and $\mathrm{m}^2_{X_1,X_2}$ are a symmetric monoidal structure on $!\_$ from the SMC $(\mathcal{L}, \&)$ to the SMC $(\mathcal{L}, \otimes)$ which is *strong*: the Seely morphisms are *isomorphisms*.

There is also a symmetric monoidal structure on $!\_$ from $(\mathcal{L}, \otimes)$ to $(\mathcal{L}, \otimes)$ given by morphisms

$$\mu^0 : 1 \to !1$$
$$\mu^2_{X_1,X_2} : !X_1 \otimes !X_2 \to !(X_1 \otimes X_2)$$

which are *not isos* in general: it is a *lax* SM structure.

μ^0 is

$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

and $\mu^2_{X_1,X_2}$ is

$$\begin{tikzcd}
!X_1 \otimes !X_2 \arrow[r, "\mathsf{m}^2_{X_1,X_2}"] & !(X_1 \,\&\, X_2) \arrow[d, "\mathsf{dig}_{X_1 \& X_2}"] \\
!(!X_1 \otimes !X_2) \arrow[d, "!(\mathsf{der}_{X_1} \otimes \mathsf{der}_{X_2})"] & !!(X_1 \,\&\, X_2) \arrow[l, "!(\mathsf{m}^2_{X_1,X_2})^{-1}"'] \\
!(X_1 \otimes X_2) &
\end{tikzcd}$$

These morphisms satisfy symmetric monoidality commutations such as

$$\begin{tikzcd}
(!X_1 \otimes !X_2) \otimes !X_3 \arrow[rr, "\alpha_{!X_1,!X_2,!X_3}"] \arrow[d, "\mu^2_{X_1,X_2} \otimes !X_3"'] & & !X_1 \otimes (!X_2 \otimes !X_3) \arrow[d, "!X_1 \otimes \mu^2_{X_2,X_3}"] \\
!(X_1 \otimes X_2) \otimes !X_3 \arrow[d, "\mu^2_{X_1 \otimes X_2, X_3}"'] & & !X_1 \otimes !(X_2 \otimes X_3) \arrow[d, "\mu^2_{X_1, X_2 \otimes X_3}"] \\
!((X_1 \otimes X_2) \otimes X_3) \arrow[rr, "!\alpha_{X_1,X_2,X_3}"] & & !(X_1 \otimes (X_2 \otimes X_3))
\end{tikzcd}$$

See the lecture notes for a complete list of these commutations.

As a consequence, we can define canonically

$$\mu^n_{X_1,\dots,X_n} : !X_1 \otimes \cdots \otimes !X_n \to !(X_1 \otimes \cdots \otimes X_n)$$

in accordance with the fact that $X_1 \otimes \cdots \otimes X_n$ makes sense *without parentheses* because $\mathcal{L}$ is a monoidal category.

Example

The last diagram tells us that, up to associativity of $\otimes$ (as specified by the α isos), there is only one way of combining the μ^2 morphisms to obtain

$$\mu^3_{X_1,X_2,X_3} : !X_1 \otimes !X_2 \otimes !X_3 \to !(X_1 \otimes X_2 \otimes X_3)$$

Lax monoidal structure in **Rel**

Remember that in **Rel**, $\top = \emptyset$ and that μ^0 is

$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

We have $\mathsf{m}^0 = \{*, [\,]\}$ and
$\mathsf{dig}_\top = \{(\Sigma M, M) \mid M \in \mathcal{M}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{fin}}(\emptyset))\}$ hence

$$\mathsf{dig}_\top = \{([\,], k[[\,]]) \mid k \in \mathbb{N}\}$$

since $\mathcal{M}_{\mathrm{fin}}(\emptyset) = \{[\,]\}$. So

$$\mu^0 = \{(*, k[*]) \mid k \in \mathbb{N}\}$$

where $km = \overbrace{m + \cdots + m}^{k}$ for any $m \in [E]$. μ^0 is not an iso!

And $\mu^2_{E_1,E_2}$ is

$$\begin{tikzcd}
!E_1 \otimes !E_2 \arrow[rr, "\mathsf{m}^2_{E_1,E_2}"] & & !(E_1 \mathbin{\&} E_2) \arrow[d, "\mathsf{dig}_{E_1 \& E_2}"] \\
!(!E_1 \otimes !E_2) \arrow[d, "!(\mathsf{der}_{E_1} \otimes \mathsf{der}_{E_2})"] & & !!(E_1 \mathbin{\&} E_2) \arrow[ll, "!(\mathsf{m}^2_{E_1,E_2})^{-1}"'] \\
!(E_1 \otimes E_2) & &
\end{tikzcd}$$

We have

$$\begin{aligned}
&!(\mathsf{der}_{E_1} \otimes \mathsf{der}_{E_2}) \\
&\qquad = \{[([a_1],[b_1]),\dots,([a_k],[b_k])],[(a_1,b_1),\dots,(a_k,b_k)]) \\
&\qquad\qquad\qquad\qquad\qquad\qquad a_1,\dots,a_k \in E \text{ and } b_1,\dots,b_k \in F\}
\end{aligned}$$

So

$$
\begin{aligned}
&!(\mathrm{der}_{E_1} \otimes \mathrm{der}_{E_2})\,!(\mathrm{m}^2_{E_1,E_2})^{-1} = \{ \\
&([([(1,a_1),(2,b_1)]),\ldots,([(1,a_k),(2,b_k)])],[(a_1,b_1),\ldots,(a_k,b_k)]) \\
&\qquad \mid a_1,\ldots,a_k \in E \text{ and } b_1,\ldots,b_k \in F\} \\
&\qquad\qquad \in \mathbf{Rel}(!!(E_1 \,\&\, E_2),!(E_1 \otimes E_2))
\end{aligned}
$$

then

$$
\begin{aligned}
&!(\mathrm{der}_{E_1} \otimes \mathrm{der}_{E_2})\,!(\mathrm{m}^2_{E_1,E_2})^{-1}\ \mathrm{dig}_{E_1\&E_2} = \{ \\
&([(1,a_1),(2,b_1),\ldots,(1,a_k),(2,b_k)],[(a_1,b_1),\ldots,(a_k,b_k)]) \\
&\qquad \mid a_1,\ldots,a_k \in E \text{ and } b_1,\ldots,b_k \in F\} \\
&\qquad\qquad \in \mathbf{Rel}(!(E_1 \,\&\, E_2),!(E_1 \otimes E_2))
\end{aligned}
$$

Finally

$$\mu^2_{E_1,E_2} = !(\mathrm{der}_{E_1} \otimes \mathrm{der}_{E_2})\,!(\mathrm{m}^2_{E_1,E_2})^{-1}\ \mathrm{dig}_{E_1 \& E_2}\ \mathrm{m}^2_{E_1,E_2} = \{ \\ (([a_1,\dots,a_k],[b_1,\dots,b_k]),[(a_1,b_1),\dots,(a_k,b_k)]) \\ \mid a_1,\dots,a_k \in E \text{ and } b_1,\dots,b_k \in F\} \\ \in \mathbf{Rel}(!(E_1 \mathbin{\&} E_2), !(E_1 \otimes E_2))$$

Computes all possible “pairings” between two multisets which have the same size.

And more generally

$$\mu^n_{E_1,\dots,E_n} \in \mathbf{Rel}(!E_1 \otimes \cdots \otimes !E_n, !(E_1 \otimes \cdots \otimes E_n)) = \{ \\ ([a^1_1,\dots,a^1_k],\dots,[a^n_1,\dots,a^n_k],[(a^1_1,\dots,a^n_1),\dots,(a^1_k,\dots,a^n_k)]) \\ \mid a^i_j \in E_i \text{ for } i = 1,\dots,n \text{ and } j = 1,\dots,k\}\,.$$

Generalized weakening and contraction

We have $\mathsf{w}_{X_1,\ldots,X_n} \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, 1)$ given by

$$!X_1 \otimes \cdots \otimes !X_n \xrightarrow{\mathsf{w}_{X_1} \otimes \cdots \otimes \mathsf{w}_{X_n}} 1 \otimes \cdots \otimes 1 \xrightarrow{\theta} 1$$

where θ is an iso obtained by combining instances of λ, ρ etc (again, by McLane's theorem, θ does not depend on the chosen combination).

and

$\mathsf{c}_{X_1,\ldots,X_n} \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, (!X_1 \otimes \cdots \otimes !X_n) \otimes (!X_1 \otimes \cdots \otimes !X_n))$

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "\mathsf{c}_{X_1} \otimes \cdots \otimes \mathsf{c}_{X_n}"] & (!X_1 \otimes !X_1) \otimes \cdots \otimes (!X_n \otimes !X_n) \arrow[d, "\theta"] \\
& (!X_1 \otimes \cdots \otimes !X_n) \otimes (!X_1 \otimes \cdots \otimes !X_n)
\end{tikzcd}$$

where θ is a combination of instances of γ and α (again the specific chosen combination is irrelevant).

in **Rel**

We have

$$\mathsf{w}_{E_1,\ldots,E_n} = \{(([\,],\ldots,[\,]),*)\}$$

and

$$\begin{aligned}
&\mathsf{c}_{E_1,\ldots,E_n} \\
&= \{((m_1 + m'_1,\ldots,m_n + m'_n),((m_1,\ldots,m_n),(m'_1,\ldots,m'_n))) \\
&\qquad \mid m_i, m'_i \in\, !E_i \text{ for } i = 1,\ldots,n\}
\end{aligned}$$

Generalized promotion

For interpreting the promotion rule of LL

$$\frac{!A_1, \ldots, !A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}$$

we need a more general kind of promotion in the model: given $s \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$ we need $s^! \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, !Y)$. It is given by:

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "\mathrm{dig}_{X_1} \otimes \cdots \otimes \mathrm{dig}_{X_n}"] & !!X_1 \otimes \cdots \otimes !!X_n \arrow[d, "\mu^n_{!X_1, \ldots, !X_n}"] \\
!Y & !(!X_1 \otimes \cdots \otimes !X_n) \arrow[l, "!s"']
\end{tikzcd}$$

In **Rel**, given $s \in \mathbf{Rel}(!E_1 \otimes \cdots \otimes !E_n, F)$ we have

$$s^! = \{(\sum_{j=1}^{k} m_j^1, \ldots, \sum_{j=1}^{k} m_j^n, [b_1, \ldots, b_k]) \\ \mid (m_j^1, \ldots, m_j^n, b_j) \in s \text{ for } j = 1, \ldots, k\}$$

Promoted morphisms are discardable and duplicable

Let $s \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$ then

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "s^!"] \arrow[dr, "\mathsf{w}_{X_1,\ldots,X_n}"'] & !Y \arrow[d, "\mathsf{w}_Y"] \\
& 1
\end{tikzcd}$$

and

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "s^!"] \arrow[d, "\mathsf{c}_{X_1,\ldots,X_n}"'] & !Y \arrow[d, "\mathsf{c}_Y"] \\
(!X_1 \otimes \cdots \otimes !X_n) \otimes (!X_1 \otimes \cdots \otimes !X_n) \arrow[r, "s^! \otimes s^!"] & !Y \otimes !Y
\end{tikzcd}$$

Promotion and "substitution"

Let $s \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$ and
$t \in \mathcal{L}(!Y_1 \otimes \cdots \otimes !Y_k \otimes !Y, Z)$. Then we have

$$\begin{tikzcd}
!Y_1 \otimes \cdots \otimes !Y_k \otimes !X_1 \otimes \cdots \otimes !X_n \arrow[r, "\mathrm{Id} \otimes s^!"] \arrow[dr, "(t\,(\mathrm{Id} \otimes s^!))^!"'] & !Y_1 \otimes \cdots \otimes !Y_k \otimes !Y \arrow[d, "t^!"] \\
& !Z
\end{tikzcd}$$

Notice that

$$t\,(\mathrm{Id} \otimes s^!) \in \mathcal{L}(!Y_1 \otimes \cdots \otimes !Y_k \otimes !X_1 \otimes \cdots \otimes !X_n, Z)$$

Promotion, dereliction and digging

Let $s \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$ then

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "s^!"] \arrow[dr, "s"'] & !Y \arrow[d, "\mathrm{der}_Y"] \\
 & Y
\end{tikzcd}$$

$$\begin{tikzcd}
!X_1 \otimes \cdots \otimes !X_n \arrow[r, "s^!"] \arrow[dr, "s^{!!}"'] & !Y \arrow[d, "\mathrm{dig}_Y"] \\
 & !!Y
\end{tikzcd}$$

The Eilenberg-Moore category of $!\_$

Given a model of LL

General idea

These structural properties of "promoted morphisms" (discardability, duplicability, substitution) can be extended to more general morphisms: those of the Eilenberg-Moore category.

$(\mathcal{L}, \dots)$, we can consider the Eilenberg Moore category $\mathcal{L}^!$ of the $(!\_, \mathrm{der}, \mathrm{dig})$ comonad, or category of *coalgebras*.

The EM category can be defined for any comonad of course, it does not use the other components of the model $\mathcal{L}$.

Definition

An object of $\mathcal{L}^!$ is a pair $P = (\underline{P}, h_P)$ where

- $\underline{P}$ is an object of $\mathcal{L}$
- and $h_P \in \mathcal{L}(\underline{P}, !\underline{P})$ such that

$$\begin{tikzcd}
\underline{P} \arrow[r, "h_P"] \arrow[dr, "\mathrm{Id}_{\underline{P}}"'] & !\underline{P} \arrow[d, "\mathrm{der}_{\underline{P}}"] \\
& \underline{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "h_P"] \arrow[d, "h_P"'] & !\underline{P} \arrow[d, "\mathrm{dig}_{\underline{P}}"] \\
!\underline{P} \arrow[r, "!h_P"] & !!\underline{P}
\end{tikzcd}$$

Morphisms in $\mathcal{L}^!$

Definition

An element of $\mathcal{L}^!(P, Q)$ is a $s \in \mathcal{L}(\underline{P}, \underline{Q})$ such that

$$\begin{tikzcd}
\underline{P} \arrow[r, "s"] \arrow[d, "h_P"'] & \underline{Q} \arrow[d, "h_Q"] \\
!\underline{P} \arrow[r, "!s"] & !\underline{Q}
\end{tikzcd}$$

LL intuition

An object P of $\mathcal{L}^!$ is an object $\underline{P}$ equipped with its own *structural rules*.

Indeed we can equip a $P \in \mathrm{Obj}(\mathcal{L}^!)$ with a weakening w_P:

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{\mathsf{w}_{\underline{P}}} 1$$

and a contraction c_P:

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{\mathsf{c}_{\underline{P}}} !\underline{P} \otimes !\underline{P} \xrightarrow{\mathrm{der}_{\underline{P}} \otimes \mathrm{der}_{\underline{P}}} \underline{P} \otimes \underline{P}$$

Fact

For any $P \in \mathrm{Obj}(\mathcal{L}^!)$, the triple $(\underline{P}, \mathsf{w}_P, \mathsf{c}_P)$ is a commutative comonoid $\mathrm{comon}(P) \in \mathrm{Obj}(\mathrm{Ccom}(\mathcal{L}))$.

Let us explain this. . .

$\ldots \mathcal{L}^!$ is cartesian!

If P and Q are objects of $\mathcal{L}^!$ then we set

$$P \otimes Q = (\underline{P} \otimes \underline{Q}, h_{P \otimes Q})$$

where $h_{P \otimes Q}$ is

$$\underline{P} \otimes \underline{Q} \xrightarrow{h_P \otimes h_Q} !\underline{P} \otimes !\underline{Q} \xrightarrow{\mu^2_{\underline{P},\underline{Q}}} !(\underline{P} \otimes \underline{Q})$$

Fact

$P \otimes Q$ is an object of $\mathcal{L}^!$.

This is based on the following commutations in $\mathcal{L}$

$$
\begin{tikzcd}
!X \otimes !Y \arrow[r, "\mu^2_{X,Y}"] \arrow[dr, "\mathrm{der}_X \otimes \mathrm{der}_Y"'] & !(X \otimes Y) \arrow[d, "\mathrm{der}_{X \otimes Y}"] \\
& X \otimes Y
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
!X \otimes !Y \arrow[rr, "\mu^2_{X,Y}"] \arrow[d, "\mathrm{dig}_X \otimes \mathrm{dig}_Y"'] & & !(X \otimes Y) \arrow[d, "\mathrm{dig}_{X \otimes Y}"] \\
!!X \otimes !!Y \arrow[r, "\mu^2_{!X,!Y}"] & !(!X \otimes !Y) \arrow[r, "!\mu^2_{X,Y}"] & !!(X \otimes Y)
\end{tikzcd}
$$

We can see 1 as an object of $\mathcal{L}^!$, taking 1 (of $\mathcal{L}$) for $\underline{1}$ and $h_1 = \mu^0 \in \mathcal{L}(1, !1)$. One can check indeed that

$$\begin{tikzcd} 1 \arrow[r, "\mu^0"] \arrow[dr, "\mathsf{Id}_1"'] & !1 \arrow[d, "\mathsf{der}_1"] \\ & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mu^0"] \arrow[d, "\mu^0"'] & !1 \arrow[d, "\mathsf{dig}_1"] \\ !1 \arrow[r, "!\mu^0"] & !!1 \end{tikzcd}$$

Fact

The object 1 of $\mathcal{L}^!$ is terminal in $\mathcal{L}^!$.

The unique element of $\mathcal{L}^!(P, 1)$ is tt_P given by

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{\mathsf{w}_{\underline{P}}} 1$$

We have projections $\mathrm{pr}_i \in \mathcal{L}^!(P_1 \otimes P_2, P_i)$, for instance pr_2 is defined as

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{h_{P_1} \otimes \underline{P_2}} !\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathrm{w}_{\underline{P_1}} \otimes \underline{P_2}} 1 \otimes \underline{P_2} \xrightarrow{\lambda_{\underline{P_2}}} \underline{P_2}$$

And given $s_i \in \mathcal{L}^!(Q, P_i)$ for $i = 1, 2$, one can define $\langle s_1, s_2\rangle^{\otimes} \in \mathcal{L}^!(Q, P_1 \otimes P_2)$ as

$$\underline{Q} \xrightarrow{h_Q} !\underline{Q} \xrightarrow{\mathsf{c}_{\underline{Q}}} !\underline{Q} \otimes !\underline{Q} \xrightarrow{\mathrm{der}_{!\underline{Q}} \otimes \mathrm{der}_{!\underline{Q}}} \underline{Q} \otimes \underline{Q} \xrightarrow{s_1 \otimes s_2} \underline{P_1} \otimes \underline{P_2}$$

Remark

It is not completely straightforward to prove that these morphisms are coalgebra morphisms (especially for the pairing).

Theorem

$(P_1 \otimes P_2, \mathrm{pr}_1, \mathrm{pr}_2)$ *is the cartesian product of* P_1 *and* P_2 *in* $\mathcal{L}^!$.

The category of commutative comonoids

We have seen that for any objects $X_1, \ldots, X_n$ of $\mathcal{L}$, the object $!X_1 \otimes \cdots \otimes !X_n$ is canonically a commutative comonoid.

We'll see that this extends to all object of $\mathcal{L}^!$.

Definition

An object of $\mathrm{Ccom}(\mathcal{L})$ is a triple $C = (\underline{C}, \mathrm{w}_C, \mathrm{c}_C)$ where $\mathrm{w}_C \in \mathcal{L}(\underline{C}, 1)$ and $\mathrm{c}_C \in \mathcal{L}(\underline{C}, \underline{C} \otimes \underline{C})$ satisfying the following commutations:

Commutative comonoid

Associativity

$$\begin{tikzcd}
\underline{C} \arrow[r, "c_C"] \arrow[d, "c_C"'] & \underline{C} \otimes \underline{C} \arrow[r, "c_C \otimes \underline{C}"] & (\underline{C} \otimes \underline{C}) \otimes \underline{C} \arrow[d, "\alpha_{\underline{C},\underline{C},\underline{C}}"] \\
\underline{C} \otimes \underline{C} \arrow[rr, "\underline{C} \otimes c_C"] & & \underline{C} \otimes (\underline{C} \otimes \underline{C})
\end{tikzcd}$$

Left neutrality

$$\begin{tikzcd}
\underline{C} \arrow[r, "\mathsf{c}_C"] \arrow[drr, "\mathsf{Id}_{\underline{C}}"'] & \underline{C} \otimes \underline{C} \arrow[r, "\mathsf{w}_C \otimes \underline{C}"] & 1 \otimes \underline{C} \arrow[d, "\lambda_{\underline{C}}"] \\
 & & \underline{C}
\end{tikzcd}$$

Commutativity

$$\begin{tikzcd}
\underline{C} \arrow[r, "\mathsf{c}_C"] \arrow[dr, "\mathsf{c}_C"'] & \underline{C} \otimes \underline{C} \arrow[d, "\gamma_{\underline{C},\underline{C}}"] \\
 & \underline{C} \otimes \underline{C}
\end{tikzcd}$$

An element of $\mathrm{Ccom}(\mathcal{L})(C, D)$ is an $s \in \mathcal{L}(\underline{C}, \underline{D})$ such that

$$\begin{tikzcd}
\underline{C} \arrow[r, "s"] \arrow[dr, "\mathrm{Id}_{\underline{C}}"'] & \underline{D} \arrow[d, "\mathsf{w}_D"] \\
 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C} \arrow[r, "s"] \arrow[d, "\mathsf{c}_C"'] & \underline{D} \arrow[d, "\mathsf{c}_D"] \\
\underline{C} \otimes \underline{C} \arrow[r, "s \otimes s"] & \underline{D} \otimes \underline{D}
\end{tikzcd}$$

Coalgebras are comonoids

Fact

For any P in $\mathcal{L}^!$, we have

$$c_P = \langle \mathsf{Id}_{\underline{P}}, \mathsf{Id}_{\underline{P}} \rangle^{\otimes} \in \mathcal{L}^!(P, P \otimes P)$$
$$w_P = \mathsf{tt}_P \in \mathcal{L}^!(P, 1)$$

Because $\mathcal{L}^!$ is cartesian, this turns P into a commutative comonoid in the SMC $(\mathcal{L}^!, \otimes)$.

Fact

In a cartesian category $\mathcal{C}$, any object has a canonical commutative comonoid structure (wrt. the monoidal structure of $\mathcal{C}$ induced by the fact that it is cartesian).

Fact

We have a functor $\mathcal{L}^! \to \mathsf{Ccom}(\mathcal{L})$ *which maps* P *to* $(\underline{P}, \mathsf{w}_P, \mathsf{c}_P)$ *and* $s \in \mathcal{L}^!(P, Q)$ *to* s*.*

$\mathcal{L}^!$ is also cocartesian

Remember that two objects X_1, X_2 of $\mathcal{L}$ have a coproduct $(X_1 \oplus X_2, \overline{\pi}_1, \overline{\pi}_2)$ with $\overline{\pi}_i \in \mathcal{L}(X_i, X_1 \oplus X_2)$.

Given objects P_1, P_2 of $\mathcal{L}^!$, we have, in $\mathcal{L}$

$$\underline{P_i} \xrightarrow{h_{P_i}} !\underline{P_i} \xrightarrow{!\overline{\pi}_i} !(\underline{P_1} \oplus \underline{P_2})$$

so we have a unique $h_{P_1 \oplus P_2} \in \mathcal{L}(\underline{P_1} \oplus \underline{P_2}, !(\underline{P_1} \oplus \underline{P_2}))$ such that

$$h_{P_1 \oplus P_2}\, \overline{\pi}_i = !\overline{\pi}_i\, h_{P_i} \quad \text{for } i = 1, 2\,.$$

Fact

$P_1 \oplus P_2 = (\underline{P_1} \oplus \underline{P_2}, h_{P_1 \oplus P_2})$ is an object of $\mathcal{L}^!$. It is the coproduct of P_1 and P_2 in $\mathcal{L}^!$.

Remark

It is also true that $\mathcal{L}$ is cartesian (with product $\&$) and cocartesian (with coproduct $\oplus$).

There is a major difference: in $\mathcal{L}^!$, the product ($\otimes$) *distributes* over the coproduct ($\oplus$) as in **Set**:

$$(P_1 \oplus P_2) \otimes Q \simeq (P_1 \otimes Q) \oplus (P_2 \otimes Q)$$

but in general

$$(X_1 \oplus X_2) \,\&\, Y \not\simeq (X_1 \,\&\, Y) \oplus (X_2 \,\&\, Y)$$

in $\mathcal{L}$.

The Kleisli category

If X is an object of $\mathcal{L}$, then $\mathrm{E}(X) = (!X, \mathrm{dig}_X)$ is an object of $\mathcal{L}^!$, indeed the two following commute by definition of a comonad:

$$
\begin{tikzcd}
!X \arrow[r, "\mathrm{dig}_X"] \arrow[dr, "\mathrm{Id}"'] & !!X \arrow[d, "\mathrm{der}_{!X}"] \\
& !X
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \arrow[r, "\mathrm{dig}_X"] \arrow[d, "\mathrm{dig}_X"'] & !!X \arrow[d, "!\mathrm{dig}_X"] \\
!!X \arrow[r, "\mathrm{dig}_{!X}"] & !!!X
\end{tikzcd}
$$

Let $s \in \mathcal{L}(X, Y)$, then $\mathrm{E}(s) = !s \in \mathcal{L}^!(\mathrm{E}(X), \mathrm{E}(Y))$ by naturality of dig.

Remark

$\mathrm{E}(X)$ is the free coalgebra generated by X.

Fact

$\mathcal{L}^!(\mathrm{E}(X), \mathrm{E}(Y)) \simeq \mathcal{L}(!X, Y)$

This bijection $\varphi : \mathcal{L}^!(\mathrm{E}(X), \mathrm{E}(Y)) \to \mathcal{L}(!X, Y)$ works as follows:

$$\varphi(s) = \mathrm{der}_Y\ s$$
$$\varphi^{-1}(t) = t^!$$

Intuition

The Kleisli category of $!\_$ is the range of the functor E, considered as a full subcategory of $\mathcal{L}^!$.

Whence the official

Definition

The Kleisli category $\mathcal{L}_!$ of $!\_$ has

- objects those of $\mathcal{L}$
- and $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$
- identity at X: $\mathrm{Id}_X^{\mathrm{Kl}} = \mathrm{der}_X \in \mathcal{L}(!X, X) = \mathcal{L}_!(X, X)$
- and composition of $s \in \mathcal{L}_!(X, Y)$ and $t \in \mathcal{L}_!(Y, Z)$ given by

$$t \circ s = t\, !s\, \mathrm{dig}_X = t\, s^!$$

$$\begin{tikzcd} !X \arrow[r, "\mathrm{dig}_X"] \arrow[rr, bend right, "s^!"'] & !!X \arrow[r, "!s"] & !Y \arrow[r, "t"] & Z \end{tikzcd}$$

Example: the category $\mathbf{Rel}_!$

The objects are the sets.

$\mathbf{Rel}_!(E, F) = \mathcal{M}_{\mathrm{fin}}(E) \times F$ and $\mathrm{Id}_E^{\mathrm{Kl}} = \{([a], a) \mid a \in E\}$.

If $s \in \mathbf{Rel}_!(E, F)$ and $t \in \mathbf{Rel}_!(F, G)$ then

$$
\begin{aligned}
t \circ s = \{(m_1 + \cdots + m_k, c) \mid \exists b_1, \ldots, b_k \in F \\
([b_1, \ldots, b_k], c) \in t \\
\text{and } (m_i, b_i) \in s \text{ for } i = 1, \ldots, k\} \,.
\end{aligned}
$$

From $\mathcal{L}$ to $\mathcal{L}_!$

We define a functor $\mathsf{D} : \mathcal{L} \to \mathcal{L}_!$ by

- $\mathsf{D}(X) = X$
- and if $s \in \mathcal{L}(X, Y)$ then
  $\mathsf{D}(s) = \mathrm{der}_Y\ s \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$.

We could call it the "dereliction functor" since it consists in forgetting that a morphism of $\mathcal{L}$ is "linear".

From $\mathcal{L}_!$ to $\mathcal{L}^!$

We define an "inclusion" functor $\mathsf{I} : \mathcal{L}_! \to \mathcal{L}^!$ by

- $\mathsf{I}(X) = (!X, \mathsf{dig}_X)$ which is an object of $\mathcal{L}^!$
- and if $s \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$ then $\mathsf{I}(s) = s^! \in \mathcal{L}^!(\mathsf{I}(X), \mathsf{I}(Y))$.

Indeed we have

$$\begin{tikzcd} !X \arrow[r, "s^!"] \arrow[d, "\mathsf{dig}_X"'] & !Y \arrow[d, "\mathsf{dig}_Y"] \\ !!X \arrow[r, "!(s^!)"] & !!Y \end{tikzcd}$$

because $!(s^!)\, \mathsf{dig}_X = s^{!!}$.

Theorem

The functor I *is full and faithful.*

This means that, for any X, Y in $\mathcal{L}$, the function

$$\begin{aligned} \varphi : \mathcal{L}_!(X, Y) &\to \mathcal{L}^!(\mathrm{I}(X), \mathrm{I}(Y)) = \mathcal{L}^!((!X, \mathrm{dig}_X), (!Y, \mathrm{dig}_Y)) \\ s &\mapsto \mathrm{I}(s) = s^! \end{aligned}$$

is surjective (full) and injective (faithful).

The inverse of φ is given by $\varphi^{-1}(t) = \mathrm{der}_Y\ t$.

Proof

Let $t \in \mathcal{L}^!(\mathsf{I}(X), \mathsf{I}(Y))$, this means

$$\begin{tikzcd} !X \arrow[r, "t"] \arrow[d, "\mathrm{dig}_X"'] & !Y \arrow[d, "\mathrm{dig}_Y"] \\ !!X \arrow[r, "!t"] & !!Y \end{tikzcd}$$

Then

$$\begin{aligned} \varphi(\mathrm{der}_Y\ t) &= (\mathrm{der}_Y\ t)^! \\ &= !(\mathrm{der}_Y\ t)\ \mathrm{dig}_X \\ &= !\mathrm{der}_Y\, !t\ \mathrm{dig}_X \\ &= !\mathrm{der}_Y\ \mathrm{dig}_Y\ t = t \end{aligned}$$

by the commutation above. For the other direction: $\mathrm{der}_Y\ s^! = s$.

Through the functor I, we can see $\mathcal{L}_!$ as a full subcategory of $\mathcal{L}^!$: the category of free $!\_$-coalgebras.

The *free coalgebra functor* $\mathsf{E} : \mathcal{L} \to \mathcal{L}^!$ is just the composit:

$$\mathsf{E} = \mathsf{I} \circ \mathsf{D}$$

Adjunctions and factorizations of $!\_$

There is an obvious *forgetful functor*

$$\begin{aligned} \mathrm{U} : \mathcal{L}^! &\to \mathcal{L} \\ P &\mapsto \underline{P} \qquad t \in \mathcal{L}^!(P, Q) \mapsto t \in \mathcal{L}(\underline{P}, \underline{Q}) \end{aligned}$$

Then we have an adjunction

$$\mathrm{U} \dashv \mathrm{E}$$
$$\mathcal{L}(\underline{P}, X) \simeq \mathcal{L}^!(P, \mathrm{E}(X)) \text{ for } P \in \mathrm{Obj}(\mathcal{L}^!) \text{ and } X \in \mathrm{Obj}(\mathcal{L})\,.$$

Fact

The associated comonad $\mathrm{U} \circ \mathrm{E}$ *coincides with* $!\_$ *: we say that* $\mathrm{U} \dashv \mathrm{E}$ *is a factorization of* $!\_$ *.*

Remark that this adjunction means that we have an even more generalized promotion: given $s \in \mathcal{L}(\underline{P}, X)$, we have $s^! \in \mathcal{L}^!(P, \mathsf{E}(X))$ that is $s^! \in \mathcal{L}(\underline{P}, !X)$ with

$$\begin{tikzcd}
\underline{P} \arrow[r, "s^!"] \arrow[d, "h_P"'] & !X \arrow[d, "\mathrm{dig}_X"] \\
!\underline{P} \arrow[r, "!(s^!)"] & !!X
\end{tikzcd}$$

actually $s^!$ is

$$\underline{P} \xrightarrow{h_P} !\underline{P} \xrightarrow{!s} !X$$

In particular if $x \in \mathrm{Pt}_{\mathcal{L}}(X) = \mathcal{L}(1, X)$ we have $s^! \in \mathrm{Pt}_{\mathcal{L}}(!X)$.

There is also a "forgetful functor"

$$\mathsf{P} = \mathsf{U} \circ \mathsf{I} : \mathcal{L}_! \to \mathcal{L}$$
$$X \mapsto\ !X \qquad s \in \mathcal{L}_!(X, Y) \mapsto s^! \in \mathcal{L}(!X, !Y)$$

and remember that we have defined $\mathsf{D} : \mathcal{L} \to \mathcal{L}_!$ ($\mathsf{D}(X) = X$ and $\mathsf{D}(s) = s\ \mathsf{der}_X$ for $s \in \mathcal{L}(X, Y)$). Then we have an adjunction

$$\mathsf{P} \dashv \mathsf{D}$$
$$\mathcal{L}(\mathsf{P}(X), Y) = \mathcal{L}_!(X, \mathsf{D}(Y)) \text{ for } X, Y \in \mathsf{Obj}(\mathcal{L})$$

Fact

$\mathsf{P} \dashv \mathsf{D}$ *is another factorization of the comonad* $!\_$ *.*

Using the fact that $(s\ \mathsf{der}_X)^! = !s$ for $s \in \mathcal{L}(X, Y)$.

In general there are a lot of possible factorizations of the comonad; in some sense $U \dashv E$ is the largest one and $P \dashv D$ is the least one.

The Kleisli category $\mathcal{L}_!$ is a CCC

$\mathcal{L}_!$ is cartesian

If $(X_i)_{i \in I}$ is a family of elements of $\mathrm{Obj}(\mathcal{L}) = \mathrm{Obj}(\mathcal{L}_!)$ then

$$(\mathop{\&}_{i \in I} X_i, (\pi_i^{\mathsf{Kl}})_{i \in I})$$

with $\pi_i^{\mathsf{Kl}} = \pi_i \; \mathrm{der}_{X_i}$ is the cartesian product of the X_i's. Given $s_i \in \mathcal{L}_!(Y, X_i)$ for each $i \in I$ then

$$\langle s_i \rangle_{i \in I} \in \mathcal{L}_!(Y, \mathop{\&}_{i \in I} X_i)$$

is the unique morphism such that $\forall i \in I \; \pi_i^{\mathsf{Kl}} \circ \langle s_j \rangle_{j \in I} = s_i$.

Given $X, Y \in \mathrm{Obj}(\mathcal{L})$, we define

$$(X \Rightarrow Y) = (!X \multimap Y)$$

Cartesian closeness, roughly:

$$\begin{aligned}
\mathcal{L}_!(Z \mathbin{\&} X, Y) &= \mathcal{L}(!(Z \mathbin{\&} X), Y) \\
&\simeq \mathcal{L}(!Z \otimes !X, Y) \quad \text{Seely} \\
&\simeq \mathcal{L}(!Z, !X \multimap Y) \quad \mathcal{L} \text{ is an SMC} \\
&= \mathcal{L}_!(Z, X \Rightarrow Y)
\end{aligned}$$

$$(X \Rightarrow Y) = (!X \multimap Y)$$

$$\mathsf{Ev} = \mathsf{ev}\ (\mathsf{der}_{!X \multimap Y} \otimes !X)\ (\mathsf{m}^2_{!X \multimap Y, X})^{-1}$$

$$\begin{array}{c}
!((!X \multimap Y) \,\&\, X) \\
\Big\downarrow {\scriptstyle (\mathsf{m}^2_{!X \multimap Y, X})^{-1}} \\
!(!X \multimap Y) \otimes !X \\
\Big\downarrow {\scriptstyle \mathsf{der}_{!X \multimap Y} \otimes !X} \\
(!X \multimap Y) \otimes !X \\
\Big\downarrow {\scriptstyle \mathsf{ev}} \\
Y
\end{array}$$

Fact

$(X \Rightarrow Y, \mathrm{Ev})$ is the hom object in $\mathcal{L}_!$. This means that, for any

$$s \in \mathcal{L}_!(Z \mathbin{\&} X, Y)$$

there is a unique $\mathrm{Cur}(s) \in \mathcal{L}_!(Z, X \Rightarrow Y)$ such that, in $\mathcal{L}_!$,

$$\begin{tikzcd}
Z \mathbin{\&} X \arrow[r, "\mathrm{Cur}(s) \& X"] \arrow[dr, "s"'] & (X \Rightarrow Y) \mathbin{\&} X \arrow[d, "\mathrm{Ev}"] \\
& Y
\end{tikzcd}$$

We have $s \in \mathcal{L}(!(Z \mathbin{\&} X), Y)$, then $s\,\mathrm{m}^2_{Z,X} \in \mathcal{L}(!Z \otimes !X, Y)$, we have

$$\mathrm{Cur}(s) = \mathrm{cur}(s\,\mathrm{m}^2_{Z,X}) \in \mathcal{L}(!Z, !X \multimap Y) = \mathcal{L}_!(Z, X \Rightarrow Y)\,.$$

Interpreting PCF in **Rel**

Reminder on cpos and fixpoints

Definition

Let $\mathcal{D}$ be a partially ordered set. A subset D of $\mathcal{D}$ is directed (*filtrant* in French) if

- D is not empty
- and $\forall x_1, x_2 \in D \, \exists x \in D \;\, x_1 \leq x$ and $x_2 \leq x$.

Remark:

- If D is directed and $x_1, \ldots, x_n \in D$ then $\exists x \in D \, \forall i \in \{1, \ldots n\} \;\; x_i \leq x$, easy induction on n. Also true for $n = 0$ by the condition $D \neq \emptyset$.
- Hence a finite directed set D has a maximal element, *i.e.* $\exists y \in D \, \forall x \in D \;\; x \leq y$.
- So directed sets are useful only when they are infinite: they generalize monotone sequences: if $x_1, x_2 \cdots \in D$ such that $\forall i \;\; x_i \leq x_{i+1}$ then $\{x_i \mid i \in \mathbb{N}\}$ is directed.

Example

If E is a set, the set $\mathcal{P}_{\mathrm{fin}}(E)$ of finite subsets of E is directed for $\subseteq$.

A cpo (complete partial order) is a partially ordered set $\mathcal{D}$ such that any directed set $D \subseteq \mathcal{D}$ has a least upper bound (lub) $\bigvee \mathcal{D} \in \mathcal{D}$:

Definition (lub)

- $\forall x \in D \; x \leq \bigvee D$
- $\forall y \in \mathcal{D} \; (\forall x \in D \; x \leq y) \Rightarrow \bigvee D \leq y$

Remark: When it exists, a lub is unique (it is defined by a universal property in $\mathcal{D}$ considered as a category: a lub is a colimit).

Let $\mathcal{D}$ and $\mathcal{E}$ be cpos and $f : \mathcal{D} \to \mathcal{E}$ be monotone.

Fact

If $D \subseteq \mathcal{D}$ is directed, then $f(D) = \{f(x) \mid x \in D\}$ is directed.

Notice that $\forall x \in D\ f(x) \leq f(\bigvee D)$ and hence $\bigvee f(D) \leq f(\bigvee D)$.

Definition

f is Scott continuous if, for any directed subset D of $\mathcal{D}$ one has $f(\bigvee D) \leq \bigvee f(D)$, that is $f(\bigvee D) = \bigvee f(D)$.

Remark: One can endow $\mathcal{D}$ and $\mathcal{E}$ with a topology such that Scott continuity coincides with ordinary topology: this is the Scott topology.

Example

Let $\mathcal{D}$ be the set of partial functions $\mathbb{N} \to \mathbb{N}$ ordered by inclusion of graphs ($f \leq g$ if for all $n \in \mathbb{N}$, if $f(n)$ is defined then $g(n)$ is defined and $g(n) = f(n)$) and let $\Sigma = \{\perp < \top\}$, both are cpos.

- The function $F : \mathcal{D} \to \Sigma$ such that, for all $f \in \mathcal{D}$

$$F(f) = \begin{cases} \top & \text{if } \exists n \in \mathbb{N}\ f(n) = f(n+1) = \cdots = f(2^n) = 0 \\ \perp & \text{otherwise} \end{cases}$$

  is monotone and Scott continuous.

- The function $G : \mathcal{D} \to \Sigma$ such that, for all $f \in \mathcal{D}$

$$G(f) = \begin{cases} \top & \text{if } \forall n \in \mathbb{N}\ f(n) \text{ defined and} \neq 0 \\ \perp & \text{otherwise} \end{cases}$$

  is monotone, but not Scott continuous.

Fact

Let $\mathcal{D}$ be a cpo which has a least element $\perp$. Let $f : \mathcal{D} \to \mathcal{D}$ be monotone and Scott continuous. Then there is $x \in \mathcal{D}$ such that

- $f(x) = x$
- *and* $\forall y \in \mathcal{D}\ f(y) = y \Rightarrow x \leq y$.

That is, x is the least fixpoint of f.

One defines $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{D}$ by $x_0 = \perp$ and $x_{n+1} = f(x_n)$. Then $\forall n \in \mathbb{N}\ x_n \leq x_{n+1}$ (easy induction on n) so $\mathcal{D} = \{x_n \mid n \in \mathbb{N}\}$ is directed.

So we can set $x = \bigvee_{n \in \mathbb{N}} x_n \in \mathcal{D}$ since $\mathcal{D}$ is a cpo.

Then by Scott continuity

$$f(x) = \bigvee_{n \in \mathbb{N}} f(x_n) = \bigvee_{n \in \mathbb{N}} f(x_{n+1}) = x\,.$$

Assume that $y \in \mathcal{D}$ and $f(y) = y$. We have $\perp \leq y$ and hence by induction $\forall n \in \mathbb{N}\ x_n \leq y$. Hence $x \leq y$.

Function induced by a morphism of $\mathcal{L}_!$

In a model $\mathcal{L}$ of LL, given $t \in \mathcal{L}_!(X, Y)$, we have a function

$$\begin{aligned} \widehat{t} : \mathrm{Pt}_{\mathcal{L}}(X) &\to \mathrm{Pt}_{\mathcal{L}}(Y) \\ x &\mapsto t\, x^! \end{aligned}$$

Remember that $\mathrm{Pt}_{\mathcal{L}}(X) = \mathcal{L}(1, X)$. This defines a functor $\mathcal{L}_! \to \mathbf{Set}$:

$$\begin{aligned} \widehat{\mathrm{der}_X}(x) &= \mathrm{der}_X\ x^! = x \\ \widehat{t}(\widehat{s}(x)) &= t\ \left(s\, x^!\right)^! = t\, s^!\, x^! = \widehat{t \circ s}(x) \end{aligned}$$

Observe that $\mathsf{Pt}_{\mathcal{L}}(\&_{i\in I} X_i) \simeq \prod_{i\in I} \mathsf{Pt}_{\mathcal{L}}(X_i)$.

In **Rel**: if $u \in \mathsf{Pt}_{\mathbf{Rel}}(E) \simeq \mathcal{P}(E)$ then we identify $u^! \in \mathsf{Pt}_{\mathbf{Rel}}(!E)$ with

$$u^{(!)} = \mathcal{M}_{\mathrm{fin}}(u)$$

Fact

Let $t \in \mathbf{Rel}_!(E,F)$*, then*

$$\begin{aligned}\widehat{t} : \mathcal{P}(E) &\to \mathcal{P}(F)\\ u &\mapsto t \cdot u^{(!)} = \{b \in F \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(u) \text{ and } (m,b) \in t\}\end{aligned}$$

$\mathcal{P}(E)$, ordered by $\subseteq$, is a cpo which has $\emptyset$ as least element and where $\bigvee D = \bigcup_{x \in D} x$.

Fact

The function $\widehat{t}$ *is monotone and Scott continuous.*

Let $t \in \mathbf{Rel}_!(E, F)$.

Let $u_1 \subseteq u_2$ in $\mathcal{P}(E)$. If $b \in \widehat{t}(u)$, there is $m \in \mathcal{M}_{\mathrm{fin}}(E)$ such that $\mathrm{supp}(m) \subseteq u_1$ and $(m, b) \in t$. Then we have $\mathrm{supp}(m) \subseteq u_2$ and hence $b \in \widehat{t}(u_2)$. So $\widehat{t}$ is monotone.

Let $D \subseteq \mathcal{P}(E)$ be directed. We prove $\widehat{t}(\bigcup D) \subseteq \bigcup \widehat{t}(D)$.

Let $b \in \widehat{t}(\bigcup D)$. Let $m \in \mathcal{M}_{\mathrm{fin}}(E)$ such that $(m, b) \in t$ and $\mathrm{supp}(m) \subseteq \bigcup D$. Let $a_1, \ldots, a_n$ be the elements of $\mathrm{supp}(m)$. For each $i \in \{1, \ldots, n\}$ let $u_i \in D$ be such that $a_i \in u_i$. Since D is directed there is $u \in D$ such that $u_i \subseteq u$ for $i = 1, \ldots, n$. We have

$$\mathrm{supp}(m) \subseteq u$$

and hence $b \in \widehat{t}(u) \subseteq \bigcup \widehat{t}(D)$.

Least fixpoints in $\mathbf{Rel}_!$

Let $t \in \mathbf{Rel}_!(E, E)$, the map

$$\widehat{t} : \mathcal{P}(E) \to \mathcal{P}(E)$$

is monotone and Scott continuous so it has a least fixpoint, namely

$$\bigcup_{n=0}^{\infty} \widehat{t}^{\,n}(\emptyset)\,.$$

Fact

Let $\mathcal{Y}(t) = \bigcup_{n=0}^{\infty} \widehat{t}^{\,n}(\emptyset)$*. It is the least subset of* E *such that:*

for any $([a_1, \ldots, a_n], a) \in t$*, if* $a_1, \ldots, a_n \in \mathcal{Y}(t)$ *then* $a \in \mathcal{Y}(t)$*.*

We want to *internalize* $\mathcal{Y}$, exhibiting $\mathcal{Y}_0 \in \mathbf{Rel}_!(!E \multimap E, E)$ such that

$$\forall t \in \mathcal{P}(!E \multimap E) \quad \mathcal{Y}(t) = \widehat{\mathcal{Y}_0}(t)$$

Idea

Define $\mathcal{Y}_0$ as the least fixpoint of a morphism

$$\mathcal{Z} \in \mathbf{Rel}_!(!(!E \multimap E) \multimap E, !(!E \multimap E) \multimap E)$$

Fact

Such a $\mathcal{Z}$ can be defined in any model $\mathcal{L}$ of LL (actually in any CCC).

We want in $\mathcal{L}$:

$$\mathcal{Z} : !(!(!X \multimap X) \multimap X) \to !(!X \multimap X) \multimap X$$

We take $\mathcal{Z} = \mathrm{cur}(\mathcal{Z}')$ for

$$\mathcal{Z}' : !(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X) \to X$$

We define $\mathcal{Z}'$ as follows:

Definition of $\mathcal{Z}'$

$$
\begin{tikzcd}
{!(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\mathrm{Id} \otimes \mathrm{c}_{!X \multimap X}"] & \\
{!(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\theta"] \arrow[ddr, bend left, "\mathcal{Z}'_1"] & \\
{!(!X \multimap X) \otimes !(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\mathrm{der}_{!X \multimap X} \otimes e^{!}"] \arrow[dr, "\mathcal{Z}'_2"] & \\
{(!X \multimap X) \otimes !X} \arrow[r, "\mathrm{ev}"] & X
\end{tikzcd}
$$

where θ is a suitable combination of instances of α and γ.

and e is

$$\begin{array}{c}
!(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X) \\
\Big\downarrow \mathrm{der} \otimes \mathrm{Id} \\
(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X) \\
\Big\downarrow \mathrm{ev} \\
X
\end{array}$$

So, in **Rel**, $(M, m, a) \in e$ iff $M = [(m, a)]$.

Computing $e^!$ in **Rel**

$e^! \in \mathbf{Rel}(!(!(!E \multimap E) \multimap E) \otimes !(!E \multimap E), !E)$

Let $M \in !(!(!E \multimap E) \multimap E)$, $m \in !(!E \multimap E)$ and $a_1, \ldots, a_k \in E$, then

$$(M, m, [a_1, \ldots, a_k]) \in e^! \Leftrightarrow \exists p_1, \ldots, p_k \in !(!E \multimap E)$$
$$M = [(p_1, a_1), \ldots, (p_k, a_k)] \text{ and } m = p_1 + \cdots + p_k$$

Computing $\mathcal{Z}$ in **Rel**

Let $M \in !(!(!E \multimap E) \multimap E)$, $m \in !(!E \multimap E)$ and $a \in E$, we have

$$
\begin{aligned}
(M, m, a) \in \mathcal{Z}' &\Leftrightarrow m = m_1 + m_2 \text{ and } (M, m_1, m_2, a) \in \mathcal{Z}'_1 \\
&\Leftrightarrow m = m_1 + m_2 \text{ and } (m_1, M, m_2, a) \in \mathcal{Z}'_2 \\
&\Leftrightarrow m = m_1 + m_2,\ (M, m_2, [a_1, \dots, a_k]) \in e^! \\
&\qquad m_1 = [c] \text{ and } ((c, [a_1, \dots, a_k]), a) \in \mathrm{ev} \\
&\Leftrightarrow m = m_1 + m_2,\ M = [(p_1, a_1), \dots, (p_k, a_k)],\ m_2 = p_1 + \dots + p_k \\
&\qquad \text{and } m_1 = [([a_1, \dots, a_k], a)] \\
&\Leftrightarrow M = [(p_1, a_1), \dots, (p_k, a_k)] \\
&\qquad \text{and } m = p_1 + \dots + p_k + [([a_1, \dots, a_k], a)]
\end{aligned}
$$

Finally

Explicit description of $\mathcal{Z}$

$$
\begin{aligned}
\mathcal{Z} = \{([(p_1, a_1), \ldots, (p_k, a_k)], (p_1 + \cdots + p_k + [([a_1, \ldots, a_k], a)], a)) \\
p_1, \ldots, p_k \in \,!(!E \multimap E) \text{ and } a_1, \ldots, a_k, a \in E\} \\
\in \mathbf{Rel}(!(!(!E \multimap E) \multimap E), !(!E \multimap E) \multimap E)
\end{aligned}
$$

Fact

Given $t \in \mathrm{Pt}_{\mathcal{L}}(!X \multimap X) \simeq \mathcal{L}_!(X, X)$ *and* $T \in \mathrm{Pt}_{\mathcal{L}}(!(!X \multimap X) \multimap X)$ *we have*

$$\widehat{\widehat{\mathcal{Z}}(T)}(t) = \widehat{t}(\widehat{T}(\widehat{t}))$$

Remember the definition of $\mathcal{Z}'$

$$
\begin{tikzcd}
{!(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\mathrm{Id} \otimes \mathrm{c}_{!X \multimap X}"] & \\
{!(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\theta"] \arrow[ddr, bend left, "\mathcal{Z}'_1"] & \\
{!(!X \multimap X) \otimes !(!(!X \multimap X) \multimap X) \otimes !(!X \multimap X)} \arrow[d, "\mathrm{der}_{!X \multimap X} \otimes e^{!}"] \arrow[dr, "\mathcal{Z}'_2"] & \\
{(!X \multimap X) \otimes !X} \arrow[r, "\mathrm{ev}"] & X
\end{tikzcd}
$$

where θ is a suitable combination of instances of α and γ.

In **Rel**

Let $T_n \in \mathbf{Rel}_!(!E \multimap E, E)$ be defined by

$$T_0 = \emptyset$$
$$T_{n+1} = \widehat{\mathcal{Z}}(T_n)\,,$$

it is a monotone sequence in $\mathcal{P}(!(!E \multimap E) \multimap E)$.

Fact

For $t \in \mathbf{Rel}_!(E, E)$, we have

$$\forall n \in \mathbb{N} \quad \widehat{T_n}(t) = \widehat{t}^{\,n}(\emptyset)$$

By induction on n. For the inductive step:

$$\widehat{T_{n+1}}(t) = \widehat{\widehat{\mathcal{Z}}(T_n)}(t) = \widehat{t}(\widehat{T_n}(t)) = \widehat{t}(\widehat{t}^{\,n}(\emptyset)) = \widehat{t}^{\,n+1}(\emptyset)$$

We set

$$\mathcal{Y}_0 = \bigcup_{n=0}^{\infty} T_n \in \mathbf{Rel}_!(!E \multimap E, E) \quad \text{the least fixpoint of } \widehat{\mathcal{Z}}$$

So that, for all $t \in \mathbf{Rel}_!(E, E)$ one has that

$$\widehat{\mathcal{Y}_0}(t) = \bigcup_{n=0}^{\infty} \widehat{t}^n(\emptyset) \quad \text{is the least fixpoint of } \widehat{t}$$

Fact

$\mathcal{Y}_0$ is the least subset of $!(!E \multimap E) \multimap E$ such that if $(m_i, a_i) \in \mathcal{Y}_0$ for $i = 1, \ldots, n$ and $a \in E$, then $(m_1 + \cdots + m_i + [([a_1, \ldots, a_n], a)], a) \in \mathcal{Y}_0$.

Example (elements of $\mathcal{Y}_0$)

- $([([], a)], a) \in \mathcal{Y}_0$ for each $a \in E$
- if $a_1, \ldots, a_n, a \in E$ then
  $([([], a_1), \ldots, ([], a_n), ([a_1, \ldots, a_n], a)], a) \in \mathcal{Y}_0$
- etc.

Natural number

We have an object

$$\mathsf{N} = \bigoplus_{i \in \mathbb{N}} 1$$

so that $\mathsf{N} = \mathbb{N}$ as a set (up to trivial iso).

Successor morphism $\overline{\mathsf{suc}} \in \mathbf{Rel}(\mathsf{N}, \mathsf{N})$ given by

$$\overline{\mathsf{suc}} = \{(n, n+1) \mid n \in \mathbb{N}\} \ .$$

If $n \in \mathbb{N}$, $\overline{n} = \{(*, n)\} \in \mathbf{Rel}(1, \mathsf{N})$.

N as an object of $\mathbf{Rel}^!$

Remember that 1 has a canonical structure of !-coalgebra (object of $\mathbf{Rel}^!$) given by

$$h_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$$

As a coproduct of copies of 1, N inherits a structure of !-coalgebra given by

$$h_{\mathsf{N}} = \{(n, k[n]) \mid k, n \in \mathbb{N} \in \mathbb{N}\}\,.$$

N as a commutative comonoid

In particular N has a structure of commutative $\otimes$-coalgebra

$$
\begin{aligned}
\mathsf{w}_{\mathsf{N}} &= \{(n, *) \mid n \in \mathbb{N}\} \in \mathbf{Rel}(\mathsf{N}, 1) \\
\mathsf{c}_{\mathsf{N}} &= \{(n, (n, n)) \mid n \in \mathbb{N}\} \in \mathbf{Rel}(\mathsf{N}, \mathsf{N} \otimes \mathsf{N})
\end{aligned}
$$

in other words: integers are freely discardable un duplicable.

A morphism for the conditional

There is also an obvious iso

$$\varphi : 1 \oplus \mathsf{N} \to \mathsf{N}$$
$$(1, *) \mapsto 0 \quad (2, n) \mapsto n + 1$$

Using these ingredients we define
$\overline{\mathsf{if}} \in \mathbf{Rel}(\mathsf{N} \otimes !E \otimes !(!\mathsf{N} \multimap E), E)$ with

$$\overline{\mathsf{if}} = \{(0, [a], [], a) \mid a \in E\}$$
$$\cup \{(n + 1, [], [(k[n], a)], a) \mid k, n \in \mathbb{N} \text{ and } a \in E\} .$$

Interpreting PCF types

We interpret types as objects of **Rel**, that is, as sets.

$$\llbracket \iota \rrbracket = \mathsf{N}$$
$$\llbracket A \Rightarrow B \rrbracket = !\llbracket A \rrbracket \multimap \llbracket B \rrbracket = \mathcal{M}_{\mathrm{fin}}(\llbracket A \rrbracket) \times \llbracket B \rrbracket$$

A context $\Gamma = (x_1 : A_1, \dots, x_l : A_l)$ is interpreted as

$$\llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \mathbin{\&} \cdots \mathbin{\&} \llbracket A_l \rrbracket$$

that we consider as an object of $\mathbf{Rel}_!$.

Interpreting PCF terms

Given a term M such that $\Gamma \vdash M : A$, we define $[\![M]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], [\![A]\!]) = \mathbf{Rel}(![\![\Gamma]\!], [\![A]\!])$, by induction on M.

- If $M = x_i$ for some $i \in \{1, \dots, l\}$, then $[\![M]\!]_\Gamma = \pi_i \ \mathrm{der}$

$$![\![\Gamma]\!] \xrightarrow{\mathrm{der}_{[\![\Gamma]\!]}} [\![\Gamma]\!] \xrightarrow{\pi_i} [\![A_i]\!]$$

- If $M = \underline{n}$ for $n \in \mathbb{N}$ then $[\![M]\!]_\Gamma = \overline{n}\, \mathrm{w}_{[\![\Gamma]\!]}$

$$![\![\Gamma]\!] \xrightarrow{\mathrm{w}_{[\![\Gamma]\!]}} 1 \xrightarrow{\overline{n}} \mathrm{N}$$

If $M = \mathsf{succ}(P)$ with $\Gamma \vdash P : \iota$, then we have $[\![P]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], \mathsf{N})$ and we set

$$[\![M]\!]_\Gamma = \overline{\mathsf{suc}}\,[\![P]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], \mathsf{N})$$

$$![\![\Gamma]\!] \xrightarrow{[\![P]\!]_\Gamma} \mathsf{N} \xrightarrow{\overline{\mathsf{suc}}} \mathsf{N}$$

If $M = \mathrm{if}(P, Q, z \cdot R)$ with $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : A$ and $\Gamma, z : \iota \vdash R : A$ then we have

$$s = [\![P]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], \mathsf{N}) \qquad [\![Q]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], [\![A]\!])$$
$$[\![R]\!]_{\Gamma, z:\iota} \in \mathbf{Rel}_!([\![\Gamma]\!] \mathbin{\&} \mathsf{N}, [\![A]\!]) = \mathbf{Rel}(![\![\Gamma]\!] \otimes !\mathsf{N}, [\![A]\!])$$

hence $t_0 = [\![Q]\!]_\Gamma^! \in \mathbf{Rel}(![\![\Gamma]\!], ![\![A]\!])$ and $t_+ = \mathrm{cur}([\![R]\!]_{\Gamma, z:\iota})^! \in \mathbf{Rel}(![\![\Gamma]\!], !(!\mathsf{N} \multimap [\![A]\!]))$

$$[\![M]\!]_\Gamma = \overline{\mathrm{if}}\,([\![P]\!]_\Gamma \otimes [\![Q]\!]_\Gamma^! \otimes \mathrm{Cur}([\![R]\!]_{\Gamma, z:\iota})^!)\ \mathsf{c} \in \mathbf{Rel}(![\![\Gamma]\!], [\![A]\!])$$

$$\begin{array}{ccccc} ![\![\Gamma]\!] & \xrightarrow{c} & ![\![\Gamma]\!] \otimes ![\![\Gamma]\!] \otimes ![\![\Gamma]\!] & \xrightarrow{s \otimes t_0 \otimes t_+} & \mathsf{N} \otimes ![\![A]\!] \otimes !(!\mathsf{N} \multimap [\![A]\!]) \\ & & & & \Big\downarrow \overline{\mathrm{if}} \\ & & & & \mathsf{N} \end{array}$$

If $M = \lambda x^B\, P$ with $\Gamma, x : B \vdash P : C$ and $A = (B \Rightarrow C)$ then $[\![P]\!]_{\Gamma,x:B} \in \mathbf{Rel}_!([\![\Gamma]\!] \,\&\, [\![B]\!], [\![C]\!])$ and we set

$$[\![M]\!]_\Gamma = \mathrm{Cur}([\![P]\!]_{\Gamma,x:B}) \in \mathbf{Rel}_!([\![\Gamma]\!], [\![B]\!] \Rightarrow [\![C]\!])\,.$$

If $M = (P)\,Q$ with $\Gamma \vdash P : B \Rightarrow A$ and $\Gamma \vdash Q : B$ then $[\![P]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], [\![B]\!] \Rightarrow [\![A]\!])$ and $[\![Q]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], [\![B]\!])$ and we set

$$\begin{aligned} [\![M]\!]_\Gamma &= \mathsf{Ev} \circ \langle [\![P]\!]_\Gamma, [\![Q]\!]_\Gamma \rangle \\ &= \mathsf{ev}\left([\![P]\!]_\Gamma \otimes [\![Q]\!]_\Gamma^!\right)\, \mathsf{c}_{[\![\Gamma]\!]} \end{aligned}$$

$$![\![\Gamma]\!] \xrightarrow{\mathsf{c}_{[\![\Gamma]\!]}} ![\![\Gamma]\!] \otimes ![\![\Gamma]\!] \xrightarrow{[\![P]\!]_\Gamma \otimes [\![Q]\!]_\Gamma^!} (![\![B]\!] \multimap [\![A]\!]) \otimes ![\![B]\!] \xrightarrow{\mathsf{ev}} [\![A]\!]$$

If $M = \mathrm{fix}(P)$ with $\Gamma \vdash P : A \Rightarrow A$, then we have $[\![P]\!]_\Gamma \in \mathbf{Rel}_!([\![\Gamma]\!], [\![A]\!] \Rightarrow [\![A]\!])$ and we set

$$\begin{aligned} [\![M]\!]_\Gamma &= \mathcal{Y}_0 \circ [\![P]\!]_\Gamma \\ &= \mathcal{Y}_0 \, [\![P]\!]_\Gamma^! \end{aligned}$$

$$![\![\Gamma]\!] \xrightarrow{[\![P]\!]_\Gamma^!} !(![\![A]\!] \multimap [\![A]\!]) \xrightarrow{\mathcal{Y}_0} [\![A]\!]$$

Substitution lemma

Lemma

Assume that $\Gamma, x : A \vdash M : B$ *and that* $\Gamma \vdash P : A$*. Then*

$$\begin{aligned}
[\![M\,[P/x]]\!]_\Gamma &= [\![M]\!]_{\Gamma,x:A} \circ \langle \mathrm{Id}_{[\![\Gamma]\!]}, [\![P]\!]_\Gamma \rangle \\
&= [\![M]\!]_{\Gamma,x:A}\,(![\![\Gamma]\!] \otimes [\![P]\!]_\Gamma^!)\; c_{[\![\Gamma]\!]}
\end{aligned}$$

$$\begin{tikzcd}
![\![\Gamma]\!] \arrow[r, "{[\![M[P/x]]\!]_\Gamma}"] \arrow[d, "c_{[\![\Gamma]\!]}"'] & {[\![B]\!]} \\
![\![\Gamma]\!] \otimes ![\![\Gamma]\!] \arrow[r, "{![\![\Gamma]\!]\otimes[\![P]\!]_\Gamma^!}"] & ![\![\Gamma]\!] \otimes ![\![A]\!] \arrow[u, "{[\![M]\!]_{\Gamma,x:A}}"']
\end{tikzcd}$$

Soundness theorem

Theorem

Assume that $\Gamma \vdash M : A$ *and that* $M \,\beta\, M'$*. Then* $[\![M']\!]_\Gamma = [\![M]\!]_\Gamma$.

The proof consists in applying equations which hold in **Rel** (actually in any model of LL with fixpoint operators and countable coproducts), and the Substitution Lemma.

Semantics of PCF in **Rel** as a typing system

We present this semantics of PCF in **Rel** as an

Intersection typing system

General idea

Consider the elements of $[\![A]\!]$ as types which can be seen as "quantitative refinements" of A.

When $\vdash M : A$, write "$a \in [\![M]\!]$" as a typing judgment

$$\vdash M : a : A$$

The typing rules are just reformulations of the above definition of the semantics of PCF in **Rel**.

Semantic typing contexts

General sequents: $\Phi \vdash M : a : A$ where
$\Phi = (x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k)$.

Underlying typing context: $\underline{\Phi} = (x_1 : A_1, \ldots, x_k : A_k)$.

If $\Gamma = (x_1 : A_1, \ldots, x_k : A_k)$ then
$0_\Gamma = (x_1 : [\,] : A_1, \ldots, x_k : k : [\,] : A_k)$.

Sum of contexts: if $\underline{\Phi} = \underline{\Psi}$ so that
$\Phi = (x_1 : m_1 : A_1 \ldots, x_k : m_k : A_k)$ and
$\Psi = (x_1 : p_1 : A_1 \ldots, x_k : p_k : A_k)$ then we define
$\Phi + \Psi = (x_1 : m_1 + p_1 : A_1 \ldots, x_k : m_k + p_k : A_k)$.
$\underline{\Phi + \Psi} = \underline{\Phi} = \underline{\Psi}$.

Convention

When we write $\Phi_0 + \Phi_1$ of $\sum_{i=1}^{k} \Phi_i$ we always assume implicitely that all the $\underline{\Phi_i}$'s are identical.

Integers

$$\frac{n \in \mathbb{N}}{0_{\Gamma} \vdash \underline{n} : n : \iota} \qquad \frac{\Phi \vdash M : n : \iota}{\Phi \vdash \mathsf{succ}(M) : n+1 : \iota}$$

$$\frac{\Phi \vdash P : 0 : \iota \qquad \Phi_0 \vdash M : a : A \qquad \underline{\Phi}, z : \iota \vdash N : A}{\Phi + \Phi_0 \vdash \mathsf{if}(P, M, z \cdot N) : a : A}$$

$$\frac{\Phi \vdash P : n+1 : \iota \qquad \underline{\Phi} \vdash M : A \qquad \Phi_+, z : k[n] : \iota \vdash N : a : A}{\Phi + \Phi_+ \vdash \mathsf{if}(P, M, z \cdot N) : a : A}$$

if $\underline{\Phi} = \underline{\Phi_0} = \underline{\Phi_+}$ and $k \in \mathbb{N}$ (possibly $k = 0$).

$$\frac{m_i = [a] \qquad m_j = [\,] \text{ if } j \neq i}{x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k \vdash x_i : a : A_i}$$

$$\frac{\Phi, x : m : A \vdash M : b : B}{\Phi \vdash \lambda x^A\, M : (m, b) : A \Rightarrow B}$$

$$\frac{\Phi \vdash M : ([a_1, \ldots, a_k], b) : A \Rightarrow B \qquad (\Phi_i \vdash N : a_i : A)_{i=1}^{k}}{\Phi + \sum_{i=1}^{k} \Phi_i \vdash (M)\, N : b : B}$$

if $\forall i\ \underline{\Phi} = \underline{\Phi_i}$.

Fixpoint

$$\frac{\Phi \vdash M : ([a_1, \ldots, a_k], a) : A \Rightarrow A \qquad (\Phi_i \vdash \mathrm{fix}(M) : a_i : A)_{i=1}^{k}}{\Phi + \sum_{i=1}^{k} \Phi_i \vdash \mathrm{fix}(M) : a : A}$$

if $\forall i \; \underline{\Phi} = \underline{\Phi_i}$.

Notice that in particular

$$\frac{\Phi \vdash M : ([\,], a) : A \Rightarrow A}{\Phi \vdash \mathrm{fix}(M) : a : A}$$

these are the leaves of the "fixpoint derivation trees".

Theorem

Assume $\Gamma \vdash M : B$ *with* $\Gamma = (x_1 : A_1, \ldots, x_k : A_k)$.

Let $m_i \in ![\![A_i]\!]$ *for* $i = 1, \ldots, k$ *and* $b \in [\![B]\!]$.

Then $(m_1, \ldots, m_k, b) \in [\![M]\!]_\Gamma$ *if and only if*
$x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k \vdash M : b : B$ *is derivable.*

The proof is a simple anlysis of the definition of $[\![M]\!]_\Gamma$ by induction on M.

Let M, M' with $\vdash M : \iota$. We know that if $M \, \beta^* \, \underline{n}$ then $[\![M]\!] = \{n\}$, that is $\vdash M : n : \iota$. The converse is true. Actually we can prove better!

Theorem

If $\vdash M : n : \iota$ then $M \, \beta^*_{\mathrm{wh}} \, \underline{n}$.

It is a normalization theorem (for β_{wh}), we prove it by the reducibility method.

Idea of the proof

2 phases in the proof:

1. By induction on A we define a relation

$$\Vdash_A \subseteq \{M \mid \ \vdash M : A\} \times [\![A]\!]$$

   in such a way that $M \Vdash_\iota n \Rightarrow M \ \beta^*_{\text{wh}} \ \underline{n}$.

2. We prove that, for all type A

$$\forall a \in [\![A]\!] \quad \vdash M : a : A \Rightarrow M \Vdash_A a \, .$$

Definition of $\Vdash_A$

By induction on A.

We say that $M \Vdash_\iota n$ if $\vdash M : \iota$ and $M \, \beta^*_{\mathrm{wh}} \, \underline{n}$.

We say that $M \Vdash_{A \Rightarrow B} ([a_1, \ldots, a_k], b)$ if $\vdash M : A \Rightarrow B$ and for all N such that $\vdash N : A$ we have

$$(\forall i \in \{1, \ldots, k\} \quad N \Vdash_A a_i) \Rightarrow (M)\, N \Vdash_B b$$

Expansion lemma

Lemma (Expansion lemma)

If $\vdash M, M' : A$ *and* $M \, \beta_{\mathrm{wh}} \, M'$ *and if* $M' \Vdash_A a$ *then* $M \Vdash_A a$.

The proof is by induction on A. If $A = \iota$, it is an obvious consequence of the definition of $\Vdash_\iota$.

Inductive step: $A = (B \Rightarrow C)$

Assume that $\vdash M, M' : B \Rightarrow C$ and $M \,\beta_{\mathrm{wh}}\, M'$ and let $a \in [\![A]\!]$ be such that $M' \Vdash_A a$.

We have $a = ([b_1, \ldots, b_k], c)$ for some $c \in [\![C]\!]$ and $b_1, \ldots, b_k \in [\![B]\!]$. We must prove that $M \Vdash_{B \Rightarrow C} ([b_1, \ldots, b_k], c)$.

So let N be such that $\vdash N : B$ and $N \Vdash_B b_i$ for $i = 1, \ldots, k$, we must prove that $(M)\, N \Vdash_C c$. We know that $(M')\, N \Vdash_C c$ since $M' \Vdash_A a$.

Since the property we want to prove holds for C (inductive hypothesis), it suffices to observe that $(M)\, N \,\beta_{\mathrm{wh}}\, (M')\, N$.

Indeed: since $M \,\beta_{\mathrm{wh}}\, M'$, M is not of shape $\lambda x^B\, P$ and hence $(M)\, N$ is not a β_{wh}-redex.

We can prove now the main statement which generalizes

$$\vdash M : a : A \Rightarrow M \Vdash_A a$$

to *open terms*, that is, terms with free variables..

Notation: $M \Vdash^!_A [a_1, \ldots, a_n]$ means that

$$\vdash M : A \text{ and } \forall i \in \{1, \ldots, n\} \;\; M \Vdash_A a_i$$

Remark:

- $M \Vdash^!_A []$ simply means that $\vdash M : A$.
- If $M \Vdash^!_A m + m'$ then $M \Vdash^!_A m$.

Theorem (Interpretation Lemma)

Assume $x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k \vdash M : a : A$.

Then for all closed terms $N_1, \ldots, N_k$ *such that* $N_i \Vdash^{!}_{A_i} m_i$ *for* $i = 1, \ldots, k$*, one has* $M\,[N_1/x_1, \ldots, N_k/x_k] \Vdash_A a$.

The proof is by induction on the derivation π of $x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k \vdash M : a : A$.

Important remark

The universal quantification on the N_i's is part of the statement that we prove by induction.

Proof of the Interpretation Lemma

π is

$$\frac{n \in \mathbb{N}}{0_\Gamma \vdash \underline{n} : n : \iota}$$

so that $M = \underline{n}$. Obviously $M\,[N_1/x_1, \ldots, N_k/x_k] = \underline{n}\;\beta^*_{\mathrm{wh}}\;\underline{n}$, that is $M\,[N_1/x_1, \ldots, N_k/x_k] \Vdash_\iota n$.

π is

$$\dfrac{\begin{array}{c}\pi_1\\ \Phi \vdash P : n : \iota\end{array}}{\Phi \vdash \mathrm{succ}(P) : n+1 : \iota}$$

where $\Phi = (x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k)$. So $M = \mathrm{succ}(P)$.

Let $N_1, \ldots, N_k$ be such that $N_i \Vdash^!_{A_i} m_i$ for $i = 1, \ldots, k$.

Notation

For any term Q, let $\widetilde{Q} = Q\,[N_1/x_1, \ldots, N_k/x_k]$.

By inductive hypothesis (applied to π_1) we know that $\widetilde{P} \Vdash_\iota n$, that is $\widetilde{P}\ \beta^*_{\mathrm{wh}}\ \underline{n}$.

Then $\widetilde{\mathrm{succ}(P)} = \mathrm{succ}(\widetilde{P})\ \beta^*_{\mathrm{wh}}\ \mathrm{succ}(\underline{n})$ by definition of β_{wh}, and $\mathrm{succ}(\underline{n})\ \beta_{\mathrm{wh}}\ \underline{n+1}$ hence $\widetilde{M}\ \beta^*_{\mathrm{wh}}\ \underline{n+1}$ that is $\widetilde{M} \Vdash_\iota \underline{n+1}$.

π is

$$\dfrac{\overset{\rho}{\Phi \vdash P : 0 : \iota} \qquad \overset{\pi_0}{\Phi_0 \vdash Q : a : A} \qquad \underline{\Phi}, z : \iota \vdash R : A}{\Phi + \Phi_0 \vdash \mathsf{if}(P, Q, z \cdot R) : a : A}$$

So we have $M = \mathsf{if}(P, Q, z \cdot R)$.

Using the notations $\Phi = (x_1 : m_1 : A_1, \dots, x_k : m_k : A_k)$ and $\Phi_0 = (x_1 : m_1^0 : A_1, \dots, x_k : m_k^0 : A_k)$ we have $\Phi + \Phi_0 = (x_1 : m_1 + m_1^0 : A_1, \dots, x_k : m_k + m_k^0 : A_k)$.

Let $N_1, \dots, N_k$ be such that $N_i \Vdash^!_{A_i} m_i + m_i^0$ for $i = 1, \dots, k$.

So we have $N_i \Vdash^{!}_{A_i} m_i$ for $i = 1, \ldots, k$.

Hence by inductive hypothesis applied to ρ we have $\widetilde{P} \Vdash_{\iota} \underline{0}$, that is $\widetilde{P}\, \beta^*_{\mathrm{wh}}\, \underline{0}$.

We have $\widetilde{M} = \mathrm{if}(\widetilde{P}, \widetilde{Q}, z \cdot \widetilde{R})$ and hence $\widetilde{M}\, \beta^*_{\mathrm{wh}}\, \mathrm{if}(\underline{0}, \widetilde{Q}, z \cdot \widetilde{R})$ by definition of β_{wh}. Hence $\widetilde{M}\, \beta^*_{\mathrm{wh}}\, \widetilde{Q}$.

We also have $N_i \Vdash^!_{A_i} m_i^0$ for $i = 1, \ldots, k$.

By inductive hypothesis applied to π_0 we have $\widetilde{Q} \Vdash_A a$ and hence $\widetilde{M} \Vdash_A a$ by the Expansion Lemma.

π is

$$\dfrac{\overset{\rho}{\Phi \vdash P : n+1 : \iota} \qquad \underline{\Phi} \vdash Q : A \qquad \overset{\pi_+}{\Phi_+, z : I[n] : \iota \vdash R : a : A}}{\Phi + \Phi_+ \vdash \mathrm{if}(P, Q, z \cdot R) : a : A}$$

So we have $M = \mathrm{if}(P, Q, z \cdot R)$.

Using the notations $\Phi = (x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k)$ and $\Phi_+ = (x_1 : m_1^+ : A_1, \ldots, x_k : m_k^+ : A_k)$ we have $\Phi + \Phi_+ = (x_1 : m_1 + m_1^+ : A_1, \ldots, x_k : m_k + m_k^+ : A_k)$.

Let $N_1, \ldots, N_k$ be such that $N_i \Vdash^!_{A_i} m_i + m_i^+$ for $i = 1, \ldots, k$.

So we have $N_i \Vdash^!_{A_i} m_i$ for $i = 1, \ldots, k$.

Hence by inductive hypothesis applied to ρ we have $\widetilde{P} \Vdash_\iota \underline{n+1}$, that is $\widetilde{P}\, \beta^*_{\mathrm{wh}}\, \underline{n+1}$.

We have $\widetilde{M} = \mathrm{if}(\widetilde{P}, \widetilde{Q}, z \cdot \widetilde{R})$ and hence $\widetilde{M}\, \beta^*_{\mathrm{wh}}\, \mathrm{if}(\underline{n+1}, \widetilde{Q}, z \cdot \widetilde{R})$ by definition of β_{wh}. Hence $\widetilde{M}\, \beta^*_{\mathrm{wh}}\, \widetilde{R}\,[\underline{n}/z]$.

We also have $N_i \Vdash^{!}_{A_i} m_i^{+}$ for $i = 1, \ldots, k$.

And $\underline{n} \Vdash_{\iota} n$.

Hence by inductive hypothesis applied to π_{+} we have $\widetilde{R}\,[\underline{n}/z] \Vdash_A a$ (whatever be the value of l) and hence $\widetilde{M} \Vdash_A a$ by the Expansion Lemma.

Remark

The $\forall$ is required in the statement proven by induction: the inductive hypothesis is applied with "parameters" $N_1, \ldots, N_k, \underline{n}$.

π is

$$\frac{m_i = [a] \qquad m_j = [\,] \text{ if } j \neq i}{x_1 : m_1 : A_1, \ldots, x_k : m_k : A_k \vdash x_i : a : A_i}$$

so $M = x_i$.

Then $\widetilde{M} = N_i$ and since we have assumed that $N_i \Vdash^!_A [a]$, we have $N_i \Vdash_A a$, that is $\widetilde{M} \Vdash_A a$.

π is

$$\dfrac{\begin{matrix}\pi_1 \\ \Phi, x : p : B \vdash P : c : C\end{matrix}}{\Phi \vdash \lambda x^A\, P : (p, c) : B \Rightarrow C}$$

so that $A = (B \Rightarrow C)$ and $M = \lambda x^B\, P$.

We have $\widetilde{M} = \lambda x^B\, \widetilde{P}$ and so we must prove that $\lambda x^B\, \widetilde{P} \Vdash_{B \Rightarrow C} (p, c)$.

So let Q be such that $Q \Vdash^{!}_{B} p$, we must prove that $(\lambda x^B\, \widetilde{P})\, Q \Vdash_C c$.

By inductive hypothesis applied to π_1, we have $\widetilde{P}\,[Q/x] \Vdash_C c$.

Since $(\lambda x^B\, \widetilde{P})\, Q \;\beta_{\mathsf{wh}}\; \widetilde{P}\,[Q/x]$ we have $(\lambda x^B\, \widetilde{P})\, Q \Vdash_C c$ by the Expansion Lemma.

π is

$$\dfrac{\begin{array}{c}\pi_0\\ \Phi \vdash P : ([b_1, \dots, b_q], c) : B \Rightarrow C\end{array} \qquad \left(\begin{array}{c}\pi_j\\ \Phi_j \vdash Q : b_j : B\end{array}\right)_{1 \leq j \leq q}}{\Phi + \sum_{j=1}^{q} \Phi_j \vdash (P)\, Q : c : C}$$

so that $M = (P)\, Q$ and $A = (B \Rightarrow C)$.

We can write $\Phi = (x_1 : m_1^0 : A_1, \dots, x_1 : m_k^0 : A_k)$ and $\Phi_j = (x_1 : m_1^j : A_1, \dots, x_1 : m_k^j : A_k)$ for $j = 1, \dots, q$. So that $\Phi + \sum_{j=1}^{q} \Phi_j = (x_1 : \sum_{j=0}^{q} m_1^j : A_1, \dots, x_k : \sum_{j=0}^{q} m_k^j : A_k)$.

Let $N_1, \dots, N_k$ be such that $N_i \Vdash^!_{A_i} \sum_{j=0}^{q} m_i^j$ for $i = 1, \dots, k$.

So we have $N_i \Vdash^!_{A_i} m_i^0$ for $i = 1, \ldots, k$.

So by inductive hypothesis applied to π_0 we have $\widetilde{P} \Vdash_{B \Rightarrow C} ([b_1, \ldots, b_q], c)$.

And for each $j \in \{1, \ldots, q\}$ we have $N_i \Vdash^!_{A_i} m_i^j$ for $i = 1, \ldots, k$.

So by inductive hypothesis applied to π_j we have $\widetilde{Q} \Vdash_B b_j$ for $j = 1, \ldots, q$, that is $\widetilde{Q} \Vdash^!_B [b_1, \ldots, b_q]$.

Therefore we have $\widetilde{M} = (\widetilde{P})\, \widetilde{Q} \Vdash_C c$.

π is

$$\dfrac{\begin{matrix}\pi_0 \\ \Phi \vdash P : ([a_1, \ldots, a_q], a) : A \Rightarrow A\end{matrix} \qquad \left(\begin{matrix}\pi_j \\ \Phi_j \vdash \mathsf{fix}(P) : a_j : A\end{matrix}\right)_{1 \le j \le q}}{\Phi + \sum_{j=1}^{q} \Phi_j \vdash \mathsf{fix}(P) : a : A}$$

so that $M = \mathsf{fix}(P)$.

We can write $\Phi = (x_1 : m_1^0 : A_1, \ldots, x_1 : m_k^0 : A_k)$ and $\Phi_j = (x_1 : m_1^j : A_1, \ldots, x_1 : m_k^j : A_k)$ for $j = 1, \ldots, q$. So that $\Phi + \sum_{j=1}^{q} \Phi_j = (x_1 : \sum_{j=0}^{q} m_1^j : A_1, \ldots, x_1 : \sum_{j=0}^{q} m_k^j : A_k)$.

Let $N_1, \ldots, N_k$ be such that $N_i \Vdash^{!}_{A_i} \sum_{j=0}^{q} m_i^j$ for $i = 1, \ldots, k$.

So we have $N_i \Vdash^{!}_{A_i} m_i^0$ for $i = 1, \ldots, k$.

So by inductive hypothesis applied to π_0 we have $\widetilde{P} \Vdash_{A \Rightarrow A} ([a_1, \ldots, a_q], a)$.

And for each $j \in \{1, \ldots, q\}$ we have $N_i \Vdash^{!}_{A_i} m_i^j$ for $i = 1, \ldots, k$.

So by inductive hypothesis applied to π_j we have $\widetilde{\mathrm{fix}(P)} \Vdash_B a_j$ for $j = 1, \ldots, q$, that is $\mathrm{fix}(\widetilde{P}) \Vdash^{!}_{B} [a_j, \ldots, a_q]$.

Hence $(\widetilde{P})\,\mathrm{fix}(\widetilde{P}) \Vdash_A a$.

Since $\widetilde{M} = \mathrm{fix}(\widetilde{P})\ \beta_{\mathrm{wh}}\ (\widetilde{P})\,\mathrm{fix}(\widetilde{P})$ we have $\widetilde{M} \Vdash_A a$ by the Expansion Lemma.

Completeness theorem for β_{wh}

We have proven

Theorem

If $\vdash M : \iota$ *and* $n \in [\![M]\!]$ *then* $M \; \beta_{\mathrm{wh}}^* \; \underline{n}$.

As a consequence

Theorem (Completeness of β_{wh})

Assume that $\vdash M : \iota$. *If* $M \sim_\beta \underline{n}$ *then* $M \; \beta_{\mathrm{wh}}^* \; \underline{n}$.

We have $[\![M]\!] = [\![\underline{n}]\!] = \{n\}$ and hence $M \; \beta_{\mathrm{wh}}^* \; \underline{n}$.

The strategy β_{wh} produces the value of any term M which has a value (for $\vdash M : \iota$).

About observational equivalence

Remember that we have defined the observational equivalence for PCF terms:

Definition

Let M_1 and M_2 be such that $\vdash M_i : A$ for $i = 1, 2$. We say that M_1 and M_2 are observationally equivalent (written $M_1 \sim M_2$) if for any term C such that $\vdash C : A \Rightarrow \iota$ one has

$$(C)\, M_1\, \beta^*_{\mathrm{wh}}\, \underline{0} \Leftrightarrow (C)\, M_2\, \beta^*_{\mathrm{wh}}\, \underline{0}\,.$$

With M_1 and M_2 such that $\vdash M_i : A$ for $i = 1, 2$, let us write

$$M_1 \sim_{\mathbf{Rel}} M_2$$

if $[\![M_1]\!] = [\![M_2]\!]$. This is an equivalence relation (the equivalence induced by the model on terms).

Theorem (Adequacy of Rel)

$$M_1 \sim_{\mathbf{Rel}} M_2 \Rightarrow M_1 \sim M_2$$

So we can use the model to prove observational equivalence.

Proof of the adequacy of **Rel**

Let M_1 and M_2 be such that $\vdash M_i : A$ for $i = 1, 2$ with $[\![M_1]\!] = [\![M_2]\!]$.

Let C be a term such that $\vdash C : A \Rightarrow \iota$ and assume that $(C)\,M_1\ \beta^*_{\mathrm{wh}}\ \underline{0}$.

Hence $[\![(C)\,M_1]\!] = \{0\}$.

But $[\![(C)\,M_1]\!] = \mathrm{Ev} \circ \langle [\![C]\!], [\![M_1]\!] \rangle$ (in $\mathbf{Rel}_!$). Hence $[\![(C)\,M_2]\!] = \{0\}$.

By the theorem we have proven this implies $(C)\,M_2\ \beta^*_{\mathrm{wh}}\ \underline{0}$.

The converse implication is proven in the same way.

Example

Take $A = (\iota \Rightarrow (\iota \Rightarrow \iota))$ and

$$M_1 = \lambda x_1^\iota\, \lambda x_2^\iota\, \mathrm{if}(x_1, \mathrm{if}(x_2, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$
$$M_2 = \lambda x_1^\iota\, \lambda x_2^\iota\, \mathrm{if}(x_2, \mathrm{if}(x_1, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})$$

Then using the semantic typing system one can prove that

$$\llbracket M_i \rrbracket = \{([0], ([0], 0))\} \cup \{([n], ([p], 1)) \mid n, p \in \mathbb{N} \text{ and not } n = p = 0\}$$

for $i = 1, 2$, hence $M_1 \sim_{\mathbf{Rel}} M_2$ and hence $M_1 \sim M_2$.

But M_1 and M_2 are β-normal: they cannot be identified by reduction.

Remark: If we have *side effects* in the language such as

- a global or local memory where one can read and write
- or input-outputs (read or write in a file etc)

then the M_1 and M_2 are no more equivalent.

Rel is not fully abstract

This proof method for $\sim$ is not complete: it is not true that, for any M_1, M_2 such that $\vdash M_1, M_2 : A$,

$$M_1 \sim M_2 \Rightarrow [\![M_1]\!] = [\![M_2]\!]$$

If a model satisfies this condition, it is said fully abstract. Let $\Omega^A = \mathrm{fix}(\lambda x^A\, x)$. Notice that $\Omega^\iota \,\beta_{\mathrm{wh}}\, \Omega^\iota$ and hence $[\![\Omega^\iota]\!] = \emptyset$.

Example

For $i = 1, 2$, consider the closed term

$$M_i = \lambda f^{\iota\Rightarrow\iota\Rightarrow\iota} \cdot \mathrm{if}((f)\,\underline{0}\,\Omega^\iota, \mathrm{if}((f)\,\Omega^\iota\,\underline{0},$$
$$\mathrm{if}((f)\,\underline{1}\,\underline{1}, \Omega^\iota, z \cdot \underline{i}), z \cdot \Omega^\iota), z \cdot \Omega^\iota)$$

of type $(\iota \Rightarrow \iota \Rightarrow \iota) \Rightarrow \iota$.

Then defining $a_i \in [\![(\iota \Rightarrow \iota \Rightarrow \iota) \Rightarrow \iota]\!]$ for $i = 1, 2$ as

$$a_i = (([([0], [\,], 0), ([\,], [0], 0), ([1], [1], 1)]), i)$$

one has $a_i \in [\![M_i]\!]$ and $a_i \notin [\![M_{3-i}]\!]$ so $[\![M_1]\!] \neq [\![M_2]\!]$.

But in coherence spaces (for instance) M_1 and M_2 are interpreted as $\emptyset$, hence $M_1 \sim M_2$. Because $([0], [\,], 0) \smile ([\,], [0], 0)$.

Probabilistic coherence spaces

General goal

Interpret program acting on uncertain data.

For instance, given

- a PCF term M such that $\vdash M : \iota \Rightarrow \iota$
- and a "term" P of type ι which reduces to $\underline{0}$ with probability $1/3$, to $\underline{4}$ with probability $1/2$ and to $\underline{7}$ with probability $1/6$,

what is the probability that $(M)\,P$ reduces to $\underline{42}$?

Moreover, the term M can also "flip coins" during its execution to make some choices.

Coefficients

We cannot restrict our attention to probabilities $\in [0, 1]$, we have to consider more general coefficients.

These coefficients will be in $\mathbb{R}_{\geq 0} = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$. No negative coefficients.

Very rarely we will consider coefficients in $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. Notice $\overline{\mathbb{R}_{\geq 0}}$, with the usual order on nulbers, is a cpo (any subset of $\overline{\mathbb{R}_{\geq 0}}$ has a least upper bound (lub) in $\overline{\mathbb{R}_{\geq 0}}$).

For multiplication to be Scott-continuous, we set $0 \times \infty = 0$.

General idea of PCS

Let I be a set of "elementary data", we want to consider subsets of $(\mathbb{R}_{\geq 0})^I$ whose elements will be considered as generalized "distributions of probabilities" over I.

Example (integers)

$I = \mathbb{N}$, we will represent the type of natural numbers as the set of all $x \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ such that $\sum_{n=0}^{\infty} x_n \leq 1$.

Why not $\sum_{n=0}^{\infty} x_n = 1$? Because we want also to consider partial programs of type ι, with probability $1 - \sum_{n=0}^{\infty} x_n$ to diverge.

Duality of PCS

Let $x, x' \in (\mathbb{R}_{\geq 0})^I$, consider x as a "probabilistic" data and x' as an observer. Then we represent the probability that the observation x' succeeds on x as

$$\langle x, x' \rangle = \sum_{i \in I} x_i x'_i$$

Intuition:

- x_i is the "probability" that x produces i
- x'_i is the weight, the significance, that the observer x' gives to value i.

So we expect that $\langle x, x' \rangle \leq 1$.

Given $\mathcal{D} \subseteq (\mathbb{R}_{\geq 0})^I$, we define

$$\mathcal{D}^\perp = \left\{x' \in (\mathbb{R}_{\geq 0})^I \mid \forall x \in \mathcal{D}\ \langle x, x' \rangle \leq 1\right\}$$

So $\mathcal{D}^\perp$ is the set of all "observations" which make sense against all the "data" of $\mathcal{D}$.

Lemma

Let $\mathcal{D}, \mathcal{E} \subseteq (\mathbb{R}_{\geq 0})^I$, then

- $\mathcal{D} \subseteq \mathcal{E} \Rightarrow \mathcal{E}^\perp \subseteq \mathcal{D}^\perp$
- *and* $\mathcal{D} \subseteq \mathcal{D}^{\perp\perp}$.

As a consequence $\mathcal{D}^{\perp\perp\perp} = \mathcal{D}^\perp$. *In other words,* $\mathcal{D}^{\perp\perp} = \mathcal{D}$ *iff* $\mathcal{D} = \mathcal{E}^\perp$ *for some* $\mathcal{E} \subseteq (\mathbb{R}_{\geq 0})^I$.

Avoiding ∞ coefficients

Notation: if $i \in I$ we use e_i for the element x of $(\mathbb{R}_{\geq 0})^I$ such that $x_j = 0$ if $j \neq i$ and $x_i = 1$.

Let $\mathcal{D} \subseteq (\mathbb{R}_{\geq 0})^I$ and assume that for some $i \in I$ we have

$$\forall x \in \mathcal{D} \quad x_i = 0\,.$$

Then $\lambda e_i \in \mathcal{D}^\perp$ for all $\lambda \in \mathbb{R}_{\geq 0}$. So if we want $\mathcal{D}^\perp$ to be complete (in the sense of complete partial orders), this will require to introduce ∞ coefficients. We prefer to avoid this.

Dually if for some $i \in I$ we have

$$\forall \lambda \in \mathbb{R}_{\geq 0} \quad \lambda e_i \in \mathcal{D}$$

then all the elements of $x' \in \mathcal{D}^{\perp}$ will satisfy $x'_i = 0$.

So we consider only sets $\mathcal{D} \subseteq (\mathbb{R}_{\geq 0})^I$ such that

$$\forall i \in I \quad 0 < \sup_{x \in \mathcal{D}} x_i < \infty\,.$$

Definition

A probabilistic coherence space (PCS) is a pair $X = (|X|, \mathrm{P}X)$ where

- $|X|$ is a set and $\mathrm{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ called the web of X
- $\mathrm{P}X^{\perp\perp} \subseteq \mathrm{P}X$ (that is $\mathrm{P}X^{\perp\perp} = \mathrm{P}X$)
- and, for all $a \in |X|$,

$$0 < \sup_{x \in \mathrm{P}X} x_a < \infty\,.$$

Then we define $X^{\perp} = (|X|, \mathrm{P}X^{\perp})$, which is also a PCS.

A PCS is down-closed and convex

Given a set I and $x, y \in (\mathbb{R}_{\geq 0})^I$, we write $x \leq y$ if $\forall i \in I\ x_i \leq y_i$. This is an order relation on $(\mathbb{R}_{\geq 0})^I$.

Lemma

Let X be a PCS and let $x \in \mathrm{P}X$. Let $y \in (\mathbb{R}_{\geq 0})^{|X|}$ be such that $y \leq x$. Then $y \in \mathrm{P}X$.

Let $x, y \in \mathrm{P}X$ and let $\lambda \in [0, 1]$. Then $\lambda x + (1 - \lambda)y \in \mathrm{P}X$.

For the first statement, let $x' \in \mathrm{P}X^{\perp}$, we have $\langle y, x' \rangle \leq \langle x, x' \rangle \leq 1$ and hence $y \in \mathrm{P}X^{\perp\perp} = \mathrm{P}X$.

For the second statement, let $x' \in \mathrm{P}X^{\perp}$. Continuity of addition and multiplication show that $\langle \lambda x + (1 - \lambda)y, x' \rangle = \lambda \langle x, x' \rangle + (1 - \lambda) \langle y, x' \rangle \leq \lambda + 1 - \lambda = 1$ hence $\lambda x + (1 - \lambda)y \in \mathrm{P}X^{\perp\perp} = \mathrm{P}X$.

A PCS is a cpo

Theorem

The poset $(\mathsf{P}X, \leq)$ *is a cpo.*

Let $D \subseteq \mathsf{P}X$ be directed. We define $x \in (\mathbb{R}_{\geq 0})^{|X|}$ by $\forall a \in |X|\ x_a = \sup_{y \in D} y_a$. We prove that $x \in \mathsf{P}X^{\perp\perp} = \mathsf{P}X$. This amounts to proving that $\forall x' \in \mathsf{P}X^{\perp}\ \langle x, x' \rangle \leq 1$. So let $x' \in \mathsf{P}X^{\perp}$.

We have

$$
\begin{aligned}
\langle x, x' \rangle &= \sum_{a \in |X|} x_a x'_a \\
&= \sup_{I \in \mathcal{P}_{\mathrm{fin}}(|X|)} \sum_{a \in I} x_a x'_a \\
&= \sup_{I \in \mathcal{P}_{\mathrm{fin}}(|X|)} \sup_{y \in D} \sum_{a \in I} y_a x'_a \quad \text{by cont. of } \times \text{ and } + \\
&= \sup_{y \in D} \sup_{I \in \mathcal{P}_{\mathrm{fin}}(|X|)} \sum_{a \in I} y_a x'_a \\
&= \sup_{y \in D} \langle y, x' \rangle \leq 1 \,.
\end{aligned}
$$

The converse is also true

It is good to know that conversely (although we will not use this property here):

Theorem

Let $P \subseteq (\mathbb{R}_{\geq 0})^I$ be such that:

- $\forall x, y \in (\mathbb{R}_{\geq 0})^I$ $(x \leq y$ *and* $y \in P) \Rightarrow x \in P$
- $\forall D \subseteq P$ D *directed* $\Rightarrow \sup D \in P$ *(remember that* $x = \sup D \in \overline{\mathbb{R}_{\geq 0}}^I$ *is given by* $x_i = \sup_{y \in D} y_i$ *for each* $i \in I$*)*
- $\forall x, y \in P$, $\forall \lambda \in [0, 1]$ $\lambda x + (1 - \lambda) y \in P$
- *and* $\forall i \in I$ $0 < \sup_{x \in P} x_i < \infty$.

Then $P^{\perp\perp} \subseteq P$ *(that is* $P^{\perp\perp} = P$*) and* (I, P) *is a PCS.*

The proof is essentially an application of the Hahn-Banach theorem.

The norm of a PCS

Given $x \in \mathsf{P}X$ we define

$$\|x\|_X = \sup_{x' \in \mathsf{P}X^\perp} \langle x, x' \rangle \leq 1$$

We have

- $\|x\|_X = 0 \Rightarrow x = 0$, indeed for each $a \in |X|$ there is $\varepsilon > 0$ such that $\varepsilon e_a \in \mathsf{P}X^\perp$ hence $\|X\|_X \geq \langle x, \varepsilon e_a \rangle = \varepsilon x_a$. So $\|x\|_X = 0 \Rightarrow \forall a \in |X|\ x_a = 0$.
- Let $\lambda \in [0,1]$, we have $\|\lambda x\|_X = \lambda \|x\|_X$.
- Let $x, y \in \mathsf{P}X$ such that $x + y \in \mathsf{P}X$. Then $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

Indeed $\|x + y\|_X = \sup_{x' \in \mathsf{P}X^\perp}(\langle x, x' \rangle + \langle y, x' \rangle) \leq \|x\|_X + \|y\|_X$.

Matrices

Let I and J be sets, an $I \times J$-matrix is an element s of $\overline{\mathbb{R}_{\geq 0}}^{I \times J}$.
Given $x \in \overline{\mathbb{R}_{\geq 0}}^{I}$ we define

$$s \cdot x = \left(\sum_{i \in I} s_{i,j} x_i \right)_{j \in J} \in \overline{\mathbb{R}_{\geq 0}}^{J}$$

application of matrix s to vector x.

If K is another set and $t \in \overline{\mathbb{R}_{\geq 0}}^{J \times K}$ we define

$$t\, s = \left(\sum_{j \in J} s_{i,j} t_{j,k} \right)_{i \in I,\, k \in K} \in \overline{\mathbb{R}_{\geq 0}}^{I \times K}$$

the product of the matrices s and t.

Morphisms of PCS

Let X and Y be PCSs. A morphism from X to Y is a $|X| \times |Y|$-matrix s such that

$$\forall x \in \mathrm{P}X \quad s \cdot x \in \mathrm{P}Y\,.$$

This implies that $s \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$ (no infinite coefficients): let $a \in |X|$ and $\varepsilon > 0$ be such that $\varepsilon e_a \in \mathrm{P}X$.

Then $s \cdot \varepsilon e_a = \varepsilon(s_{a,b})_{b \in |Y|} \in \mathrm{P}Y \subseteq (\mathbb{R}_{\geq 0})^{|Y|}$.

The category of PCSs

$\mathbf{Pcoh}(X, Y)$ the set of these morphisms.

Identity morphism $\mathrm{Id}_X \in (\mathbb{R}_{\geq 0})^{|X| \times |X|}$ given by

$$(\mathrm{Id}_X)_{a,a'} = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

Let $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$, then $t\,s \in \mathbf{Pcoh}(X, Z)$.

Indeed let $x \in \mathrm{P}X$, we have $s \cdot x \in \mathrm{P}Y$, hence $(t\,s) \cdot x = t \cdot (s \cdot x) \in \mathrm{P}Z$.

Morphisms as functions

Fact

The morphisms of **Pcoh** *are fully determined by their functional behaviour:*

Let $s, s' \in \mathbf{Pcoh}(X, Y)$*.*

$$(\forall x \in \mathrm{P}X \;\; s \cdot x = s' \cdot x) \Rightarrow s = s'$$

Assume that $\forall x \in \mathrm{P}X \;\; s \cdot x = s' \cdot x$. Let $a \in |X|$ and $b \in |Y|$. Let $\varepsilon > 0$ be such that $\varepsilon e_a \in \mathrm{P}X$. We have

$$(s \cdot \varepsilon e_a)_b = (s' \cdot \varepsilon e_a)_b$$

that is $s_{a,b} = s'_{a,b}$ for all $a \in |X|$, $b \in |Y|$ since $(s \cdot \varepsilon e_a)_b = \varepsilon s_{a,b}$.

Characterizing linear maps on PCS

Fact

Let $s \in \mathbf{Pcoh}(X, Y)$, *then the function* $\widetilde{s} : \mathrm{P}X \to \mathrm{P}Y$ *defined by* $\widetilde{s}(x) = s \cdot x$ *satisfies*

- *if* $x(1), x(2) \in \mathrm{P}X$ *are such that* $x(1) + x(2) \in \mathrm{P}X$ *then* $\widetilde{s}(x(1) + x(2)) = \widetilde{s}(x(1)) + \widetilde{s}(x(2))$ *and as a consequence* $\widetilde{s}$ *is monotone (because* $x(1) \leq x(2) \Leftrightarrow \exists x \in \mathrm{P}X \; x(1) + x = x(2)$*)*
- *if* $x \in \mathrm{P}X$ *and* $\lambda \in [0, 1]$ *then* $\widetilde{s}(\lambda x) = \lambda \widetilde{s}(x)$
- *and* $\widetilde{s}$ *is Scott continuous: for any* $D \subseteq \mathrm{P}X$ *directed,* $\widetilde{s}(\sup D) \leq \sup_{x \in D} \widetilde{s}(x)$.

Conversely for any function $f : \mathrm{P}X \to \mathrm{P}Y$ *with these properties, there is an* $s \in \mathbf{Pcoh}(X, Y)$ *such that* $f = \widetilde{s}$ *(and this s is unique).*

From relations to matrices

Given $u \subseteq I \times J$, that is $u \in \mathbf{Rel}(I, J)$, we define $\mathrm{mat}(u) \in (\mathbb{R}_{\geq 0})^{I \times J}$ (the incidence matrix of u) by

$$\mathrm{mat}(u)_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in u \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{mat}(\mathrm{Id}_I) = Id$ where Id_I is the diagonal relation. And also, if $u \subseteq I \times J$ and $u \subseteq J \times K$ are graphs of bijections, then

$$\mathrm{mat}(v\, u) = \mathrm{mat}(v)\, \mathrm{mat}(u)$$

where $v\, u$ is composition in **Rel** and $\mathrm{mat}(v)\, \mathrm{mat}(u)$ is composition of matrices.

Isomorphisms of PCSs

A priori an iso in **Pcoh** could be a complicated matrix.

A *strong iso* from X to Y is a bijection $\varphi : |X| \to |Y|$ such that

$$\forall x \in (\mathbb{R}_{\geq 0})^{|X|} \quad x \in \mathsf{P}X \Leftrightarrow \mathrm{mat}(\varphi) \cdot x \in \mathsf{P}Y$$

considering φ as a relation from $|X|$ to $|Y|$.

And then φ^{-1} is a strong iso from Y to X with $\mathrm{mat}(\varphi^{-1}) = \mathrm{mat}(\varphi)^{-1}$.

Theorem

Any iso of PCS is a strong iso.

Exercise!

Terminology

We use the wods "strong iso" to speak about φ (the bijection) or about $\mathrm{mat}(\varphi)$ (the matrix), depending on the context.

An important equation

Let $x \in \overline{\mathbb{R}_{\geq 0}}^{I}$ and $y \in \overline{\mathbb{R}_{\geq 0}}^{J}$, we define $x \otimes y \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$ by $(x \otimes y)_{i,j} = x_i y_j$.

Lemma

Let $x \in \overline{\mathbb{R}_{\geq 0}}^{I}$, $y' \in \overline{\mathbb{R}_{\geq 0}}^{J}$ and $s \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$. Then

$$\langle s \cdot x, y' \rangle = \langle s, x \otimes y' \rangle = \sum_{i \in I,\, j \in J} s_{i,j} x_i y'_j \,.$$

$X \multimap Y$ is a PCS

Let X and Y be PCSs and $s \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$. We have

$$\begin{aligned} s \in \mathbf{Pcoh}(X, Y) &\Leftrightarrow \forall x \in \mathrm{P}X,\ \forall y' \in \mathrm{P}Y^{\perp}\ \langle s \cdot x, y' \rangle \leq 1 \\ &\Leftrightarrow \forall x \in \mathrm{P}X,\ \forall y' \in \mathrm{P}Y^{\perp}\ \langle s, x \otimes y' \rangle \leq 1 \end{aligned}$$

Let $X \multimap Y$ be $(|X| \times |Y|, \mathbf{Pcoh}(X, Y))$, we have just seen that

$$\mathrm{P}(X \multimap Y) = \left\{ x \otimes y' \mid x \in \mathrm{P}X \text{ and } y' \in \mathrm{P}Y^{\perp} \right\}^{\perp}$$

Therefore $\mathrm{P}(X \multimap Y) = \mathrm{P}(X \multimap Y)^{\perp\perp}$.

Let $a \in |X|$ and $b \in |Y|$. We can find $\varepsilon > 0$ such that $\varepsilon e_a \in \mathsf{P}X$ and $\varepsilon e_b \in \mathsf{P}Y^\perp$. Let also $M \in \mathbb{R}_{\geq 0}$ be such that
$\forall x \in \mathsf{P}X\ x_a \leq M$ and $\forall y' \in \mathsf{P}Y^\perp\ y'_b \leq M$.

We have $\varepsilon^2 e_{a,b} = \varepsilon e_a \otimes \varepsilon e_b$ and hence
$\forall s \in \mathsf{P}(X \multimap Y)\ \langle s, \varepsilon^2 e_{a,b}\rangle \leq 1$, that is
$\forall s \in \mathsf{P}(X \multimap Y)\ s_{a,b} \leq \varepsilon^{-2}$.

We have $M^{-2} e_{a,b} \in \mathsf{P}(X \multimap Y)$. Indeed, let $x \in \mathsf{P}X$ and
$y' \in \mathsf{P}Y^\perp$, we have $\langle M^{-2} e_{a,b}, x \otimes y'\rangle = M^{-2} x_a y'_b \leq M^{-2} M^2 = 1$.

This shows that $X \multimap Y$ is a PCS.

Transpose of a matrix

Lemma

The swap bijection $\gamma : |X| \times |Y| \to |Y| \times |X|$ *such that* $\gamma(a,b) = (b,a)$ *is a strong iso from* $X \multimap Y$ *to* $Y^\perp \multimap X^\perp$.

It maps $t \in \mathbf{Pcoh}(X,Y)$ *to* $t^\perp \in \mathbf{Pcoh}(Y^\perp, X^\perp)$ *given by* $t^\perp_{b,a} = t_{a,b}$, *the transpose of the matrix* t.

$\_^\perp$ is a functor $\mathbf{Pcoh} \to \mathbf{Pcoh}^{\mathrm{op}}$: $\mathrm{Id}_X^\perp = \mathrm{Id}_X$ and $(t\,s)^\perp = s^\perp\,t^\perp$.

This functor is involutive: $X^{\perp\perp} = X$ and $t^{\perp\perp} = t$.

Lemma

$\forall x \in \mathsf{P}X, \forall y' \in \mathsf{P}Y^\perp \quad \langle t \cdot x, y' \rangle = \langle x, t \cdot y' \rangle$.

Indeed $\langle t \cdot x, y' \rangle = \langle x, t \cdot y' \rangle = \sum_{a \in |X|,\, b \in |Y|} t_{a,b} x_a y'_b$.

Tensor product of PCS

Definition

$1 = (\{*\}, \{(*, \lambda) \mid \lambda \in [0,1]\})$, we shall simply write $\mathrm{P}1 = [0,1]$.

$X \otimes Y = (X \multimap Y^{\perp})^{\perp}$.

So $|X \otimes Y| = |X| \times |Y|$ and
$\mathrm{P}(X \otimes Y) = \{x \otimes y \mid x \in \mathrm{P}X \text{ and } y \in \mathrm{P}Y\}^{\perp\perp}$.

Lemma

Let X_1, X_2 and Y be PCSs. Let
$t \in (\mathbb{R}_{\geq 0})^{(|X_1| \times |X_2|) \times |Y|} = (\mathbb{R}_{\geq 0})^{|X_1 \otimes X_2 \multimap Y|}$.

We have $t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$ *iff*
$\forall x(1) \in \mathrm{P}X_1,\ x(2) \in \mathrm{P}X_2\ \ t \cdot (x(1) \otimes x(2)) \in \mathrm{P}Y$

Assume first that $t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$. Let $x(1) \in \mathrm{P}X_1$ and $x(2) \in \mathrm{P}X_2$. Then we have $x(1) \otimes x(2) \in \mathrm{P}(X_1 \otimes X_2)$ and hence $t \cdot (x(1) \otimes x(2)) \in \mathrm{P}Y$.

Conversely assume that
$\forall x(1) \in \mathsf{P}X_1,\ x(2) \in \mathsf{P}X_2\ t \cdot (x(1) \otimes x(2)) \in \mathsf{P}Y$.

We prove that $t^\perp \in \mathbf{Pcoh}(Y^\perp, (X_1 \otimes X_2)^\perp)$. So let $y' \in \mathsf{P}Y^\perp$, we prove that $t^\perp \cdot y' \in \mathsf{P}(X_1 \otimes X_2)^\perp$.

We have $(X_1 \otimes X_2)^\perp = X_1 \multimap X_2^\perp$. It suffices to prove that $\forall x(1) \in \mathsf{P}X_1\ (t^\perp \cdot y') \cdot x(1) \in \mathsf{P}X_2^\perp$. So it suffices to prove that

$$\forall x(1) \in \mathsf{P}X_1,\ x(2) \in \mathsf{P}X_2 \quad \langle (t^\perp \cdot y') \cdot x(1), x(2) \rangle \leq 1\,.$$

We have

$$\begin{aligned}
\langle (t^{\perp} \cdot y') \cdot x(1), x(2) \rangle &= \sum_{a_1 \in |X_1|,\, a_2 \in |X_2|,\, b \in |Y|} t_{(a_1, a_2), b}\, x(1)_{a_1} x(2)_{a_2} y'_b \\
&= \langle t \cdot (x(1) \otimes x(2)), y' \rangle \\
&\leq 1
\end{aligned}$$

By our assumption about t.

So $t^{\perp} \in \mathbf{Pcoh}(Y^{\perp}, (X_1 \otimes X_2)^{\perp})$ and hence
$t = t^{\perp\perp} \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$.

Functoriality of $\otimes$ in PCSs

Let $t(i) \in \overline{\mathbb{R}_{\geq 0}}^{I_i \times J_i}$ for $i = 1, 2$.

We define $t(1) \otimes t(2) \in \overline{\mathbb{R}_{\geq 0}}^{(I_1 \times I_2) \times (J_1 \times J_2)}$ by

$$(t(1) \otimes t(2))_{(i_1, i_2), (j_1, j_2)} = t(1)_{i_1, j_1} t(2)_{i_2, j_2}$$

Lemma

Given $x(i) \in \overline{\mathbb{R}_{\geq 0}}^{I_i}$ for $i = 1, 2$, we have

$$(t(1) \otimes t(2)) \cdot (x(1) \otimes x(2)) = (t(1) \cdot x(1)) \otimes (t(2) \cdot x(2))$$

Easy computation

Fact

Let $s(i) \in \mathbf{Pcoh}(X_i, Y_i)$ *for* $i = 1, 2$. *Then* $s(1) \otimes s(2) \in \mathbf{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.

Indeed, by the previous lemma, it suffices to prove that

$$\forall x(1) \in \mathrm{P}X_1,\ x(2) \in \mathrm{P}X_2 \quad (s(1) \otimes s(2)) \cdot (x(1) \otimes x(2)) \in \mathrm{P}(Y_1 \otimes Y_2)$$

This results from

$$(s(1) \otimes s(2)) \cdot (x(1) \otimes x(2)) = (s(1) \cdot x(1)) \otimes (s(2) \cdot x(2))$$

and $s(i) \in \mathbf{Pcoh}(X_i, Y_i)$.

We have proven:

Lemma

$\otimes$ *is a functor* $\mathbf{Pcoh}^2 \to \mathbf{Pcoh}$.

Indeed $\mathrm{Id}_{X_1} \otimes \mathrm{Id}_{X_2} = \mathrm{Id}_{X_1 \otimes X_2}$.

And if $s(i) \in \mathbf{Pcoh}(X_i, Y_i)$ and $t(i) \in \mathbf{Pcoh}(Y_i, Z_i)$ for $i = 1, 2$, then

$$(t(1)\, s(1)) \otimes (t(2)\, s(2)) = (t(1) \otimes t(2))\, (s(1) \otimes s(2))$$

Lemma

Let X_1, X_2 and Y be PCSs. Then the bijection

$$\alpha : |(X_1 \otimes X_2) \multimap Y| \to |X_1 \multimap (X_2 \multimap Y)|$$
$$((a_1, a_2), b) \mapsto (a_1, (a_2, b))$$

is a strong iso from $(X_1 \otimes X_2) \multimap Y$ to $X_1 \multimap (X_2 \multimap Y)$.

We need to prove that

$$\mathrm{mat}(\alpha) \in \mathbf{Pcoh}(X_1 \otimes X_2 \multimap Y, X_1 \multimap (X_2 \multimap Y)) .$$

So let $t \in \mathsf{P}(X_1 \otimes X_2 \multimap Y)$, we have to prove that $\mathrm{mat}(\alpha) \cdot t \in \mathsf{P}(X_1 \multimap (X_2 \multimap Y))$.

Given $x(i) \in \mathsf{P}X_i$ for $i = 1, 2$, we have to prove that $((\mathrm{mat}(\alpha) \cdot t) \cdot x(1)) \cdot x(2) \in \mathsf{P}Y$.

This results from

$$((\mathrm{mat}(\alpha) \cdot t) \cdot x(1)) \cdot x(2) = t \cdot (x(1) \otimes x(2))$$

and $t \in \mathsf{P}(X_1 \otimes X_2 \multimap Y)$.

Conversely we must prove that

$$\mathrm{mat}(\alpha^{-1}) \in \mathbf{Pcoh}(X_1 \multimap (X_2 \multimap Y), X_1 \otimes X_2 \multimap Y)$$

so let $t \in \mathrm{P}(X_1 \multimap (X_2 \multimap Y))$ and let us prove that $\mathrm{mat}(\alpha^{-1}) \cdot t \in \mathrm{P}(X_1 \otimes X_2 \multimap Y)$.

By the last lemma, it suffices to prove that for all $x(1) \in \mathrm{P}X_1$ and $x(2) \in \mathrm{P}X_2$ we have $(\mathrm{mat}(\alpha^{-1}) \cdot t) \cdot (x(1) \otimes x(2)) \in \mathrm{P}Y$.

This results from the assumption that $t \in \mathrm{P}(X_1 \multimap (X_2 \multimap Y))$ and from

$$(\mathrm{mat}(\alpha^{-1}) \cdot t) \cdot (x(1) \otimes x(2)) = (t \cdot x(1)) \cdot x(2)\,.$$

So $\mathrm{mat}(\alpha)^\perp = \mathrm{mat}(\alpha^{-1})$ is a strong iso from $(X_1 \multimap (X_2 \multimap Y))^\perp = X_1 \otimes \left(X_2 \otimes Y^\perp\right)$ to $(X_1 \otimes X_2 \multimap Y)^\perp = (X_1 \otimes X_2) \otimes Y^\perp$.

Taking $Y = X_3^\perp$, this shows that α is a strong iso from $(X_1 \otimes X_2) \otimes X_3$ to $X_1 \otimes (X_2 \otimes X_3)$.

We have obvious strong isos λ from $1 \otimes X$ to X given by $\lambda(*, a) = a$, ρ from $X \otimes 1$ to X and γ from $X_1 \otimes X_2$ to $X_2 \otimes X_1$ (given by $\gamma(a_1, a_2) = (a_2, a_1)$).

In that way we turn **Pcoh** into a symmetric monoidal category. Notice that α, λ, ρ and γ are defined *exactly* as in **Rel**. So the commutation of the coherence diagrams holds.

Monoidal closeness

Given PCSs X and Y we define $\mathrm{ev} \in (\mathbb{R}_{\geq 0})^{((X \multimap Y) \otimes X) \multimap Y}$ by

$$\mathrm{ev}_{((a,b),a'),b'} = \begin{cases} 1 & \text{if } a = a' \text{ and } b = b' \\ 0 & \text{otherwise.} \end{cases}$$

By this definitions, it follows that if $t \in \mathrm{P}(X \multimap Y)$ and $x \in \mathrm{P}X$, then

$$\mathrm{ev} \cdot (t \otimes x) = t \cdot x \in \mathrm{P}Y \, .$$

It follows that $\mathrm{ev} \in \mathbf{Pcoh}((X \multimap Y) \otimes X, Y)$ by the usual lemma.

Then $(X \multimap Y, \mathrm{ev})$ is the linear hom object of X and Y. Indeed, given $s \in \mathbf{Pcoh}(Z \otimes X, Y)$, define $t = \mathrm{cur}(s) \in (\mathbb{R}_{\geq 0})^{|Z \multimap (X \multimap Y)|}$ by

$$\mathrm{cur}(s)_{c,(a,b)} = s_{(c,a),b} \, .$$

Then

$$\forall z \in \mathrm{P}Z,\ x \in \mathrm{P}X \quad (\mathrm{cur}(s) \cdot z) \cdot x = s \cdot (z \otimes x) \in \mathrm{P}Y$$

and hence

1. $\forall z \in \mathrm{P}Z\ \mathrm{cur}(s) \cdot z \in \mathrm{P}(X \multimap Y)$
2. and $t = \mathrm{cur}(s) \in \mathbf{Pcoh}(Z, X \multimap Y)$.

Pcoh is $*$-autonomous

We take $\perp = 1$, that is $\perp = (\{*\}, [0,1])$.

Then the standard morphism

$$\eta_X = \mathrm{cur}(\mathrm{ev}\ \gamma) \in \mathbf{Pcoh}(X, (X \multimap \perp) \multimap \perp)$$

is a strong iso (the underlying bijection maps a to $((a,*),*)$).

Simply because we have a strong iso $\theta : X^{\perp} \to (X \multimap \perp)$: as a bijection on the webs, $\theta(a) = (a, *)$. Indeed we have

$$(\mathsf{mat}(\theta) \cdot x') \cdot x = \langle x, x' \rangle = \sum_{a \in |X|} x_a x'_a$$

for all $x, x' \in (\mathbb{R}_{\geq 0})^{|X|}$.

And hence $x' \in \mathsf{P}X^{\perp}$ iff $\mathsf{mat}(\theta) \cdot x' \in \mathsf{P}(X \multimap \perp)$.

Then the fact that η is a strong iso comes from $X^{\perp\perp} = X$ which holds by definition of a PCS.

Cartesian product

Let $(X_i)_{i\in I}$ be a collection of PCSs. We define $X = \&_{i\in I} X_i$ as follows:

- $|X| = \bigcup_{i\in I} \{i\} \times |X_i| = \&_{i\in I} |X_i|$ (in **Rel**)
- and, given $x \in (\mathbb{R}_{\geq 0})^{|X|}$, $x \in \mathrm{P}X$ iff $\forall i \in I$ $\mathrm{mat}(\pi_i) \cdot x \in \mathrm{P}X_i$.

Remember that $\pi_i \in \mathbf{Rel}(\&_{j\in I} |X_j|, |X_i|)$ is the i-th projection of the cartesian product in **Rel**.

$\pi_i = \{((i, a), a) \mid a \in |X_i|\}$.

By this definition we have that $\mathsf{P}X \simeq \prod_{i\in I} \mathsf{P}X_i$ (isomorphic as partially ordered sets), by the mapping $x \mapsto (\mathrm{mat}(\pi_i) \cdot x)_{i\in I}$.

If follows that for all $d = (i, a) \in |X|$

$$0 < \sup_{x\in \mathsf{P}X} x_d = \sup_{y\in \mathsf{P}X_i} y_a < \infty\,.$$

Fact

$$\mathsf{P}(\mathop{\&}_{i\in I} X_i) = \left\{\mathrm{mat}(\overline{\pi}_i)\cdot x' \mid i \in I \textit{ and } x' \in \mathsf{P}X_i^{\perp}\right\}^{\perp}$$

This is simply because $\langle x, \mathrm{mat}(\overline{\pi}_i)\cdot x'\rangle = \langle \mathrm{mat}(\pi_i)\cdot x, x'\rangle$.

It follows that $X^{\perp\perp} = X$ and hence $X = \&_{i\in X_i} X_i$ is a PCS.

Observe also that by definition of this PCS, we have

$$\forall i \in I \quad \mathrm{mat}(\pi_i) \in \mathbf{Pcoh}(\mathop{\&}_{j\in I} X_j, X_i)$$

From now on we write π_i instead of $\mathrm{mat}(\pi_i)$.

Fact

$(\&_{i\in I} X_i, (\pi_i)_{i\in I})$ is the cartesian product of the X_i's in **Pcoh***.*

We set $X = \&_{i\in I} X_i$ as above.

Let $t(i) \in \mathbf{Pcoh}(Y, X_i)$ for each $i \in I$, let $t \in (\mathbb{R}_{\geq 0})^{|Y|\times|X|}$ be defined by

$$\forall b \in |Y|, \forall i \in I, \forall a \in |X_i| \quad t_{b,(i,a)} = (t(i))_{b,a}\,.$$

Then

$$\forall y \in \mathrm{P}Y, \forall i \in I \quad \mathrm{mat}(\pi_i) \cdot (t \cdot y) = t(i) \cdot y \in \mathrm{P}X_i\,.$$

That is $\forall y \in \mathrm{P}Y\ \ t \cdot y \in \mathrm{P}X$ and hence $t \in \mathbf{Pcoh}(Y, \&_{i\in I} X_i)$.

It is clear that this t is the unique element of $\mathbf{Pcoh}(Y, \&_{i\in I} X_i)$ such that

$$\forall i \in I \quad \mathrm{mat}(\pi_i)\, t = t(i)$$

which shows that $(\&_{i\in I} X_i, (\pi_i)_{i\in I})$ is the cartesian product of the X_i's in **Pcoh**.

As usual we write $t = \langle t(i)\rangle_{i\in I}$.

Coproducts

Since **Pcoh** is $*$-autonomous it has coproducts $(\bigoplus_{i\in I} X_i, (\overline{\pi}_i)_{i\in I})$ with

$$\bigoplus_{i\in I} X_i = (\mathop{\&}_{i\in I} X_i^{\perp})^{\perp}$$

and already defined injections.

$\overline{\pi}_i$ is the matrix associated with the i-th injection in **Rel**:

$$\{(a,(i,a)) \mid a \in |X_i|\}$$

so that $\overline{\pi}_i = \pi_i^{\perp}$ (as relations and as matrices).

Fact

$$\mathrm{P}\Big(\bigoplus_{i \in I} X_i\Big) = \left\{ x \in \mathrm{P}\Big(\mathop{\&}_{i \in I} X_i\Big) \mid \sum_{i \in I} \|\pi_i \cdot x\|_{X_i} \leq 1 \right\}$$

Proof in the lecture notes.

Example

- $\mathrm{P}(1 \,\&\, 1) \simeq \{(x_1, x_2) \mid x_1, x_2 \in [0,1]\}$
- $\mathrm{P}(1 \oplus 1) \simeq \{(x_1, x_2) \mid x_1, x_2 \in [0,1] \text{ and } x_1 + x_2 \leq 1\}$ (probabilistic booleans)
- $\mathrm{P}((1 \oplus 1) \,\&\, (1 \oplus 1)) \simeq$ $\{(x_1, x_2, x_3, x_4) \mid \forall i \; x_i \in [0,1],\; x_1 + x_2 \leq 1 \text{ and } x_3 + x_4 \leq 1\}$
- $$\begin{aligned}\mathrm{P}((1 \,\&\, 1) \oplus (1 \,\&\, 1)) \simeq \{(x_1, x_2, x_3, x_4) \mid \forall i \; x_i \in [0,1], \\ x_1 + x_3 \leq 1, x_1 + x_4 \leq 1,\; x_2 + x_3 \leq 1 \text{ and } x_2 + x_4\}\end{aligned}$$

Exponential

Given $x \in (\mathbb{R}_{\geq 0})^I$ and $m \in \mathcal{M}_{\mathrm{fin}}(I)$ (finite multiset of elements of I), we define

$$x^m = \prod_{i \in I} x_i^{m(i)} \in \mathbb{R}_{\geq 0} \,.$$

In other words, if $m = [i_1, \ldots, i_k]$:

$$x^m = \prod_{h=1}^{k} x_{i_h}$$

Definition of $!X$

Then we define $x^{(!)} \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(I)}$ by

$$x^{(!)}_m = x^m$$

for each $m \in \mathcal{M}_{\mathrm{fin}}(I)$.

Finally, given a PCS X we define $!X$ by $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and

$$\mathrm{P}(!X) = \left\{ x^{(!)} \mid x \in \mathrm{P}X \right\}^{\perp\perp} .$$

Hence by definition $\mathrm{P}(!X)^{\perp\perp} = \mathrm{P}(!X)$.

We must prove that $\forall m \in \mathcal{M}_{\mathrm{fin}}(|X|) \quad 0 < \sup_{u \in \mathsf{P}(!X)} u_m < \infty$.

Let $m = [a_1, \ldots, a_k] \in \mathcal{M}_{\mathrm{fin}}(|X|)$. For each $i \in \{1, \ldots, k\}$ we can find $\varepsilon_i > 0$ such that $\varepsilon_i e_{a_i} \in \mathsf{P}X$ for $i = 1, \ldots, k$. Then let $\varepsilon > 0$ be such that $\varepsilon \leq \varepsilon_i$ for $i = 1, \ldots, k$.

Then $\varepsilon e_{a_i} \in \mathsf{P}X$ for each i and hence $x = \frac{\varepsilon}{k+1} \sum_{i=1}^{k} e_{a_i} \in \mathsf{P}X$ (we use $k+1$ instead of k to avoid division by 0).

Then $x_m^{(!)} = x^m = \frac{\varepsilon^k}{(k+1)^k} > 0$ and since $x^{(!)} \in \mathsf{P}(!X)$ we have $\sup_{u \in \mathsf{P}(!X)} u_m > 0$.

Similarly let $M \in \mathbb{R}_{\geq 0}$ be such that
$\forall x \in \mathrm{P}X,\ \forall i \in \{1, \ldots, k\}\ \ x_{a_i} \leq M$.

Let $x \in \mathrm{P}X$, we have

$$\langle x^{(!)}, \frac{1}{M^k} e_m \rangle = \frac{1}{M^k} x^m = \frac{1}{M^k} \prod_{i=1}^{k} x_{a_i} \leq 1$$

Hence $\frac{1}{M^k} e_m \in \mathrm{P}(!X)^{\perp}$.

Therefore $\forall u \in \mathrm{P}(!X)\ \langle u, \frac{1}{M^k} e_m \rangle \leq 1$, that is
$\forall u \in \mathrm{P}(!X)\ u_m \leq M^k$.

Fact

$!X$ is a PCS.

Analytic functions in **Pcoh**

Let $t \in \mathbf{Pcoh}(!X, Y)$. If $x \in \mathsf{P}X$ then $x^{(!)} \in \mathsf{P}(!X)$ and hence

$$t \cdot x^{(!)} \in \mathsf{P}Y$$

We define $\widehat{t} : \mathsf{P}X \to \mathsf{P}Y$ by $\widehat{t}(x) = t \cdot x^{(!)}$.

Fact

Let $t \in (\mathbb{R}_{\geq 0})^{|!X \multimap Y|}$. One has $t \in \mathsf{P}(!X \multimap Y)$ iff $\forall x \in \mathsf{P}X \;\; t \cdot x^{(!)} \in \mathsf{P}Y$.

If $t \in (\mathbb{R}_{\geq 0})^{|!X \multimap Y|}$ we have $t \cdot x^{(!)} \in \mathrm{P}Y$ because $x^{(!)} \in \mathrm{P}(!X)$. Conversely assume that $\forall x \in \mathrm{P}X \;\; t \cdot x^{(!)} \in \mathrm{P}Y$. We prove that $t^{\perp} \in \mathbf{Pcoh}(Y^{\perp}, \mathrm{P}(!X)^{\perp})$.

Notice first that

$$\mathrm{P}(!X)^{\perp} = \left\{x^{(!)} \mid x \in \mathrm{P}X\right\}^{\perp\perp\perp} = \left\{x^{(!)} \mid x \in \mathrm{P}X\right\}^{\perp} .$$

Let $y' \in \mathrm{P}Y^{\perp}$, we prove that $t \cdot y' \in \mathrm{P}(!X)^{\perp}$. So let $x \in \mathrm{P}X$, it suffices to prove that $\langle t^{\perp} \cdot y', x^{(!)} \rangle \leq 1$.

This comes from $\langle t^{\perp} \cdot y', x^{(!)} \rangle = \langle y', t \cdot x^{(!)} \rangle$ and from our assumption about t.

Fact (functional characterization)

Let $s, t \in \mathbf{Pcoh}(!X, Y)$. If $\widehat{s} = \widehat{t}$ then $s = t$.

Idea of the proof: we can express the values of s_m and t_m using the derivatives of the function $\widehat{s} = \widehat{t}$ at 0. Since the derivatives depend only on the function, this shows that $s_m = t_m$. See the lecture notes.

Remark: This means that the morphisms of $\mathbf{Pcoh}_!$ can be considered as functions. As we shall see, composition in $\mathbf{Pcoh}_!$ coincides with composition of the corresponding functions.

A function $f : \mathrm{P}X \to \mathrm{P}Y$ such that there is an $s \in \mathbf{Pcoh}(!X, Y)$ is called an *analytic function*. Then s is the power series of f.

Example (analytic function on 1)

What is an $s \in \mathbf{Pcoh}(!1, 1) = \mathrm{P}(!1 \multimap 1)$?

First we can identify $|!1 \multimap 1|$ with $\mathbb{N}$, so $s \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$.

The condition $s \in \mathbf{Pcoh}(!1, 1)$ means that $\forall x \in \mathrm{P}1\ s \cdot x^{(!)} \in \mathrm{P}1$, that is $\forall x \in [0, 1]\ \sum_{n=0}^{\infty} s_n x^n \in [0, 1]$. That is $\sum_{n \in \mathbb{N}} s_n \leq 1$.

$f(x) = x^k$ (for $k \in \mathbb{N}$) is analytic, $f(x) = \frac{1}{7} + \frac{1}{3}x(2) + \frac{8}{21}x^7$ is analytic. The function $f : [0, 1] \to [0, 1]$ defined by $f(x) = 2x - x^2$ is not analytic (although it is monotone and Scott continuous). The function $f(x) = 1 - \sqrt{1 - x}$ is analytic.

Example (analytic function on the booleans)

What is an $s \in \mathbf{Pcoh}(!(1 \oplus 1), 1) = \mathrm{P}(!(1 \oplus 1) \multimap 1)$?

We can identify $|!(1 \oplus 1) \multimap 1|$ with $\mathbb{N} \times \mathbb{N}$, so $s \in (\mathbb{R}_{\geq 0})^{\mathbb{N} \times \mathbb{N}}$.

Then the condition $s \in \mathbf{Pcoh}(!(1 \oplus 1), 1)$ can be written:
$\forall \lambda \in [0, 1]\ \ \sum_{n,k \in \mathbb{N}} s_{n,k} \lambda^n (1 - \lambda)^k \leq 1$.

For each $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ we have $\lambda^n (1 - \lambda)^n \leq 1/4^n$ and hence if we set for instance

$$s_{n,k} = \begin{cases} 2^n & \text{if } n = k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

then $s \in \mathbf{Pcoh}(!(1 \oplus 1), 1)$. So the function $f(x, y) = \sum_{n=1}^{\infty} 2^n x^n y^n$ is analytic.

Notice that the coefficients of f are unbounded. This example shows why the coefficients have to be in $\mathbb{R}_{\geq 0}$ and not only in $[0, 1]$.

Analytic functions of several arguments

Fact

Let $s \in (\mathbb{R}_{\geq 0})^{|!X_1 \otimes \cdots \otimes !X_k \multimap Y|}$.

One has $s \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ *iff for all* $x(1) \in \mathsf{P}X_1, \ldots, x(k) \in \mathsf{P}X_k$ *one has*

$$s \cdot (x(1)^{(!)} \otimes \cdots \otimes x(k)^{(!)}) \in \mathsf{P}Y .$$

Fact

Let $s, t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$*. If for all* $x(1) \in \mathrm{P}X_1, \ldots, x(k) \in \mathrm{P}X_k$ *one has*

$$s \cdot (x(1)^{(!)} \otimes \cdots \otimes x(k)^{(!)}) = t \cdot (x(1)^{(!)} \otimes \cdots \otimes x(k)^{(!)})$$

then $s = t$*.*

Use the previous (and monoidal closeness of **Pcoh**) in an easy induction on k.

Notation: $\widehat{s}(x(1), \ldots, x(k)) = s \cdot (x(1)^{(!)} \otimes \cdots \otimes x(k)^{(!)})$. It is k-ary analytic function.

Linear maps are analytic

If $s \in \mathbf{Pcoh}(X, Y)$ then the associated linear function $f = \widetilde{s} : \mathsf{P}X \to \mathsf{P}Y$ given by $f(x) = s \cdot x$ is analytic.

The associated power series $t \in \mathbf{Pcoh}(!X, Y)$ is given by

$$t_{m,b} = \begin{cases} s_{a,b} & \text{if } m = [a] \\ 0 & \text{otherwise.} \end{cases}$$

Monotonicity and Scott continuity

Fact

Let $f : \mathsf{P}X \to \mathsf{P}Y$ be analytique, and let $s \in \mathbf{Pcoh}(!X, Y)$ be such that $f = \widehat{s}$ (the power series of f).

Then f is monotone and Scott continuous.

Observe that $f(x) = \widetilde{s}(x^{(!)})$ and we know that $\widetilde{s}$ is monotone and Scott continuous. So it suffices to prove that the function

$$\begin{aligned} \delta : \mathsf{P}X &\to \mathsf{P}(!X) \\ x &\mapsto x^{(!)} \end{aligned}$$

is monotone and Scott continuous.

Easy: it suffices to check that for each $m \in |!X|$ the map $x \to x^m$ from $\mathsf{P}X$ to $\mathbb{R}_{\geq 0}$ is monotone and Scott continuous. This comes from the monotonicity and Scott continuity of multiplication in $\mathbb{R}_{\geq 0}$.

The exponential of a morphism

Given $m \in \mathcal{M}_{\mathrm{fin}}(I)$ and $p \in \mathcal{M}_{\mathrm{fin}}(J)$, we define $L(m, p)$ as the set of all *pairings* of m and p: multisets $r \in \mathcal{M}_{\mathrm{fin}}(I \times J)$ such that

$$\forall i \in I \qquad \sum_{j \in \mathcal{M}_{\mathrm{fin}}(J)} r(i, j) = m(i)$$

$$\forall j \in J \qquad \sum_{i \in \mathcal{M}_{\mathrm{fin}}(I)} r(i, j) = p(j)\,.$$

Notice that if $r \in L(m, p)$ then $\#r = \#m = \#p$ (where $\#m = \sum_{i \in I} m(i)$). So if $L(m, p) \neq \emptyset$ we must have $\#m = \#p$.

If $m \in \mathcal{M}_{\mathrm{fin}}(I)$, we set $m! = \prod_{i \in I} m(i)!$.

Given $r \in L(m, p)$, we set

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p!}{r!} = \prod_{j \in J} \frac{p(j)!}{\prod_{i \in I} r(i,j)!}$$

Notice that $\frac{p(j)!}{\prod_{i \in I} r(i,j)!} \in \mathbb{N}$ because $p(j) = \sum_{i \in I} r(i,j)$. For instance $(10 = 2 + 2 + 3 + 3)$

$$\frac{10!}{2!^2 3!^2} = \frac{10!}{2^4 3^2} = 25200$$

Multinomial coefficient.

Remark: Let $n, n_1, \ldots, n_k \in \mathbb{N}$ be such that $n = n_1 + \cdots + n_k$, then the multinomial coefficient

$$\frac{n!}{n_1! \cdots n_k!}$$

is the number of sets $\{I_1, \ldots, I_k\}$ of k pairwise disjoint subsets of $\{1, \ldots, n\}$ such that $I_1 \cup \cdots \cup I_k = \{1, \ldots, n\}$.

See the lecture notes for a similar combinatorial interpretation of $\begin{bmatrix} p \\ r \end{bmatrix}$ for $r \in L(m, p)$.

Example

$I = \{1, 2, 3\}$, $J = 1, 2$, $m = 5[1] + 3[2] + 5[3]$ and $p = 8[1] + 5[2]$.

Let
$r = 3[(1,1)] + 2[(2,1)] + 3[(3,1)] + 2[(1,2)] + [(2,2)] + 2[(3,2)]$,
we have $r \in L(m, p)$ and

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p!}{r!} = \frac{8! \times 5!}{3! \times 2! \times 3! \times 2! \times 1! \times 2!} = 16800\,.$$

Definition of $!s$

Let $s \in (\mathbb{R}_{\geq 0})^{I \times J}$. Then we define $!s \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(I) \times \mathcal{M}_{\mathrm{fin}}(J)}$.

We set

$$!s_{m,p} = \sum_{r \in L(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} s^r$$

Notice that $L(m,p)$ is a finite set, so this sum is finite.

Remember that $s^r = \prod_{i \in I, j \in J} s_{i,j}^{r(i,j)}$.

Fact

$$\forall x \in (\mathbb{R}_{\geq 0})^I \quad !s \cdot x^{(!)} = (s \cdot x)^{(!)} \,.$$

This is proven by a simple computation (see the lecture notes). As a consequence:

Fact

For all $s \in \mathbf{Pcoh}(X, Y)$ *we have* $!s \in \mathbf{Pcoh}(!X, !Y)$*.*

Indeed, by the crucial property above it suffices to prove that $\forall x \in \mathrm{P}X \ \ !s \cdot x^{(!)} \in \mathrm{P}(!Y)$. This comes from $s \cdot x \in \mathrm{P}Y$ and from $!s \cdot x^{(!)} = (s \cdot x)^{(!)}$.

Dereliction

Let $\mathrm{der}_X \in (\mathbb{R}_{\geq 0})^{|!X| \times |X|}$ be given by

$$\mathrm{der}_{X\,m,a} = \begin{cases} 1 & \text{if } m = [a] \\ 0 & \text{otherwise} \end{cases}$$

that is $\mathrm{der}_X = \mathrm{mat}(\mathrm{der}_{|X|})$.

Then we have $\forall x \in \mathrm{P}X\ \mathrm{der}_X \cdot x^{(!)} = x \in \mathrm{P}X$ and hence $\mathrm{der}_X \in \mathbf{Pcoh}(!X, X)$ by the crucial property again.

Digging

Let $\mathrm{dig}_X \in (\mathbb{R}_{\geq 0})^{|!X| \times |!!X|}$ be given by

$$\mathrm{dig}_{X\,m,M} = \begin{cases} 1 & \text{if } m = \Sigma M \\ 0 & \text{otherwise} \end{cases}$$

that is $\mathrm{dig}_X = \mathrm{mat}(\mathrm{dig}_{|X|})$. Remember that if $M = [m_1, \ldots, m_k] \in |!!X|$ then $\Sigma M = m_1 + \cdots + m_k \in |!X|$.

Then we have $\forall x \in \mathrm{P}X$ $\mathrm{dig}_X \cdot x^{(!)} = x^{(!)(!)} \in \mathrm{P}(!!X)$ and hence $\mathrm{der}_X \in \mathbf{Pcoh}(!X, !!X)$ by the crucial property again.

Naturality of der

We have to prove that if $s \in \mathbf{Pcoh}(X, Y)$ then

$$\begin{array}{ccc} !X & \xrightarrow[!s]{} & !Y \\ \downarrow{\scriptstyle \mathrm{der}_X} & & \downarrow{\scriptstyle \mathrm{der}_Y} \\ X & \xrightarrow{s} & Y \end{array}$$

Let $s(1) = \mathrm{der}_Y \; !s$ and $s(2) = s \; \mathrm{der}_X$. By one of the lemmas above, it suffices to prove that $\forall x \in \mathrm{P}X$,

$$s(1) \cdot x^{(!)} = s(2) \cdot x^{(!)}.$$

This is easy:

$$s(1) \cdot x^{(!)} = \mathsf{der}_Y \cdot (!s \cdot x^{(!)}) = \mathsf{der}_Y \cdot (s \cdot x)^{(!)} = s \cdot x$$

and

$$s(2) \cdot x^{(!)} = s \cdot (\mathsf{der}_X \cdot x^{(!)}) = s \cdot x\,.$$

All commutations of naturality and comonadicity are proven in the same way.

Another example

$$\begin{tikzcd}
!X \arrow[r, "\mathrm{dig}_X"] \arrow[d, "\mathrm{dig}_X"'] & !!X \arrow[d, "\mathrm{dig}_{!X}"] \\
!!X \arrow[r, "!\mathrm{dig}_X"] & !!!X
\end{tikzcd}$$

We take $x \in \mathrm{P}X$, we have

$$\begin{aligned}
(\mathrm{dig}_{!X}\ \mathrm{dig}_X) \cdot x^{(!)} &= \mathrm{dig}_{!X} \cdot (\mathrm{dig}_X \cdot x^{(!)}) \\
&= \mathrm{dig}_{!X} \cdot x^{(!)(!)} = x^{(!)(!)(!)} \\
(!\mathrm{dig}_X\ \mathrm{dig}_X) \cdot x^{(!)} &= !\mathrm{dig}_X \cdot x^{(!)(!)} = x^{(!)(!)(!)}
\end{aligned}$$

Strong monoidality of the comonad

The bijections

$$\begin{aligned} \mathrm{m}^0 : |1| &\to |!\top| \\ * &\mapsto [] \\ \mathrm{m}^2_{|X_1|,|X_2|} : |!X_1 \otimes !X_2| &\to !(|X_1 \mathbin{\&} X_2|) \\ (m(1), m(2)) &\mapsto 1 \cdot m(1) + 2 \cdot m(2) \end{aligned}$$

where $i \cdot [a_1, \ldots, a_k] = [(i, a_1), \ldots, (i, a_k)]$ induce strong isos

$$\begin{aligned} \mathrm{mat}(\mathrm{m}^0) &\in \mathbf{Pcoh}(1, !\top) \\ \mathrm{mat}(\mathrm{m}^2_{|X_1|,|X_2|}) &\in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \mathbin{\&} X_2)) \end{aligned}$$

simply denoted as m^0 and $\mathrm{m}^2_{X_1,X_2}$.

All required diagrams are satisfied, let us check for instance

$$\begin{tikzcd}
{!X \otimes !Y} \arrow[rr, "\mathsf{m}^2_{X,Y}"] \arrow[dd, "\mathsf{dig}_X \otimes \mathsf{dig}_Y"'] & & {!(X \mathbin{\&} Y)} \arrow[d, "\mathsf{dig}_{X \& Y}"] \\
& & {!!(X \mathbin{\&} Y)} \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\
{!!X \otimes !!Y} \arrow[rr, "\mathsf{m}^2_{!X,!Y}"] & & {!(!X \mathbin{\&} !Y)}
\end{tikzcd}$$

Observe first that $\widehat{\mathsf{m}^2_{X,Y}}(x, y)$, that is $\mathsf{m}^2_{X,Y} \cdot \left(x^{(!)} \otimes y^{(!)}\right)$, is equal to $\langle x, y \rangle^{(!)}$.

Let $s = \mathrm{m}^2_{!X,!Y}\ (\mathrm{dig}_X \otimes \mathrm{dig}_Y)$ and $t = !\langle !\pi_1, !\pi_2\rangle\ \mathrm{dig}_{X\&Y}\ \mathrm{m}^2_{X,Y}$, it suffices to prove

$$\forall x \in \mathsf{P}X, y \in \mathsf{P}Y \quad s \cdot \left(x^{(!)} \otimes y^{(!)}\right) = t \cdot \left(x^{(!)} \otimes y^{(!)}\right).$$

We have

$$\begin{aligned} s \cdot \left(x^{(!)} \otimes y^{(!)}\right) &= \mathrm{m}^2_{X,Y} \cdot \left(x^{(!)(!)} \otimes y^{(!)(!)}\right) \\ &= \langle x^{(!)}, x^{(!)}\rangle^{(!)} \end{aligned}$$

and

$$
\begin{aligned}
t \cdot \left(x^{(!)} \otimes y^{(!)}\right) &= (!\langle !\pi_1, !\pi_2\rangle \ \mathrm{dig}_{X\&Y}) \cdot \langle x, y\rangle^{(!)} \\
&= !\langle !\pi_1, !\pi_2\rangle \cdot \langle x, y\rangle^{(!)(!)} \\
&= (\langle !\pi_1, !\pi_2\rangle \cdot \langle x, y\rangle^{(!)})^{(!)} \\
&= \langle !\pi_1 \cdot \langle x, y\rangle^{(!)}, !\pi_2 \cdot \langle x, y\rangle^{(!)}\rangle^{(!)} \\
&= \langle (\pi_1 \cdot \langle x, y\rangle)^{(!)}, (\pi_2 \cdot \langle x, y\rangle)^{(!)}\rangle^{(!)} \\
&= \langle x^{(!)}, y^{(!)}\rangle^{(!)}
\end{aligned}
$$

Conclusion: **Pcoh** is a model of classical LL!

The associated cartesian closed category

It is the category $\mathbf{Pcoh}_!$:

- Objects: the PCSs.
- $\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$
- Identity is $\mathsf{Id}^{\mathsf{Kl}}_X = \mathsf{der}_X \in \mathbf{Pcoh}_!(X, X)$ so that $\widehat{\mathsf{Id}^{\mathsf{Kl}}}(x) = \mathsf{der}_X \cdot x^{(!)} = x$. That is $\widehat{\mathsf{Id}^{\mathsf{Kl}}}$ is the identity function.

- And if $s \in \mathbf{Pcoh}_!(X, Y)$ and $t \in \mathbf{Pcoh}_!(Y, Z)$ then $t \circ s = t\, s^!$ so that

$$\begin{aligned}\widehat{t \circ s}(x) &= t \cdot (s^! \cdot x^{(!)}) \\ &= t \cdot (s \cdot x^{(!)})^{(!)} \\ &= \widehat{t}(\widehat{s}(x))\end{aligned}$$

  that is $\widehat{t \circ s} = \widehat{t} \circ \widehat{s}$.

This is very important: composition (and identities) in $\mathbf{Pcoh}_!$ coincides with composition (and identities) of functions, when considering the morphisms of $\mathbf{Pcoh}_!$ as functions.

$\mathbf{Pcoh}_!$ as a category of functions.

This means that we have a faithful (but not full!) functor $\mathcal{U} : \mathbf{Pcoh}_! \to \mathbf{Set}$ which maps X to $\mathsf{P}X$ and $s \in \mathbf{Pcoh}_!(X, Y)$ to $\widehat{s}$.

If $(X_i)_{i \in I}$ is a family of objects of **Pcoh** then

$$\mathcal{U}(\mathop{\&}_{i \in I} X_i) \simeq \prod_{i \in I} \mathcal{U}(X_i)$$

More precisely $\mathcal{U}$ preserves cartesian products.

$\mathbf{Pcoh}_!$ is a CCC with $(X \Rightarrow Y) = (!X \multimap Y)$ and $\mathsf{Ev} \in \mathbf{Pcoh}_!((X \Rightarrow Y) \mathbin{\&} X, Y)$ is

$$\begin{tikzcd}
!((X \Rightarrow Y) \mathbin{\&} X) \arrow[d, "(\mathsf{m}^2_{X \Rightarrow Y, X})^{-1}"] \\
!(!X \multimap Y) \otimes !X \arrow[d, "\mathsf{der} \otimes !X"] \\
(!X \multimap Y) \otimes !X \arrow[d, "\mathsf{ev}"] \\
Y
\end{tikzcd}$$

It follows that, if $s \in \mathsf{P}(X \Rightarrow Y)$ and $x \in \mathsf{P}X$

$$
\begin{aligned}
\widehat{\mathsf{Ev}}(\langle s, x\rangle) &= (\mathsf{ev}\ (\mathsf{der} \otimes !X)) \cdot ((\mathsf{m}^2_{X \Rightarrow Y, X})^{-1} \cdot \langle s, x\rangle^{(!)}) \\
&= (\mathsf{ev}\ (\mathsf{der} \otimes !X)) \cdot \left(s^{(!)} \otimes x^{(!)}\right) \\
&= \mathsf{ev} \cdot \left(s \otimes x^{(!)}\right) \\
&= \widehat{s}(x)
\end{aligned}
$$

And if $s \in \mathbf{Pcoh}_!(Z \mathbin{\&} X, Y)$ then $\mathrm{Cur}(s) \in \mathbf{Pcoh}_!(Z, X \Rightarrow Y)$ is characterized by the fact that for each $z \in \mathsf{P}Z$, the element $t = \widehat{\mathrm{Cur}(s)}(z)$ of $\mathsf{P}(X \Rightarrow Y)$ is characterized by

$$\forall x \in \mathsf{P}X \quad \widehat{t}(x) = \widehat{s}(\langle z, x\rangle)$$

In other words evaluation and curryfication are defined exactly as in the CCC **Set**.

Contraction, weakening

As in any model of LL, we have a weakening and a contraction morphism

$$\mathsf{w}_X \in \mathbf{Pcoh}(!X, 1) \qquad \mathsf{c}_X \in \mathbf{Pcoh}(!X, !X \otimes !X)$$

We have, for all $x \in \mathsf{P}X$:

$$\begin{aligned} \mathsf{w}_X \cdot x^{(!)} &= 1 \\ \mathsf{c}_X \cdot x^{(!)} &= x^{(!)} \otimes x^{(!)} \end{aligned}$$

If $y \in \mathbf{Pcoh}(1, Y)$ (that is $y \in \mathrm{P}Y$) then

$$\widehat{(y\,\mathsf{w}_X)}(x) = y$$

and if $s \in \mathbf{Pcoh}(!X \otimes !X, Y)$

$$\widehat{(s\ \mathsf{c}_X)}(x) = \widehat{s}(x, x)$$

Integers

Remember we have defined $\mathsf{N} = (\mathbb{N}, \mathsf{PN})$ with

$$\mathsf{PN} = \left\{ x \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \mid \sum_{n \in \mathbb{N}} x_n \leq 1 \right\} .$$

That is $\mathsf{N} = \oplus_{n \in \mathbb{N}} 1$.

So we have a strong iso $1 \oplus \mathsf{N} \simeq \mathsf{N}$ induced by the following bijection

$$\begin{aligned} \theta : |1 \oplus \mathsf{N}| &\to |\mathsf{N}| \\ (1, *) &\mapsto 0 \\ (2, n) &\mapsto n + 1 \end{aligned}$$

In particular $\overline{\mathsf{suc}} = \mathrm{mat}(\theta)\,\overline{\pi}_2 \in \mathbf{Pcoh}(\mathsf{N}, \mathsf{N})$ characterized by $\widetilde{\overline{\mathsf{suc}}}(u)_0 = 0$ and $\widetilde{\overline{\mathsf{suc}}}(u)_{n+1} = x_n$. In other words $\widetilde{\overline{\mathsf{suc}}}(u) = \sum_{n=0}^{\infty} u_n e_{n+1}$.

Remember that if $s \in \mathbf{Pcoh}(X, Y)$ and $x \in \mathsf{P}X$, $\widetilde{s}(x) = s \cdot x$ that is $\widetilde{s}$ is the linear function induced by s).

As in **Rel** we can define

$$\overline{\mathsf{if}} \in \mathbf{Pcoh}(!\mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X), X)$$

characterized by

$$\widehat{\overline{\mathsf{if}}}(u, x, s) = u_0 x + \sum_{n=0}^{\infty} u_{n+1} \widehat{s}(e_n).$$

Its matrix is given by

$$\overline{\mathsf{if}}_{m,p,q,a} = \begin{cases} 1 & \text{if } m = [0],\ p = [a] \text{ and } q = [] \\ 1 & \text{if } m = [n+1],\ p = [] \text{ and } q = [(k[n], a)] \\ & \qquad\qquad \text{for some } n, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for $m \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$, $p \in \mathcal{M}_{\mathrm{fin}}(|X|)$,
$q \in \mathcal{M}_{\mathrm{fin}}(|!\mathsf{N} \multimap X|) = \mathcal{M}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{fin}}(\mathbb{N}) \times |X|)$ and $a \in |X|$.

Least fixed points of analytic functions

Given $s \in \mathbf{Pcoh}_!(Y, Y)$, we know that the function $\widehat{s} : \mathrm{P}Y \to \mathrm{P}Y$ is Scott continuous so $\widehat{s}$ has a least fixed point $\sup_{n\in\mathbb{N}} \widehat{s}^n(0) \in \mathrm{P}X$.

Remember that

$$\begin{aligned} \widehat{s} : \mathrm{P}Y &\to \mathrm{P}Y \\ x &\mapsto s \cdot y^{(!)} = \left(\sum_{m \in \mathcal{M}_{\mathrm{fin}}(|Y|)} s_{m,a} y^m \right)_{a\in|Y|} \end{aligned}$$

Least fixed point operator

As in **Rel** we can apply this to $Y = ((X \Rightarrow X) \Rightarrow X)$ and to the morphism $\mathcal{Z} \in \mathbf{Pcoh}_!((X \Rightarrow X) \Rightarrow X, (X \Rightarrow X) \Rightarrow X)$ such that, for $S \in \mathrm{P}((X \Rightarrow X) \Rightarrow X)$

$$T = \widehat{\mathcal{Z}}(S) \in \mathrm{P}((X \Rightarrow X) \Rightarrow X) = \mathbf{Pcoh}_!(X \Rightarrow X, X)$$

satisfies that, for all $s \in \mathrm{P}(X \Rightarrow X)$,

$$\widehat{T}(s) = \widehat{s}(\widehat{S}(s)) \,.$$

The fact that $\mathcal{Z}$ is a morphism in $\mathbf{Pcoh}_!$ comes from the cartesian closeness of that category.

Fact

Then $\mathcal{Y}$, the least fixed point of $\mathcal{Z}$, satisfies

$$\mathcal{Y} \in \mathsf{P}\,Y = \mathbf{Pcoh}_!(X \Rightarrow X, X)$$

and

$$\forall s \in \mathbf{Pcoh}_!(X, X) \quad \widehat{\mathcal{Y}}(s) = \sup_{n \in \mathbb{N}} \widehat{s}^n(0)$$

It is not obvious at all, at first sight, that the map $s \mapsto \sup_{n \in \mathbb{N}} \widehat{s}^n(0)$ is analytic!

What comes next?

We can now use this model to interpret an extension of PCF with a random primitive for instance a constant which reduces to $\underline{0}$ with probability $1/2$ and to $\underline{1}$ with probability $1/2$.

For this language, reduction will be probabilist: if $\vdash M : \iota$, M has a probability $p_n \in [0,1]$ to reduce to $\underline{n}$, for each $n \in \mathbb{N}$.

We will also have a denotational semantics: $[\![M]\!] \in \mathrm{PN}$.

Adequacy: $\forall n \in \mathbb{N} \quad p_n = [\![M]\!]_n$.

Next week: part III of MPRI 2-02, by Michele Pagani.