The Scott model of Linear Logic is the extensional collapse of its relational model

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Abstract

We show that the extensional collapse of the relational model of linear logic is the model of prime-algebraic complete lattices, a natural extension to linear logic of the well known Scott semantics of the lambda-calculus.
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Introduction

Linear Logic arose from denotational investigations of second order intuitionistic logic by Girard (system F [Gir86]). He observed that the qualitative domains[1] used for interpreting system F can be assumed to be generated by a binary relation on a set of vertices (the *web*): such a structure is called a *coherence space*[2]. The category of coherence spaces, with linear maps (stable maps preserving arbitrary existing unions) as morphisms, has remarkable symmetry properties that led him to the sequent calculus of LL, and then to proof-nets [Gir87] and to the Geometry of Interaction.

Scott semantics of LL. In spite of Barr's observation [Bar79] that the category of complete lattices and linear maps is $*$-autonomous, it was a common belief in the Linear Logic community that the standard Scott semantics of the lambda-calculus (Scott domains and continuous maps) cannot provide models of classical linear logic. Huth showed however in [Hut93] that prime-algebraic complete lattices and lub-preserving maps provide a model of classical LL whose associated cartesian closed category (CCC for short) — the Kleisli category of the "!" comonad — is a full sub-CCC of the category of Scott domains and continuous maps. These ideas are further developed in [HJK00] in the general setting of complete lattices which satisfy a linear finite approximation condition (FS-lattices). A few years later, Winskel rediscovered the same model in a semantical investigation of concurrency [Win98] (see also the beginning of [Win04] for instance). As a particular case of a more general profunctor construction, he showed indeed that the category whose objects are preordered sets and where

[1] Qualitative domains can be seen as particular dI-domains [Ber78].
[2] The pure lambda-calculus, or the Turing-complete functional language PCF [Plo77], can also be interpreted in coherence spaces.

the morphisms from a preorder S to a preorder T are the functions from the set $\mathcal{I}(S)$ of downward closed subsets of S to the set $\mathcal{I}(T)$ which preserve arbitrary unions is a model of classical LL. This category is equivalent to Huth's model, but we prefer Winskel's approach, as it insists on considering preorders (and not lattices) as objects: preorders are similar to the webs of coherence spaces, to the sets of the relational model, and represent the prime elements of the corresponding lattices. Moreover, the LL constructions are easier to describe in terms of preorders than in terms of lattices. It is fair to mention also that Krivine [Kri90, Kri93] used the same construction (set $\mathcal{I}(S)$ of "initial segments" of a preorder S) for describing models of the pure lambda-calculus and mentioned that these preorders give rise to a model of LL, with linear negation corresponding to taking the opposite preorder.

Relational semantics. On the other hand, when one applies the Occam's Razor Principle to the coherence space semantics, one is led to interpreting formulas as sets (the webs, without any structure) and proofs as relations between these sets. Something tricky happens during this process: since coherence vanishes, one cannot restrict the set interpreting an "of course" to contain only finite cliques as Girard did in [Gir86], the best one can do is take all finite subsets. But then, the dereliction relation (from $!X$ to X), which is the set of all pairs $(\{a\}, a)$ where $a \in X$, is no longer a natural transformation. This problem can easily be solved by replacing finite sets with finite multisets, but the effect of this choice is that the corresponding Kleisli category is no longer well-pointed. One defines in that way the *relational semantics of linear logic*, which is certainly its simplest (and, maybe, most canonical) denotational model.

Coefficients. One way of turning the CCC associated with the relational model into a well-pointed category is by enriching it with coefficients: instead of taking subset of $X \times Y$ as morphisms from X to Y, take elements of $C^{X \times Y}$, where C is a suitable set (or class) of coefficients; a canonical choice consists in taking $C = \mathbf{Set}$, the class of all sets. An element of $\mathbf{Set}^{X \times Y}$ should be considered as a matrix whose rows are indexed by the elements of Y, and columns by the elements of X: this is basically the idea of Girard's *quantitative semantics* [Gir88], which is presented as a model of intuitionistic logic, but is indeed a model of LL (Girard wrote this paper before he discovered LL), see [Has02]. It is also an instance of the already mentioned profunctor constructions [Win98].

Finite coefficients belonging to more standard algebraic structures (rigs, fields, etc.) can also be considered, but this requires adding some structure to these sets for guaranteeing the convergence of the sums which appear when multiplying matrices, see [Ehr02, Ehr05, DE11]: the effect of such additional structure is that objects are equipped with a topology for which the (generally infinite) sums involved in multiplying matrices converge.

Extensional collapse of the relational model. The other way of making the relational model well-pointed is by performing an *extensional collapse*. This operation is easily understood in the type hierarchy associated with the cartesian closed Kleisli category of the finite multiset comonad on the category of sets and relations: each type A is interpreted by its relational interpretation $[A]$ (a simple set), together with a partial equivalence relation (PER) $\sim_A$ on $\mathcal{P}([A])$. When A is the type $B \Rightarrow C$, an element of $\mathcal{P}([A])$ is a morphism from B to C, and two such morphisms f and g are $\sim_{B\Rightarrow C}$-equivalent if, for any x, y such that $x \sim_A y$, one has $f(x) \sim_B g(y)$. In other words, this PER is a logical relation[3] (a notion introduced by Tait in [Tai67]), and the extensional collapse of this type hierarchy is obtained by quotienting each set $\mathcal{P}([A])$ by the PER $\sim_A$ (one considers, when forming the quotient, only the elements x of $\mathcal{P}([A])$ such that $x \sim_A x$, which are often called *invariant* elements).

Content of the paper. We prove that this extensional collapse of the relational model coincides precisely with the Scott model of preorders. The first problem we have to face is to give a precise and convincing meaning to this statement. We start from the work of Bucciarelli [Buc97], recasting it in a categorical setting: given a CCC $\mathcal{C}$ and a well-pointed CCC $\mathcal{E}$, we want to express what it means for $\mathcal{E}$ to "be" (we'll say to "represent") the extensional collapse of $\mathcal{C}$. For this, we introduce two categorical constructions.

- The *homogeneous collapse category* $\mathsf{e}(\mathcal{C})$, whose objects are pairs $(U, \sim)$ where U is an object of $\mathcal{C}$ and $\sim$ is a partial equivalence relation (PER) on the points of U (that is on $\mathcal{C}(\top, U)$ where $\top$ is the terminal object of $\mathcal{C}$). The morphisms are those of $\mathcal{C}$ which preserve this additional structure, and it is easy to see that this category is a CCC. The important point in this definition is that the object of morphisms from $(U, \sim)$ to $(V, \sim)$ is $(W, \sim_W)$ where W is the object of morphisms from U to V in $\mathcal{C}$ and the relation $\sim_W$ is defined as a logical relation.
- The *heterogeneous collapse category* $\mathsf{e}(\mathcal{C}, \mathcal{E})$, whose objects are triples $(U, E, \Vdash)$ where U is an object of $\mathcal{C}$, E is an object of $\mathcal{E}$ and $\Vdash \subseteq \mathcal{C}(\top, U) \times \mathcal{E}(\top, E)$ should be understood as a realizability predicate: $x \Vdash \zeta$ means intuitively that ζ represents the "extensional behavior" of x. The morphisms are the pairs (f, φ) of morphisms which preserve the relation $\Vdash$, and again, it is easy to check that this category is a CCC. Again, the important point is that, when constructing the object of morphisms, $\Vdash$ is defined as a logical relation.

These two constructions are possible for any CCCs $\mathcal{C}$ and $\mathcal{E}$. We say that $\mathcal{E}$ represents the extensional collapse of $\mathcal{C}$ if

- $\mathsf{e}(\mathcal{C}, \mathcal{E})$ contains a "sufficiently large" (in a reasonable sense, to be made precise later) sub-CCC $\mathcal{H}$ whose objects $(U, E, \Vdash)$ are *modest*, meaning that $\Vdash$ is a *partial surjection* from $\mathcal{C}(\top, U)$ to $\mathcal{E}(\top, E)$, and therefore induces a PER on $\mathcal{C}(\top, U)$ (observe that $\mathcal{E}(\top, E)$ can be considered as the quotient of $\mathcal{C}(\top, U)$ by this PER)
- and the functor $\mathcal{H} \to \mathsf{e}(\mathcal{C})$ which maps $(U, E, \Vdash)$ to $(U, \sim)$, where $\sim$ is the PER induced by $\Vdash$ (and maps a morphism (f, φ) to f), is a CCC functor (that is, preserves the CCC structure on the nose).

The nice feature of this definition is that it is compatible with the standard one (based on type hierarchies) and that it can easily be extended, for instance, to a simple and general definition of what it means for a model of the pure lambda-calculus to represent the extensional collapse of another one.

[3] Logicians would speak of a binary reducibility predicate.

It would be nice of course to have a similar definition of the extensional collapse of a categorical model of LL, and not only of CCCs, but since the definition of such a model is already rather complicated, we prefer not to address this issue. Instead, we perform the CCC constructions defined above concretely, in a completely linear setting, obtaining both CCCs $\mathsf{e}(\mathcal{C})$ and $\mathcal{H}$ as Kleisli constructions of suitable exponential comonads: in the present paper, $\mathcal{C}$ is the Kleisli category $\mathbf{Rel}_!$ associated with the LL model of sets and relations, and $\mathcal{E}$ is the Kleisli category $\mathbf{ScottL}_!$ associated with the LL model of preorders and linear maps between the associated complete lattices.

After having introduced the necessary preliminary material, we first build in Section 2.2 a linear version of the category $\mathsf{e}(\mathbf{Rel}_!)$. More precisely, we define a model of LL denoted as $\mathbf{PerL}$, whose objects are called PER-objects: they are sets equipped with a PER on their powersets. The Kleisli category $\mathbf{PerL}_!$ is isomorphic to $\mathsf{e}(\mathbf{Rel}_!)$ (or, more precisely, to a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!)$).

Then, in Section 3, we describe the Scott model $\mathbf{ScottL}$ of LL. The objects are preordered sets, and a morphism from S to T is a linear map (that is, a map preserving all unions) from $\mathcal{I}(S)$ (the set of all downward-closed subsets of S) to $\mathcal{I}(T)$. As far as sets are concerned, the multiplicative and additive constructions in this model coincide with those of the model $\mathbf{Rel}$ (more things have to be said about the associated preorders: for instance, $S^{\perp}$ is the set S equipped with the opposite of the preorder of S). As to the exponential, the natural choice would be to define $!S$ as the set of finite subsets of S with a suitable preorder: with that choice, the Kleisli category $\mathbf{ScottL}_!$ is a sub-CCC of the CCC of complete lattices and Scott-continuous functions. But we can obtain the same effect by defining $!S$ as the set of all finite multisets of elements of S, endowed with a similarly defined preorder relation which does not take multiplicities into account, and this will greatly simplify our constructions. Indeed, with this choice, the set interpreting an LL formula in $\mathbf{Rel}$ coincides with the set interpreting the same formula in $\mathbf{ScottL}$ (remember that this set is equipped with a preorder).

In Section 4, we introduce the linear version of the "heterogeneous category" $\mathcal{H}$ of the construction described above. An object should be a triple $(X, S, \Vdash)$ where X is a set, S is a preordered set and $\Vdash \subseteq \mathcal{P}(X) \times \mathcal{I}(S)$ (which has to be a partial surjection). By our choice above for the definition of $!S$, we can assume $X = S$, so as a first simplification, we can assume our objects to be pairs $(S, \Vdash)$ where S is a preordered set and $\Vdash \subseteq \mathcal{P}(S) \times \mathcal{I}(S)$ has to be a partial surjection. A careful analysis shows that, when $x \Vdash u$, we must have $u = \downarrow x$ (the downward closure of x in S), so that, for defining the partial surjection $\Vdash$, we only need to know its domain D. So an object of our category will be a pair (S, D) where $D \subseteq \mathcal{P}(S)$. What condition should D satisfy? As usual, it should be equal to its double dual for a suitable notion of duality: here, we say that $x, x' \subseteq S$ are dual if $x' \cap \downarrow x \neq \emptyset \Rightarrow x' \cap x \neq \emptyset$, that is x' cannot separate x from its downward closure. We show that these objects (called "preorders with projections"), with suitable linear morphisms, form a model of linear logic $\mathbf{PpL}$, whose associated Kleisli category $\mathbf{PpL}_!$ can be considered as a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, of which all objects are modest. And we show that $\mathbf{ScottL}_!$ represents the extensional collapse of $\mathbf{Rel}_!$ in the sense explained above. We actually exhibit a functor from $\mathbf{PpL}$ to $\mathbf{PerL}$ which preserves the structure of LL model and which induces the required CCC functor from $\mathbf{PpL}_!$ to $\mathbf{PerL}_!$.

In the course of these constructions, we also build models of the pure lambda-calculus, using notions of inclusions between the various structures we consider, organizing them into complete partially ordered classes, and using the fact that the logical constructions (tensor product, orthogonality etc) are continuous wrt. these inclusions. This provides a simple representation of the extensional collapse of the reflexive object in $\mathbf{Rel}_!$ we introduced in [BEM07], as a reflexive object in the CCC of complete lattices and continuous maps.

1 Preliminaries

1.1 Notations

A finite multiset p of elements of S is a map $p : S \to \mathbb{N}$ such that $p(a) = 0$ for almost all $a \in S$. We write $a \in p$ for $p(a) > 0$, and use $\mathsf{supp}(p)$ for the support of p which is the set $\{a \in S \mid a \in p\}$. We use $p + q$ for the pointwise sum of multisets, and 0 for the empty multiset. We denote by $\mathcal{M}_{\text{fin}}(S)$ the set of all finite multisets of elements of S.

Given a category $\mathcal{C}$ and two morphisms $f \in \mathcal{E}(E, F)$ and $x \in \mathcal{C}(\top, E)$ (where $\top$ is the terminal object of $\mathcal{C}$ that we assume to exist), we write $f(x)$ instead of $f \circ x$ because we consider x as a "point" (an "element") of E.

1.2 Cartesian closed categories and models of the pure lambda-calculus

We briefly recall that a category $\mathcal{C}$ is a CCC if each finite family $(E_i)_{i\in I}$ of objects of $\mathcal{C}$ has a cartesian product $\&_{i\in I} E_i$ (in particular, it has a terminal object $\top$) together with projections $\pi_j \in \mathcal{C}(\&_{i\in I} E_i, E_j)$ such that, for any family $(f_i)_{i\in I}$ with $f_i \in \mathcal{C}(F, E_i)$ there is an unique morphism $\langle f_i\rangle_{i\in I} \in \mathcal{C}(F, \&_{i\in I} E_i)$ such that $\pi_j \circ \langle f_i\rangle_{i\in I} = f_j$ for each j and if, given two objects E and F of $\mathcal{C}$, there is a pair $(E \Rightarrow F, \mathsf{Ev})$, called the *object of morphisms from E to F*, together with an evaluation morphism $\mathsf{Ev} \in \mathcal{C}((E \Rightarrow F) \mathbin{\&} E, F)$ such that, for any $f \in \mathcal{C}(G \mathbin{\&} E, F)$, there is an unique $\mathsf{Cur}(f) \in \mathcal{C}(G, E \Rightarrow F)$ such that $\mathsf{Ev} \circ (\mathsf{Cur}(f) \mathbin{\&} \mathsf{Id}_E) = f$.

Given two CCCs $\mathcal{C}$ and $\mathcal{D}$, a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ will be said to be a *cartesian closed functor* if it preserves the cartesian closed structure on the nose. This means that $\mathcal{F}(\&_{i\in I} E_i) = \&_{i\in I} \mathcal{F}(E_i)$, $\mathcal{F}(\pi_i) = \pi_i$, $\mathcal{F}(E \Rightarrow F) = \mathcal{F}(E) \Rightarrow \mathcal{F}(F)$ and $\mathcal{F}(\mathsf{Ev}) = \mathsf{Ev}$.

A *reflexive object* in a CCC $\mathcal{C}$ is a triple $(H, \mathsf{app}, \mathsf{lam})$ where H is an object of $\mathcal{C}$, $\mathsf{app} \in \mathcal{C}(H, H \Rightarrow H)$ and $\mathsf{lam} \in \mathcal{C}(H \Rightarrow H, H)$ satisfy $\mathsf{app} \circ \mathsf{lam} = \mathsf{Id}_{H\Rightarrow H}$. One says moreover that $(H, \mathsf{app}, \mathsf{lam})$ is *extensional*[4] if $\mathsf{lam} \circ \mathsf{app} = \mathsf{Id}_H$. If $(H, \mathsf{app}, \mathsf{lam})$ is a reflexive object in $\mathcal{C}$ and if $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a CCC functor, then $(\mathcal{F}(H), \mathcal{F}(\mathsf{app}), \mathcal{F}(\mathsf{lam}))$ is a reflexive object in $\mathcal{D}$, which is extensional if $(H, \mathsf{app}, \mathsf{lam})$ is extensional.

Let $(H, \mathsf{app}, \mathsf{lam})$ be a reflexive object in the CCC $\mathcal{C}$. Then, given any lambda-term M and any repetition-free list of variables $\vec{x} = x_1, \dots, x_n$ which contains all the free variables of M (such a list will be said to be *adapted* to

[4]This notion of extensionality, which corresponds to the η conversion rule of the lambda-calculus, should not be confused with the notion of extensionality we are dealing with in this paper, which is related to the categorical notion of well-pointedness.

M), one defines $[M]^H_{\vec{x}} \in \mathcal{C}(H^n, H)$ by induction on M ($[x_i]^H_{\vec{x}} = \pi_i$, $[\lambda x\, N]^H_{\vec{x}} = \mathsf{lam} \circ \mathsf{Cur}([N]^H_{\vec{x},x})$ and $[(N)\, P]^H_{\vec{x}} = \mathsf{Ev} \circ \langle \mathsf{app} \circ [N]^H_{\vec{x}}, [P]^H_{\vec{x}} \rangle$). If M and M' are β-equivalent and $\vec{x}$ is adapted to M and M', we have $[M]^H_{\vec{x}} = [M']^H_{\vec{x}}$. If $(H, \mathsf{app}, \mathsf{lam})$ is extensional, we have $[M]^H_{\vec{x}} = [M']^H_{\vec{x}}$ when M and M' are $\beta\eta$-equivalent.

If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a CCC functor then, for any lambda-term M, we have $\mathcal{F}([M]^H_{\vec{x}}) = [M]^{\mathcal{F}(H)}_{\vec{x}}$ where $[M]^{\mathcal{F}(H)}_{\vec{x}}$ is the interpretation of M in the reflexive object $(\mathcal{F}(H), \mathcal{F}(\mathsf{app}), \mathcal{F}(\mathsf{lam}))$.

1.3 Seely categories and LL-functors

We introduce now the notion of categorical model of LL that we'll use in this paper. There are several ways to axiomatize such categories, and for a complete description of these notions, and comparisons between them, we refer to [Mel09]. From that paper, we use the notion of *Seely category* originally called "new-Seely category" in [Bie95].

1.3.1 $*$-autonomous categories. A monoidal category is a category $\mathcal{C}$ (where we denote the composition of morphisms by simple juxtaposition: if $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, then $g\, f \in \mathcal{C}(X, Z)$) together with a bifunctor $\otimes : \mathcal{C}^2 \to \mathcal{C}$, an object $1 \in \mathcal{C}$ and natural isomorphisms $\lambda_X : 1 \otimes X \to X$, $\rho_X : X \otimes 1 \to X$, $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and these isomorphisms are required to satisfy coherence commutative diagrams that we do not recall here (see [Mac71]).

A symmetric monoidal category is a monoidal category together with a natural isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ such that $\sigma_{Y,X}\, \sigma_{X,Y} = \mathsf{Id}_{X \otimes Y}$ which has also to satisfy other commutations (again, see [Mac71]).

A symmetric monoidal closed category (SMCC for short) is a symmetric monoidal category $\mathcal{C}$ such that, for each object X, the functor $Y \mapsto X \otimes Y$ has a right adjoint $Y \mapsto (X \multimap Y)$. Let X, Y and Z be objects of $\mathcal{C}$, we have a linear evaluation morphism $\mathsf{ev} \in \mathcal{C}((X \multimap Z) \otimes X, Z)$, and, given a morphism $f \in \mathcal{C}(Y \otimes X, Z)$, we have a morphism $\lambda(f) \in \mathcal{C}(Y, X \multimap Z)$. Monoidal closeness boils down to the following three equations:

$$\begin{aligned} \mathsf{ev}\, (\lambda(f) \otimes \mathsf{Id}_X) &= f \\ \lambda(f)\, h &= \lambda(f\, (h \otimes \mathsf{Id}_X)) \quad \text{where } h \in \mathcal{C}(Y', Y) \\ \lambda(\mathsf{ev}) &= \mathsf{Id}_{X \multimap Z}\ . \end{aligned}$$

In particular, we have a morphism $\eta_X = \lambda(\mathsf{ev}\, \sigma) \in \mathcal{C}(X, (X \multimap Z) \multimap Z)$ which is natural in X.

Last, a $*$-autonomous category is an SMCC $\mathcal{C}$ together with an object $\perp$ such that the canonical natural morphism $\eta_X : X \to ((X \multimap \perp) \multimap \perp)$ is an isomorphism.

Therefore, in a $*$-autonomous category $\mathcal{C}$, there is a contravariant functor $X \mapsto X^\perp = (X \multimap \perp)$ which is actually an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}$. Given $f \in \mathcal{C}(X, Y)$, we denote as $f^\perp$ the associated morphism $Y^\perp \to X^\perp$. Through this isomorphism, we can define another symmetric monoidal category structure on $\mathcal{C}$ whose binary operation (the "co-tensor product" or *par*) is defined by $X \parr Y = (X^\perp \otimes Y^\perp)^\perp$ so that we have in particular $X \multimap Y = X^\perp \parr Y$ up to a natural isomorphism).

In a cartesian $*$-autonomous category $\mathcal{C}$, we denote a terminal object as $\top$ and a choice of cartesian product of the a finite family $(X_i)_{i\in I}$ of objects is denoted as $\&_{i\in I} X_i$, with projections $\pi_i \in \mathcal{C}(\&_{j\in I} X_j, X_i)$. Given a family $f_i \in \mathcal{C}(Y, X_i)$ of morphisms, the unique morphism $f \in \mathcal{C}(Y, \&_{i\in I} X_i)$ such that $\pi_i\, f = f_i$ for each $i \in I$ is denoted as $\langle f_i\rangle_{i\in I}$.

Then $\mathcal{C}$ is also cocartesian with initial object $0 = \top^\perp$ and cocartesian product (also known as direct sum) $\oplus$.

1.3.2 Seely categories. A *Seely category* consists of

- a cartesian $*$-autonomous category $\mathcal{C}$;
- a comonad $!\_ : \mathcal{C} \to \mathcal{C}$ which is monoidal from $(\mathcal{C}, \top, \&)$ to $(\mathcal{C}, \otimes, 1)$ (counit denoted as $\mathsf{d}_X : !X \to X$ and called *dereliction*, comultiplication denoted as $\mathsf{p}_X : !X \to !!X$ and called *digging*, monoidality isomorphisms $\mu_{X,Y} : !X \otimes !Y \to !(X \,\&\, Y)$, $\mu_1 : 1 \to !\top$, often called *Seely isomorphisms* though they were noticed first by Girard, see [Gir87]) such that the following diagram commutes (it expresses a coherence condition relating the isomorphism μ and the natural transformation p)

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "\mu_{X,Y}"] \arrow[dd, "\mathsf{p}_X \otimes \mathsf{p}_Y"] & !(X \,\&\, Y) \arrow[d, "\mathsf{p}_{X\&Y}"] \\
 & !!(X \,\&\, Y) \arrow[d, "!\langle !\pi_1, !\pi_2\rangle"] \\
!!X \otimes !!Y \arrow[r, "\mu_{!X,!Y}"] & !(!X \,\&\, !Y)
\end{tikzcd} \tag{1}$$

This monoidal structure induces a lax monoidal structure on the functor $!\_$ from the monoidal category $(\mathcal{C}, \otimes, 1)$ to itself: this monoidal structure consists of a morphism $\mathsf{m}_1 : 1 \to !1$ and of a natural transformation $\mathsf{m}_{X,Y} : !X \otimes !Y \to !(X \otimes Y)$ that we give now explicitly. We define m_1 as the following composition of morphisms:

$$1 \xrightarrow{\mu_1} !\top \xrightarrow{\mathsf{p}_\top} !!\top \xrightarrow{!(\mu_1{}^{-1})} !1$$

and $\mathsf{m}_{X,Y}$ as

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "\mu_{X,Y}"] & !(X \,\&\, Y) \arrow[r, "\mathsf{p}_{X\&Y}"] & !!(X \,\&\, Y) \arrow[r, "!(\mu_{X,Y}{}^{-1})"] & !(!X \otimes !Y) \arrow[d, "!(\mathsf{d}_X \otimes \mathsf{d}_Y)"] \\
 & & & !(X \otimes Y)
\end{tikzcd}$$

1.3.3 Associated Kleisli CCC. Let $\mathcal{C}$ be a Seely category (we use the notations above for the monoidal and exponential structures). The Kleisli category of the comonad $!\_$ — simply called *Kleisli category of* $\mathcal{C}$ in the sequel — is defined as follows: it is the category $\mathcal{C}_!$ whose objects are those of $\mathcal{C}$ and $\mathcal{C}_!(X, Y) = \mathcal{C}(!X, Y)$. The identity morphism is $\mathsf{d}_X \in \mathcal{C}_!(X, X)$. Given

$f \in \mathcal{C}_!(X,Y)$ and $g \in \mathcal{C}_!(Y,Z)$ is defined as the following composition of morphisms in $\mathcal{C}$:

$$!X \xrightarrow{\mathsf{p}_X} !!X \xrightarrow{!f} !Y \xrightarrow{g} Z$$

that we denote as $g \circ f$. In that way, one defines a category which is cartesian closed: the cartesian product of the family $(X_i)_{i\in I}$ is $X = \&_{i\in I} X_i$ with projections $\pi_i \, \mathsf{d}_X$ and tupling $\langle f_i\rangle_{i\in I} \in \mathcal{C}_!(Y,X)$ for a family of morphisms $f_i \in \mathcal{C}_!(Y,X_i) = \mathcal{C}(!Y,X_i)$. Then the object of morphisms from X to Y in $\mathcal{C}_!$ is $X \Rightarrow Y = !X \multimap Y$ with evaluation $\mathsf{Ev} \in \mathcal{C}_!((X \Rightarrow Y) \,\&\, X, Y)$ given as the following composition of morphisms in $\mathcal{C}$

$$\begin{tikzcd}
!((!X \multimap Y) \,\&\, X) \arrow[r, "\mu^{-1}"] & !(!X \multimap Y) \otimes !X \arrow[r, "\mathsf{d}_{!X\multimap Y}"] & (!X \multimap Y) \otimes !X \arrow[d, "\mathsf{ev}"] \\
& & Y
\end{tikzcd}$$

Given $f \in \mathcal{C}_!(Z \,\&\, X, Y)$, the "curryfied" morphism $\Lambda(f) \in \mathcal{C}_!(Z, X \Rightarrow Y)$ is simply $\Lambda(f) = \lambda(f \, \mu_{Z,X}{}^{-1})$.

1.3.4 LL-functors. Given two Seely categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is an *LL-functor* if it commutes on the nose with all the structures defined above, eg. $F(X \otimes_{\mathcal{C}} Y) = F(X) \otimes_{\mathcal{D}} F(Y)$, $F(\mathsf{d}^{\mathcal{C}}_X) = \mathsf{d}^{\mathcal{D}}_X$ etc. Then one has $F([A]^{\mathcal{C}}_I) = [A]^{\mathcal{D}}_{F\circ I}$ and $F([\pi]^{\mathcal{C}}_I) = [\pi]^{\mathcal{D}}_{F\circ I}$ for all formula A and proof π of LL, where I is a valuation from type atoms to objects of $\mathcal{C}$.

Such an LL-functor F induces a cartesian closed functor (still denoted with F) from $\mathcal{C}_!$ to $\mathcal{D}_!$.

1.4 Intuitionistic extensional collapse

We present a categorical version of the extensional collapse of a model of the typed lambda-calculus which is based on [Buc97].

From the usual intuitionistic viewpoint, the extensional collapse is a *logical relation*. More specifically, consider the hierarchy of simple types based on some type atoms α, $\beta\ldots$, and intuitionistic implication $\Rightarrow$. Consider a cartesian closed category $\mathcal{C}$ (with terminal object $\top$, cartesian product $\&$ and function space $\Rightarrow$). Given a valuation I from type atoms to objects of $\mathcal{C}$, we have an interpretation of types $[A]_I \in \mathcal{C}$. The *extensional collapse* of this interpretation is a type-indexed family of partial equivalence relations $(\sim_A)$, where $\sim_A \subseteq \mathcal{C}(\top, [A]_I)^2$. This relation is defined by induction on types.

- At each basic type α, the relation $\sim_\alpha$ coincides with equality on $\mathcal{C}(\top, I(\alpha))$.
- Then, given $f, g \in \mathcal{C}(\top, [A \Rightarrow B]_I) = \mathcal{C}(\top, [A]_I \Rightarrow [B]_I) \simeq \mathcal{C}([A]_I, [B]_I)$, one has $f \sim_{A\Rightarrow B} g$ if, for all $x, y \in \mathcal{C}(\top, [A]_I)$ such that $x \sim_A y$, one has $f(x) \sim_B g(y)$ (where we recall that we write $f(x)$ instead of $f \circ x$ when the source of x is the terminal object).

By induction on types, one proves easily that $\sim_A$ is a PER on $\mathcal{C}(\top, [A]_I)$ for each type A. Since the family of PERs $(\sim_A)$ is defined as a logical relation, it is compatible with the syntax of the simply typed lambda-calculus, in the sense that, if M is a closed term of type A, its semantics $[M]_I \in \mathcal{C}(\top, [A]_I)$ satisfies $[M]_I \sim_A [M]_I$. This property can be extended to *functional* enriched versions of the simply typed lambda-calculus (such as PCF) under some mild assumptions on $\mathcal{C}$ and I.

1.4.1 Representing the collapse as an interpretation. Let $\mathcal{E}$ be another cartesian closed category, that we assume to be well-pointed (meaning that, if $\varphi, \psi \in \mathcal{E}(E, F)$ satisfy $\varphi(\zeta) = \psi(\zeta)$ for all $\zeta \in \mathcal{E}(\top, E)$, then $\varphi = \psi$). Let J be a valuation of type atoms in $\mathcal{E}$ and, for each type atom α, let $\Vdash_\alpha \subseteq \mathcal{C}(\top, I(\alpha)) \times \mathcal{E}(\top, J(\alpha))$ be a *bijection* (to be understood as expressing an equality relation between the elements of the two models at ground types). Then we define $\Vdash_A \subseteq \mathcal{C}(\top, [A]_I) \times \mathcal{E}(\top, [A]_J)$ for all types A as a logical relation (called the *heterogeneous relation*), that is

$$f \Vdash_{A \Rightarrow B} \psi \Leftrightarrow (\forall x, \zeta \;\; x \Vdash_A \zeta \Rightarrow f(x) \Vdash_B \varphi(\zeta)) \,.$$

If $\Vdash_A$ is surjective for all types A (that is $\forall \zeta \in \mathcal{E}(\top, [A]_J)\, \exists x \in \mathcal{C}(\top, [A]_I)\; x \Vdash_A \zeta$), then all the relations $\Vdash_A$ are functional (in the sense that if $x \Vdash_A \zeta$ and $x \Vdash_A \zeta'$, then $\zeta = \zeta'$). This is easy to check by induction on types and is due to the well-pointedness of $\mathcal{E}$.

We say that $\Vdash_A$ is a *representation of the collapse* of the interpretation I by the interpretation J if, for all types A, $\Vdash_A$ is surjective (and bijective when $A = \alpha$ is a basic type) and one has

$$\forall x, y \in \mathcal{C}(\top, [A]_I) \quad x \sim_A y \Leftrightarrow (\exists \zeta \in \mathcal{E}(\top, [A]_J)\; x \Vdash_A \zeta \text{ and } y \Vdash_A \zeta) \,.$$

This means that, at each type A, the relation $\Vdash_A$ induces a bijection between $\mathcal{E}(\top, [A]_J)$ and the quotient[5] $\mathcal{C}(\top, [A]_I)/{\sim_A}$.

Assume that $\Vdash_A$ is such a representation. Since it is defined as a logical relation, we have $[M]_I \Vdash_A [M]_J$ for each closed lambda-term of type A, we have $[M]_I \sim_A [N]_I$ iff $[M]_J = [N]_J$ for all closed terms M and N of type A.

1.4.2 Categorical presentation. There is another, more conceptual way to describe the situation above. We prefer this approach because it consists in building new CCCs using $\mathcal{C}$ and $\mathcal{E}$. Indeed, one of the main goals of this paper is to show that these CCCs can be defined in another and much more informative way: we prove that they arise as Kleisli categories of categorical models of linear logic (see Section 1.3).

First one defines the *collapse category* $\mathsf{e}(\mathcal{C})$ of $\mathcal{C}$. Its objects are pairs $U = (\ulcorner U \urcorner, \sim_U)$ where $\ulcorner U \urcorner$ is an object of $\mathcal{C}$ and $\sim_U \subseteq \mathcal{C}(\top, \ulcorner U \urcorner)^2$ is a PER. Given two objects U and V of $\mathsf{e}(\mathcal{C})$, the elements of $\mathsf{e}(\mathcal{C})(U, V)$ are the morphisms $f \in \mathcal{C}(\ulcorner U \urcorner, \ulcorner V \urcorner)$ such that

$$\forall x, x' \in \mathcal{C}(\top, \ulcorner U \urcorner) \quad x \sim_U x' \Rightarrow f(x) \sim_V f(x') \,.$$

If the category $\mathcal{C}$ is cartesian, then so is $\mathsf{e}(\mathcal{C})$ (with cartesian products defined in the most obvious way). And if $\mathcal{C}$ is cartesian closed, so is $\mathsf{e}(\mathcal{C})$. Given two objects U and V of $\mathcal{C}$, one defines $U \Rightarrow V = (\ulcorner U \urcorner \Rightarrow \ulcorner V \urcorner, \sim_{U \Rightarrow V})$ with $f \sim_{U \Rightarrow V} f'$ iff $f(x) \sim_Y f'(x')$ for all $x, x' \in \mathcal{C}(\top, \ulcorner U \urcorner)$ such that $x \sim_U x'$

[5] When quotienting a set by a PER, one considers only the elements of the set which are equivalent to themselves.

(for $f, f' \in \mathcal{C}(\top, \ulcorner U \Rightarrow V \urcorner) \simeq \mathcal{C}(\ulcorner U \urcorner, \ulcorner V \urcorner)$). The evaluation morphism $\mathsf{Ev} \in \mathsf{e}(\mathcal{C})((U \Rightarrow V)\ \&\ U, V)$ is the evaluation morphism of the category $\mathcal{C}$, which is also a morphism in $\mathsf{e}(\mathcal{C})$. We say that an object U of $\mathsf{e}(\mathcal{C})$ is *discrete* if $\sim_U$ coincides with equality.

Similarly, one defines the *heterogeneous category* $\mathsf{e}(\mathcal{C}, \mathcal{E})$ of $\mathcal{C}$ and $\mathcal{E}$. Its objects are triples $X = (\ulcorner X \urcorner, \llcorner X \lrcorner, \Vdash_X)$ where $\ulcorner X \urcorner$ is an object of $\mathcal{C}$, $\llcorner X \lrcorner$ is an object of $\mathcal{E}$ and $\Vdash_X \subseteq \mathcal{C}(\top, \ulcorner X \urcorner) \times \mathcal{E}(\top, \llcorner X \lrcorner)$. A morphism θ from X to Y in that category is a pair $(\ulcorner \theta \urcorner, \llcorner \theta \lrcorner)$ where $\ulcorner \theta \urcorner \in \mathcal{C}(\ulcorner X \urcorner, \ulcorner Y \urcorner)$ and $\llcorner \theta \lrcorner \in \mathcal{E}(\llcorner X \lrcorner, \llcorner Y \lrcorner)$ satisfy $\ulcorner \theta \urcorner(x) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$ for all (x, ζ) such that $x \Vdash_X \zeta$.

Again, if both categories $\mathcal{C}$ and $\mathcal{E}$ are cartesian, so is $\mathsf{e}(\mathcal{C}, \mathcal{E})$, and if they are cartesian closed, so is $\mathsf{e}(\mathcal{C}, \mathcal{E})$, with $X \Rightarrow Y$ defined as follows: $\ulcorner X \Rightarrow Y \urcorner = \ulcorner X \urcorner \Rightarrow \ulcorner Y \urcorner$, $\llcorner X \Rightarrow Y \lrcorner = \llcorner X \lrcorner \Rightarrow \llcorner Y \lrcorner$ and, given $f \in \mathcal{C}(\top, \ulcorner X \Rightarrow Y \urcorner) \simeq \mathcal{C}(\ulcorner X \urcorner, \ulcorner Y \urcorner)$ and $\varphi \in \mathcal{E}(\top, \llcorner X \Rightarrow Y \lrcorner) \simeq \mathcal{C}(\llcorner X \lrcorner, \llcorner Y \lrcorner)$, we have $f \Vdash_{X \Rightarrow Y} \varphi$ if $f(x) \Vdash_Y \varphi(\zeta)$ for all (x, ζ) such that $x \Vdash_X \zeta$.

Let us say that an object X of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ is *modest*[6] if the relation $\Vdash_X$ is a *partial surjection* from $\mathcal{C}(\top, \ulcorner X \urcorner)$ to $\mathcal{E}(\top, \llcorner X \lrcorner)$. Let $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ be the full subcategory of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ whose objects are the modest objects. If $\mathcal{C}$ and $\mathcal{E}$ are cartesian, then $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ is a sub-cartesian category of $\mathsf{e}(\mathcal{C}, \mathcal{E})$. But in general, $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ is not cartesian closed. It can be noticed that, if X and Y are objects of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ which are modest (so that, again, $X \Rightarrow Y$ is well defined but not necessarily modest) and if $\Vdash_{X \Rightarrow Y}$ is surjective, then $\Vdash_{X \Rightarrow Y}$ is functional, and hence $X \Rightarrow Y$ is modest.

There is a cartesian closed "second projection" functor $\sigma : \mathsf{e}(\mathcal{C}, \mathcal{E}) \to \mathcal{E}$ (it maps an object X to $\llcorner X \lrcorner$ and a morphism θ to $\llcorner \theta \lrcorner$). There is also a functor $\varepsilon : \mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E}) \to \mathsf{e}(\mathcal{C})$ which maps an object X to $(\ulcorner X \urcorner, \sim_{\varepsilon(X)})$, where $x_1 \sim_{\varepsilon(X)} x_2$ if $x_1 \Vdash_X \zeta$ and $x_2 \Vdash_X \zeta$ for some (necessarily unique) ζ. Given $\theta \in \mathsf{e}(\mathcal{C}, \mathcal{E})(X, Y)$, we set $\varepsilon(\theta) = \ulcorner \theta \urcorner$. Indeed, let $x_1, x_2 \in \mathcal{C}(\top, \ulcorner X \urcorner)$ such that $x_1 \sim_{\varepsilon(X)} x_2$ (with $\zeta \in \mathcal{E}(\top, \llcorner X \lrcorner)$ such that $x_1 \Vdash_X \zeta$ and $x_2 \Vdash_X \zeta$), we have $\ulcorner \theta \urcorner(x_1) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$ and $\ulcorner \theta \urcorner(x_2) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$, and hence $\ulcorner \theta \urcorner(x_1) \sim_Y \ulcorner \theta \urcorner(x_2)$, so that $\ulcorner \theta \urcorner \in \mathsf{e}(\mathcal{C})(\varepsilon(X), \varepsilon(Y))$.

We say that the category $\mathcal{E}$ *represents the extensional collapse of the category* $\mathcal{C}$ if there exists a sub-CCC $\mathcal{H}$ of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ such that

- each object of $\mathcal{H}$ is modest;
- the functor $\varepsilon : \mathcal{H} \to \mathsf{e}(\mathcal{C})$ is cartesian closed
- and, for any[7] *discrete* object U of $\mathsf{e}(\mathcal{C})$, there is an object X of $\mathcal{H}$ such that $\varepsilon(X) = U$ (so that $\ulcorner X \urcorner = U$ and $\Vdash_X$ is a bijection).

1.4.3 Connection between the two definitions.

The motivation for this definition is that, in that situation, if I is a type valuation in $\mathcal{C}$ then, for each ground type α, we can find an object $J(\alpha)$ of $\mathcal{E}$ such that $K(\alpha) = (I(\alpha), J(\alpha), \Vdash_\alpha)$ is an object of $\mathcal{H}$, for some bijection $\Vdash_{K(\alpha)}$. We can extend K into an interpretation of types $[A]_K$ in the CCC $\mathcal{H}$ which satisfies $[A]_K = ([A]_I, [A]_J, \Vdash_A)$ where $\Vdash_A$ coincides with the heterogeneous logical relation defined in Section 1.4.1. Then our assumption that $\mathcal{E}$ represents the

[6]This is compatible with the standard terminology of realizability, see e.g. [AC98].

[7]We actually don't need this property for all discrete Us, but only for those which are intended to represent the basic types of the functional language we have in mind. For the sake of simplicity, we adopt this stronger hypothesis.

extensional collapse of $\mathcal{C}$ implies that $\Vdash_A$ is a representation of the extensional collapse of I by J, in the sense of Section 1.4.1.

The benefit of this abstraction is that the concept of a CCC $\mathcal{E}$ representing the extensional collapse of a CCC $\mathcal{C}$ is quite flexible and independent of any type hierarchy given *a priori*. For instance, it provides a natural definition of the extensional collapse of a model of the pure lambda-calculus.

1.4.4 Extensional collapse of a reflexive object. Assume indeed that $\mathcal{E}$ represents the extensional collapse of $\mathcal{C}$ in the sense above, with $\mathcal{H}$ as heterogeneous collapse CCC. Let $(Z, \mathsf{app}, \mathsf{lam})$ be a reflexive object in $\mathcal{H}$. Then $(\varepsilon(Z), \ulcorner\mathsf{app}\urcorner, \ulcorner\mathsf{lam}\urcorner)$ is a reflexive object in $\mathsf{e}(\mathcal{C})$, $(\ulcorner Z\urcorner, \ulcorner\mathsf{app}\urcorner, \ulcorner\mathsf{lam}\urcorner)$ is a reflexive object in $\mathcal{C}$ and $(\llcorner Z\lrcorner, \llcorner\mathsf{app}\lrcorner, \llcorner\mathsf{lam}\lrcorner)$ is a reflexive object in $\mathcal{E}$.

In that case, we say that the reflexive object $(\llcorner Z\lrcorner, \llcorner\mathsf{app}\lrcorner, \llcorner\mathsf{lam}\lrcorner)$ is a representation of the extensional collapse of the reflexive object $(\ulcorner Z\urcorner, \ulcorner\mathsf{app}\urcorner, \ulcorner\mathsf{lam}\urcorner)$.

Remark: The precise logical meaning of this definition is not completely clear yet since logical relations are defined by induction on types whereas here we are in an untyped setting. In this paper, we'll give a representation of the extensional collapse of the relational model of the lambda-calculus introduced in [BEM07] (in the sense above), and these two models will clearly be quite different. However, both models induce the same equational theory on lambda-terms (namely, the theory $\mathcal{H}^*$, according to which two terms M and M' are equivalent if, for any context C, the term $C[M]$ has a head normal form iff the term $C[M']$ has a head normal form). With the notations above, this means that, when restricted to the interpretations of lambda-terms, the relation $\sim_Z$ is just equality. Extending for instance the lambda-calculus with a *parallel composition* construction based on the *mix* rule of Linear Logic as in [DK00, BEM09], the situation becomes more interesting and the theories induced by the two models on the language are distinct.

2 The collapse partial equivalence relation

In this section, we first define the Seely category **Rel** of sets and relations which is a quite simple and canonical model of linear logic. Then we define a Seely category whose objects are sets equipped with a PER on their powersets (the *collapse category* of **Rel**) and prove that the associated Kleisli category is isomorphic to $\mathsf{e}(\mathbf{Rel}_!)$ (see Section 1.4.2).

2.1 The category of sets and relations

The Seely category **Rel** that we describe now underlies many well known models of linear logic (coherence spaces etc). As far as we know, it appears implicitly for the first time in [Gir88], and it is a typical piece of folklore of linear logic: it would be almost impossible to say who mentioned for the first time explicitly that it is a model of LL. We are almost sure that Girard was aware of that fact when he wrote [Gir87], and that he didn't mention it, considering it as too degenerate for deserving attention.

2.1.1 Linear structure. The category of sets and relations **Rel** has sets as objects, and, given two sets E and F, the set of morphisms from E to F

is $\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$. Composition is defined in the standard relational way: the composition of $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, G)$ is $t\,s \in \mathbf{Rel}(E, G)$. The identity morphism is the diagonal relation $\mathsf{Id} \in \mathbf{Rel}(E, E)$. This category has a quite simple monoidal structure: the tensor product is $E \otimes F = E \times F$ and the unit of the tensor is $1 = \{*\}$. This tensor product is a functor: given $s_i \in \mathbf{Rel}(E_i, F_i)$ for $i = 1, 2$, then $s_1 \otimes s_2 = \{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in s_i \text{ for } i = 1, 2\}$. Equipped with this tensor product, $\mathbf{Rel}$ is symmetric monoidal closed (the associativity, neutrality and symmetry isomorphisms are defined in the usual obvious way), with an object of linear morphisms $E \multimap F = E \times F$ and linear evaluation morphism $\mathsf{ev} \in \mathbf{Rel}((E \multimap F) \otimes E, F)$ given by $\mathsf{ev} = \{(((a, b), a), b) \mid a \in E \text{ and } b \in F\}$.

The symmetric monoidal closed category $\mathbf{Rel}$ is a $*$-autonomous category, with dualizing object $\perp = 1$, and the corresponding duality is trivial: $E^\perp = E$ and, given $s \in \mathbf{Rel}(E, F)$, the relation $s^\perp \in \mathbf{Rel}(F, E)$ is the "transpose" of s, that is $s^\perp = \{(b, a) \mid (a, b) \in s\}$. We have $E \parr F = E \multimap F = E \otimes F = E \times F$ in this model.

Remark: This category is a "degenerate model" of LL in the sense that it identifies $\otimes$ and $\parr$. We showed in [BE01] how this model can be enriched with various structures *without modifying the interpretation of proofs*, making $\otimes$ and $\parr$ non-isomorphic operations. This can be considered as one of the most striking features of LL: this logical system is so robust that it survives (in the sense that proofs are not trivialized) in such a degenerate framework.

Given $s \in \mathbf{Rel}(E, F)$ and $x \subseteq E$, one sets $s\,x = \{b \mid \exists a \in x \text{ and } (a, b) \in s\}$.

The category $\mathbf{Rel}$ is cartesian. The cartesian product of a family $(E_i)_{i \in I}$ of sets is $\&_{i \in I} E_i = \bigcup_{i \in I}(\{i\} \times E_i)$, with projections $\pi_j = \{((j, a), a) \mid a \in E_j\} \in \mathbf{Rel}(\&_{i \in I} E_i, E_j)$. Given a family of morphisms $s_i \in \mathbf{Rel}(F, E_i)$, the corresponding morphism $\langle s_i \rangle_{i \in I} \in \mathbf{Rel}(F, \&_{i \in I} E_i)$ is given by $\langle s_i \rangle_{i \in I} = \{(b, (i, a)) \mid i \in I \text{ and } (b, a) \in s_i\}$. The terminal object is $\top = \emptyset$.

The exponential comonad is $!E = \mathcal{M}_{\mathrm{fin}}(E)$, see in Section 1.1 our notations for finite multisets. The action of this functor on morphisms is defined as follows: $!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid (a_i, b_i) \in s \text{ for } i = 1, \dots, n\} \in \mathbf{Rel}(!E, !F)$ for $s \in \mathbf{Rel}(E, F)$. Dereliction is given by $\mathsf{d}_E = \{([a], a) \mid a \in S\} \in \mathbf{Rel}(!E, E)$ and digging by $\mathsf{p}_E = \{(m_1 + \cdots + m_n, [m_1, \dots, m_n]) \mid n \in \mathbb{N} \text{ and } m_1, \dots, m_n \in !E\} \in \mathbf{Rel}(!E, !!E)$. Given $x \subseteq E$, one defines $x^! = \mathcal{M}_{\mathrm{fin}}(x)$. Observe that, as usual, $!s\,x^! = (s\,x)^!$, $\mathsf{d}_E\,x^! = x$ and $\mathsf{p}_E\,x^! = x^{!!}$.

The Seely isomorphism $1 \simeq !\top$ identifies $*$ and $[]$, and the Seely isomorphism $!E \otimes !F \simeq !(E \& F)$ maps the element $([a_1, \dots, a_l], [b_1, \dots, b_r])$ of $!E \otimes !F$ to $[(1, a_1), \dots, (1, a_l), (2, b_1), \dots, (2, b_r)] \in !(E \& F)$.

All these data define a Seely category in the sense of Section 1.3.

2.1.2 The associated CCC.

Remember from Section 1.3.3 that the Kleisli category $\mathbf{Rel}_!$ is cartesian closed. Given a set E, a point of E in $\mathbf{Rel}_!$ is by definition a morphism in $\mathbf{Rel}(!\top, E)$, that is, a subset of E. The terminal object is $\top$, the cartesian product of $(E_i)_{i \in I}$ is $E = \&_{i \in I} E_i$, with projections $\pi_i \circ \mathsf{d}_E$ (still denoted as π_i). The object of morphisms $E \Rightarrow F$ is $!E \multimap F$, with evaluation map (keeping implicit the Seely isomorphism)

$$\mathsf{Ev} = \{((([(m, b)], m), b) \mid m \in !E \text{ and } b \in F\} \in \mathbf{Rel}(!(!E \multimap F) \otimes !E, F)\,.$$

Applying a morphism $s \in \mathbf{Rel}_!(E, F) = \mathbf{Rel}(!E, F)$ to a point $x \subseteq E$ consists in composing s with x (considered as a morphism from $\top$ to E) in $\mathbf{Rel}_!$; the result is

$$s(x) = s\,x^! = \{b \mid \exists m\ (m, b) \in s \text{ and } \mathsf{supp}(m) \subseteq x\}\,.$$

The category $\mathbf{Rel}_!$ is not well pointed, in the sense that two distinct morphisms $s_1, s_2 \in \mathbf{Rel}_!(E, F)$ can satisfy $\forall x \subseteq E\ s_1(x) = s_2(x)$; take for instance $s_1 = \{([a], b)\}$ and $s_2 = \{([a, a], b)\}$.

The purpose of the collapse PER is precisely to make it explicit when two such morphisms should be identified. This depends of course on the PERs E and F themselves are equipped with: the collapse PER is a *logical relation*. We'll present this construction as a new category.

2.1.3 Inclusions. Let E and F be two sets such that $E \subseteq F$. Then we denote by $\eta_{E,F}$ and $\rho_{E,F}$ the relations

$$\eta_{E,F} = (E \times F) \cap \mathsf{Id}_E \quad \text{and} \quad \rho_{E,F} = (F \times E) \cap \mathsf{Id}_E\ .$$

Observe that $\rho_{E,F} \circ \eta_{E,F} = \mathsf{Id}_E$.

We denote by $\mathbf{RelC}$ the class of all sets, ordered by inclusion. This is a partially ordered class, which is complete in the sense that any family $(E_\gamma)_{\gamma\in\Gamma}$ of elements of $\mathbf{RelC}$ admits a least upper bound. We'll consider actually only directed families (that is, where Γ is a directed poset, and $\gamma \leq \delta \Rightarrow E_\gamma \subseteq E_\delta$).

2.2 The linear collapse category

We equip now the objects of $\mathbf{Rel}$ with a partial equivalence relation whose purpose is to identify morphisms which yield equivalent values when applied to equivalent arguments. In that way, we define a new Seely category $\mathbf{PerL}$, and we'll see that its Kleisli CCC $\mathbf{PerL}_!$ is a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!)$, see Section 1.4.2.

2.2.1 Pre-PERs, PER objects and morphisms of PER objects. Let E be a set. Given a binary relation B on $\mathcal{P}(E)$, we define another binary relation $B^\perp$ on $\mathcal{P}(E)$, called the *dual* of B, as follows:

$$x'\ B^\perp\ y' \quad \text{if} \quad \forall x, y \in \mathcal{P}(E)\ x\ B\ y \Rightarrow (x \cap x' \neq \emptyset \Leftrightarrow y \cap y' \neq \emptyset)\,.$$

Consider $x \subseteq E$ as a datum of type E and $x' \subseteq E$ as an observation of type E, we can say that the observation x' succeeds on x if $x \cap x' \neq \emptyset$. Intuitively, $x\ B\ y$ means that the data x and y are observationally equivalent. So two observations $x', y' \subseteq E$ are equivalent (in the sense of B) when they simultaneously succeed or fail on equivalent data: this is exactly the definition of $x'\ B^\perp\ y'$.

As usual, one has $B \subseteq C \Rightarrow C^\perp \subseteq B^\perp$ and $B \subseteq B^{\perp\perp}$ (as subsets of $\mathcal{P}(E)^2$). We say that the relation B is a *pre-PER* if it is symmetric and satisfies $x\ B\ y \Rightarrow x\ B\ x$. Clearly, any PER is a pre-PER and if B is a pre-PER, then $B^\perp$ is a PER; it is of course for this reason that we introduce the notion of pre-PER.

A *PER-object* is a pair $U = (|U|, \sim_U)$, where $|U|$ is a set and $\sim_U$ is a binary relation on $\mathcal{P}(|U|)$ which is a pre-PER such that $\sim_U^{\perp\perp} = \sim_U$. This simply means that, given $x, y \subseteq |U|$, one has $x \sim_U y$ as soon as $x \cap x' \neq \emptyset \Leftrightarrow y \cap y' \neq \emptyset$, for all $x', y' \subseteq |U|$ such that $x' \sim_U^\perp y'$. By this condition, $\sim_U$ is automatically a

PER (indeed, $\sim_U$ is pre-PER, hence $\sim_U^\perp$ is a PER, and therefore $\sim_U = \sim_U^{\perp\perp}$ is a PER).

Let **PerL** be the category whose objects are the PER-objects, and where a morphism from U to V is a relation $t \subseteq |U| \times |V|$ such, for all $x, y \in \mathcal{P}(|X|)$, if $x \sim_X y$ then $t\,x \sim_Y t\,y$.

Remark: Let U be a PER-object and $\mathcal{A} \subseteq \mathcal{P}(|U|)$ such that $\forall x_1, x_2 \in \mathcal{A}\ x_1 \sim_U x_2$. Then $\forall x \in \mathcal{A}\ x \sim_C \bigcup \mathcal{A}$. Indeed, let $x'_1, x'_2 \subseteq |U|$ be such that $x'_1 \sim_{U^\perp} x'_2$. If $x \cap x'_1 \neq \emptyset$, then $x \cap x'_2 \neq \emptyset$ because $x \sim_U x$, and hence $\bigcup \mathcal{A} \cap x'_2 \neq \emptyset$. Conversely, if $\bigcup \mathcal{A} \cap x'_2 \neq \emptyset$, there is some $y \in \mathcal{A}$ such that $y \cap x'_2 \neq \emptyset$ and we conclude since $x \sim_U y$. So each equivalence class of $\sim_U$ has a maximal element, which is the union of all the elements of the class. These particular elements x of $\mathcal{P}(|U|)$ are characterized by the two following properties:

- $x \sim_U x$
- and $\forall y \in \mathcal{P}(|U|) \quad y \sim_U x \Rightarrow y \subseteq x$.

Lemma 1 *Let U be a PER-object and let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be families of elements of $\mathcal{P}(|U|)$ be such that $x_i \sim_U y_i$ for each $i \in I$. Then $\bigcup_{i \in I} x_i \sim_U \bigcup_{i \in I} y_i$.*

The proof is straightforward. In particular $\emptyset \sim_U \emptyset$, for any PER-object U.

2.2.2 Orthogonality and strong isomorphisms.

We define the PER-object $U^\perp$ by $|U^\perp| = |U|$ and $\sim_{U^\perp} = \sim_U^\perp$, so that $U^{\perp\perp} = U$.

Lemma 2 *Given two PER-objects U and V, any bijection $\theta : |U| \to |V|$ such that, for all $x, y \in \mathcal{P}(|X|)$, one has $x \sim_U y$ iff $\theta(x) \sim_V \theta(y)$ is an isomorphism from U to V in* **PerL**.

Such a bijection will be called a *strong isomorphism* from U to V.

The proof of the lemma is straightforward verification. Of course, θ^{-1} is a strong isomorphism from V to U.

Observe that any strong isomorphism θ from U to V is also a strong isomorphism from $U^\perp$ to $V^\perp$. Indeed, let $x'_1, x'_2 \subseteq |U|$. Assume first that $x'_1 \sim_{U^\perp} x'_2$ and let us show that $\theta(x'_1) \sim_{V^\perp} \theta(x'_2)$. So let $y_1, y_2 \subseteq |V|$ be such that $y_1 \sim_V y_2$. We have $\theta(x'_1) \cap y_1 \neq \emptyset \Leftrightarrow x'_1 \cap \theta^{-1}(y_1) \neq \emptyset$ and we conclude since θ^{-1} is a strong isomorphism from V to U. The converse implication $\theta(x'_1) \sim_{V^\perp} \theta(x'_2) \Rightarrow x'_1 \sim_{U^\perp} x'_2$ is proven similarly.

2.2.3 Monoidal structure.

We define $U \otimes V$ as follows. We take $|U \otimes V| = |U| \times |V|$, and $\sim_{U \otimes V} = E^{\perp\perp}$ where

$$E = \{(x_1 \times y_1, x_2 \times y_2) \mid x_1 \sim_U x_2 \text{ and } y_1 \sim_U y_2\} \subseteq \mathcal{P}(|U \otimes V|)^2 .$$

Since this relation E is a pre-PER (but not a PER *a priori*, since one cannot recover x and y from $x \times y$ when one of these two sets is empty), the relation $\sim_{U \otimes V}$ is a PER, and $U \otimes V$ so defined is a PER-object. We define $U \multimap V = (U \otimes V^\perp)^\perp$.

Remember that, if t is a binary relation, then $t^\perp = \{(b, a) \mid (a, b) \in t\}$.

Lemma 3 *One has $|U \multimap V| = |U| \times |V|$. If $t_1, t_2 \in \mathcal{P}(|U \multimap V|)$, one has $t_1 \sim_{U \multimap V} t_2$ iff for all $x_1, x_2 \subseteq |U|$ such that $x_1 \sim_U x_2$, one has $t_1\, x_1 \sim_Y t_2\, x_2$. Moreover, one has $t_1 \sim_{U \multimap V} t_2 \Leftrightarrow t_1^\perp \sim_{V^\perp \multimap U^\perp} t_2^\perp$.*

Proof. This is due to the fact that, for any $t \subseteq |U \multimap V|$, $x \subseteq |U|$ and $y' \subseteq |V|$, one has $t \cap (x \times y') \neq \emptyset \Leftrightarrow (t\, x) \cap y' \neq \emptyset$ □

So the morphisms from U to V are exactly the $t \in \mathcal{P}(|U \multimap V|)$ such that $t \sim_{U \multimap V} t$. Moreover, if $t \in \mathbf{PerL}(U, V)$ then $t^\perp \in \mathbf{PerL}(V^\perp, U^\perp)$.

Lemma 4 *The obvious bijection λ from $|U \otimes V \multimap W|$ to $|U \multimap (V \multimap W)|$ defines a strong isomorphism between the PER-objects $U \otimes V \multimap W$ and $U \multimap (V \multimap W)$. In particular, for $s_1, s_2 \in \mathcal{P}(|U \otimes V \multimap W|)$, one has $s_1 \sim_{U \otimes V \multimap W} s_2$ iff for any $x_1, x_2 \in \mathcal{P}(|U|)$ and $y_1, y_2 \in \mathcal{P}(|V|)$ such that $x_1 \sim_U x_2$ and $y_1 \sim_U y_2$, one has $s_1\,(x_1 \times y_1) \sim_W s_2\,(x_2 \times y_2)$.*

Proof. Let $t_1, t_2 \subseteq \mathcal{P}(U \otimes V \multimap W)$. Assume first that $t_1 \sim_{U \otimes V \multimap W} t_2$, we want to prove that $\lambda(t_1) \sim_{U \multimap (V \multimap W)} \lambda(t_2)$. But this is clear since, if $x_1, x_2 \subseteq |U|$ and $y_1, y_2 \subseteq |V|$ satisfy $x_1 \sim_U x_2$ and $y_1 \sim_V y_2$, then we have $x_1 \times y_2 \sim_{U \otimes V} x_2 \times y_2$, and therefore $(\lambda(t_1)\, x_1)\, y_1 = t_1\,(x_1 \times y_1) \sim_W t_2\,(x_2 \times y_2) = (\lambda(t_2)\, x_2)\, y_2$. Assume conversely that $\lambda(t_1) \sim_{U \multimap (V \multimap W)} \lambda(t_2)$, we prove that $t_1 \sim_{U \otimes V \multimap W} t_2$. For this, we proceed as above, showing that $t_1^\perp \sim_{W^\perp \multimap (U \otimes V)^\perp} t_2^\perp$ and applying Lemma 3. □

Lemma 5 *The obvious bijection $\alpha : |(U \otimes V) \otimes W| \to |U \otimes (V \otimes W)|$ is an isomorphism of PER-objects from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$.*

Proof. By Section 2.2.2, it suffices to prove that α is an isomorphism from $((U \otimes V) \otimes W)^\perp$ to $(U \otimes (V \otimes W))^\perp$, and this results from Lemma 4. □

Given $s \in \mathbf{PerL}(U_1, U_2)$ and $t \in \mathbf{PerL}(V_1, V_2)$, one defines $s \otimes t \subseteq |U_1 \otimes V_1| \times |U_2 \otimes V_2|$ by $s \otimes t = \{((a_1, b_1), (a_2, b_2)) \mid (a_1, a_2) \in s \text{ and } (b_1, b_2) \in t\}$. Then one shows using Lemma 4 that $s \otimes t \in \mathbf{PerL}(U_1 \otimes V_1, U_2 \otimes V_2)$, and one checks that the category $\mathbf{PerL}$ equipped with this $\otimes$ binary functor, together with the associativity isomorphism of Lemma 5 (as well as the symmetry isomorphism etc.) is a symmetric monoidal category, which is closed (with $U \multimap V$ as object of linear morphisms from U to V) by Lemma 4. The linear evaluation morphism is ev, as defined in Section 2.1.

$\mathbf{PerL}$ is $*$-autonomous, with $\perp = (\{*\}, =)$ as dualizing object.

2.2.4 Additive structure. Given a family $(U_i)_{i \in I}$ of PER-objects, one defines $U = \&_{i \in I} U_i$ by setting $|U| = \bigcup_{i \in I}(\{i\} \times |U_i|)$, and by saying that, for any $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in \mathcal{P}(|U|)$ (identifying this latter set with a product), one has $x \sim_U y$ if one has $x_i \sim_{U_i} y_i$ for all $i \in I$. Using the fact that $\emptyset \sim_V \emptyset$ in any PER-object V, one shows that $\sim_U^\perp = \sim_{\&_{i \in I} U_i^\perp}$ and it follows that U is a PER-object. It is routine to check that $\&_{i \in I} U_i$ so defined is the cartesian product of the U_is in the category $\mathbf{PerL}$, and that this cartesian product is also a coproduct. When U is a PER-object and I is a set, we denote with U^I the product $\&_{i \in I} U_i$ where $U_i = U$ for each U.

In particular, $\mathbf{PerL}$ has a terminal object $\top$, given by $|\top| = \emptyset$ and $\emptyset \sim_\top \emptyset$. Observe that this is the only PER-object with an empty web.

2.2.5 Exponentials. Given a PER-object U, we define $!U$ by $|!U| = \mathcal{M}_{\mathrm{fin}}(|U|)$, and $\sim_{!U} = E^{\perp\perp}$ where

$$E = \{(x_1^!, x_2^!) \mid x_1, x_2 \in \mathcal{P}(|U|)\ x_1 \sim_U x_2\}$$

where we recall that $x^! = \mathcal{M}_{\mathrm{fin}}(x)$. Since E is a pre-PER (and actually a PER, because x can be recovered from $x^!$ using *dereliction*: $x = \{a \mid [a] \in x^!\}$), the relation $\sim_{!U}$ is a PER. We recall that, if $s \subseteq |!U \multimap V|$ and $x \subseteq |U|$, then we denote with $s(x)$ the subset $s\,x^!$ of $|Y|$, see Section 2.1.

Lemma 6 *Let U and V be PER-objects and let $s_1, s_2 \subseteq |!U \multimap V|$. One has $s_1 \sim_{!U \multimap V} s_2$ iff*

$$\forall x_1, x_2 \subseteq |U| \quad x_1 \sim_U x_2 \Rightarrow s_1(x_1) \sim_V s_2(x_2)\,.$$

Proof. The $\Rightarrow$ direction is trivial. For the converse, one assumes that the stated condition holds, and one checks that $s_1^\perp \sim_{V^\perp \multimap (!U)^\perp} s_2^\perp$, and for this purpose, it suffices to apply Lemma 3. □

Given $s \in \mathbf{PerL}(U, V)$, one defines $!s \subseteq |!U| \times |!V|$ as in the standard relational model by setting

$$!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N},\ (a_i, b_i) \in s \text{ for } i = 1, \dots, n\}\,.$$

Then, since $!s\,x^! = (s\,x)^!$, we have $!s_1 \sim_{!U \multimap !V} !s_2$ as soon as $s_1 \sim_{U \multimap V} s_2$ (by Lemma 6); in particular, if $s \in \mathbf{PerL}(U, V)$, one has $!s \in \mathbf{PerL}(!U, !V)$ and so the operation $s \mapsto !s$ is an endofunctor on $\mathbf{PerL}$.

One defines $\mathsf{d}_U \subseteq |!U| \times |U|$ as $\mathsf{d}_U = \{([a], a) \mid a \in |U|\}$, and since $\mathsf{d}_U\, x^! = x$ for all $x \subseteq |U|$, we get easily $\mathsf{d}_U \in \mathbf{PerL}(!U, U)$. Similarly, one defines $\mathsf{p}_U \subseteq |!U| \times |!!U|$ as $\mathsf{p}_U = \{(m_1 + \cdots + m_k, [m_1, \dots, m_k]) \mid m_1, \dots, m_k \in |!U|\}$. Since $\mathsf{p}_U\, x^! = x^{!!}$, we get $\mathsf{p}_U \in \mathbf{PerL}(!U, !!U)$. The naturality in U of these morphisms is clear (it holds in the relational model), and $!\_$ equipped with these two natural transformations is a comonad. Moreover, the Seely isomorphism also holds in this setting.

2.2.6 Seely isomorphism and cartesian closeness. Let U and V be PER-objects. Let $\theta : |!(U \mathbin{\&} V)| \to |!U \otimes !V|$ be the usual bijection defined by

$$\theta([(1, a_1), \dots, (1, a_l), (2, b_1), \dots, (2, b_r)]) = ([a_1, \dots, a_l], [b_1, \dots, b_r])$$

Using Lemma 6, one shows easily that $\theta \in \mathbf{PerL}(!(U \mathbin{\&} V), !U \otimes !V)$ (as a relation). For showing that $\theta^{-1} \in \mathbf{PerL}(!U \otimes !V, !(U \mathbin{\&} V))$, one applies Lemma 4 and then Lemma 6, twice. This shows that θ is a strong isomorphism of PER-objects.

So the category of PER-objects (together with the monoidal and exponential structure explained above) is a Seely category with Seely isomorphism θ^{-1}, see Section 1.3.

We know that the associated Kleisli category $\mathbf{PerL}_!$ is cartesian closed. The object of morphisms from U to V is $U \Rightarrow V = !U \multimap V$ and we have seen that the associated PER $\sim_{U \Rightarrow V}$ is such that, given two elements s_1 and s_2 of $\mathbf{PerL}_!(U, V)$, one has $s_1 \sim_{U \Rightarrow V} s_2$ iff $s_1(x_1) \sim_V s_2(x_2)$ for all $x_1, x_2 \subseteq |U|$ such that $x_1 \sim_U x_2$. The evaluation morphism is Ev, as defined in Section 2.1.2.

2.2.7 The inclusion functor into the collapse category of $\mathbf{Rel}_!$.

Any PER-object $U = (|U|, \sim_U)$ is an object of the category $\mathsf{e}(\mathbf{Rel}_!)$ since $\sim_U$ is a PER on $\mathcal{P}(|U|) = \mathbf{Rel}_!(\top, |U|)$. We have $\mathbf{PerL}_!(U,V) = \mathsf{e}(\mathbf{Rel}_!)(U,V)$ by Lemma 6 and both categories $\mathbf{PerL}_!$ and $\mathsf{e}(\mathbf{Rel}_!)$ have the same identity morphisms and composition laws, which are those of the category $\mathbf{Rel}_!$. Therefore, $\mathbf{PerL}_!$ is a full subcategory of $\mathsf{e}(\mathbf{Rel}_!)$ and we denote as q the corresponding inclusion functor. It is clear moreover that the functor q is a cartesian closed functor.

2.3 The partially ordered class of PER-objects

Let U and V be PER-objects. We say that U is a subobject of V and write $U \sqsubseteq V$ if $|U| \subseteq |V|$, and moreover $\eta_{|U|,|V|} \in \mathbf{PerL}(U,V)$ and $\rho_{|U|,|V|} \in \mathbf{PerL}(V,U)$. This is an adaptation of the concept of embedding-retraction pair (see [Sco76]) to the present setting. We'll introduce several similar notions in the sequel.

One has $U \sqsubseteq V$ iff conditions are satisfied

$$\forall x_1, x_2 \subseteq |U| \quad x_1 \sim_U x_2 \Rightarrow x_1 \sim_V x_2$$

and

$$\forall y_1, y_2 \subseteq |V| \quad y_1 \sim_V y_2 \Rightarrow y_1 \cap |U| \sim_U y_2 \cap |U| \,.$$

Observe that $\sqsubseteq$ a partial order relation on PER-objects and let $\mathbf{PerC}$ be the partially ordered class of PER-objects ordered by $\sqsubseteq$.

One of the main features of this definition is that linear negation is covariant with respect to the subobject partial order. We retrieve of course the crucial property of embedding-retraction pairs: function space becomes a covariant operation wrt. both parameters when considered as a functor acting on such pairs.

Lemma 7 *If $U \sqsubseteq V$ then $U^\perp \sqsubseteq V^\perp$.*

Proof. We have $|U^\perp| = |U| \subseteq |V| = |V^\perp|$. Moreover $\eta^\perp_{|U|,|V|} = \rho_{|U|,|V|}$ and $\rho^\perp_{|U|,|V|} = \eta_{|U|,|V|}$. The result follows. □

2.3.1 Completeness.

Lemma 8 *Let Γ be a directed set and let $(U_\gamma)_{\gamma\in\Gamma}$ be a directed family of PERs (meaning that $\gamma \le \delta \Rightarrow U_\gamma \sqsubseteq U_\delta$). We define $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$ by $|U| = \bigcup_{\gamma\in\Gamma} |U_\gamma|$ and, for $x_1, x_2 \subseteq |U|$, $x_1 \sim_U x_2$ iff $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$. Then U is a PER-object. Moreover $U^\perp = \bigsqcup_{\gamma\in\Gamma} U_\gamma^\perp$.*

Proof. Let $U' = \bigsqcup_{\gamma\in\Gamma} U_\gamma^\perp$, it will be enough to show that $U = U'^\perp$. Let $x_1, x_2 \subseteq |U|$. Assume first that $x_1 \sim_U x_2$ and let us show that $x_1 \sim_{U'^\perp} x_2$. So let $x_1', x_2' \subseteq |U|$ be such that $x_1' \sim_{U'} x_2'$ and assume that $x_1 \cap x_1' \neq \emptyset$. Let $\gamma \in \Gamma$ be such that $x_1 \cap x_1' \cap |U_\gamma| \neq \emptyset$. By definition of U and U', we have $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ and $x_1' \cap |U_\gamma| \sim_{U_\gamma^\perp} x_2' \cap |U_\gamma|$, and therefore $x_2 \cap x_2' \cap |U_\gamma| \neq \emptyset$, and hence $x_2 \cap x_2' \neq \emptyset$ as required. Assume next that $x_1 \sim_{U'^\perp} x_2$ and let us show that $x_1 \sim_U x_2$. So let $\gamma \in \Gamma$ and let us prove that $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$. So let $x'_1, x'_2 \subseteq |U_\gamma|$ be such that $x'_1 \sim_{U_\gamma^\perp} x'_2$ and assume that $(x_1 \cap |U_\gamma|) \cap x'_1 \neq \emptyset$, that is $x_1 \cap x'_1 \neq \emptyset$.

We show that $x'_1 \sim_{U'} x'_2$. Let $\delta \in \Gamma$ and let us show that $x'_1 \cap |U_\delta| \sim_{U_\delta^\perp} x'_2 \cap |U_\delta|$. So let $\varepsilon \in \Gamma$ be such that $\gamma, \delta \leq \varepsilon$. Let $y_1, y_2 \subseteq |U_\delta|$ be such that $y_1 \sim_{U_\delta} y_2$ and $x'_1 \cap |U_\delta| \cap y_1 \neq \emptyset$. Since $U_\delta \sqsubseteq U_\varepsilon$ and $U_\delta^\perp \sqsubseteq U_\varepsilon^\perp$ (by Lemma 7), we have $x'_1 \sim_{U_\varepsilon^\perp} x'_2$ and $y_1 \sim_{U_\varepsilon} y_2$. Therefore $x'_2 \cap y_2 \neq \emptyset$, that is $x'_2 \cap |U_\delta| \cap y_2 \neq \emptyset$ (since $y_2 \subseteq |U_\delta|$) as required.

Since $x_1 \sim_{U'^\perp} x_2$ and $x'_1 \sim_{U'} x'_2$, we have $x_2 \cap x'_2 \neq \emptyset$, that is $(x_2 \cap |U_\gamma|) \cap x'_2 \neq \emptyset$ (since $x'_2 \subseteq |U_\gamma|$) as required. □

Lemma 9 *If $(U_\gamma)_{\gamma\in\Gamma}$ is a directed family of PER-objects, then $\bigsqcup_{\gamma\in\Gamma} U_\gamma$ is its lub in* **PerC**.

Proof. For showing that $U_\delta \sqsubseteq \bigsqcup_{\gamma\in\Gamma} U_\gamma$, one must show that, if $x_1 \sim_{U_\delta} x_2$, then $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for any given $\gamma \in \Gamma$; one picks some $\varepsilon \in \Gamma$ such that $\gamma, \delta \leq \varepsilon$ and one proceeds as in the proof of Lemma 8. Let V be a PER-object an assume that $U_\gamma \sqsubseteq V$ for all $\gamma \in \Gamma$, we must show that $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma \sqsubseteq V$. Let first $x_1, x_2 \subseteq |U|$ and assume that $x_1 \sim_U x_2$, and let us prove that $x_1 \sim_V x_2$. So let $y'_1, y'_2 \subseteq |V|$ be such that $y'_1 \sim_{V^\perp} y'_2$, and assume that $x_1 \cap y'_1 \neq \emptyset$. Let $\gamma \in \Gamma$ be such that $x_1 \cap y'_1 \cap |U_\gamma| \neq \emptyset$. Since $U_\gamma^\perp \sqsubseteq V^\perp$ by Lemma 7, we have $y'_1 \cap |U_\gamma| \sim_{U_\gamma^\perp} y'_2 \cap |U_\gamma|$ and hence $x_2 \cap y'_2 \cap |U_\gamma| \neq \emptyset$ and so $x_2 \cap y'_2 \neq \emptyset$. Let now $y_1, y_2 \subseteq |V|$ be such that $y_1 \sim_V y_2$ and let us show that $y_1 \cap |U| \sim_U y_2 \cap |U|$, that is $y_1 \cap |U_\gamma| \sim_U y_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$, which holds since $U_\gamma \sqsubseteq V$ by assumption. □

2.3.2 Variable PER-objects and fixpoints thereof.

A functor (that is, a "monotone" class function) $\Phi : \mathbf{PerC}^n \to \mathbf{PerC}$ which commutes with the lubs of directed families (of n-tuples) of PER-objects will be said to be continuous, or to be a *variable PER-object* (the terminology is borrowed from [Gir86]). Let $\Psi : \mathbf{PerC} \to \mathbf{PerC}$ be a variable PER-object. Then Ψ has a least fixpoint $\mathsf{fix}(\Psi) = \bigsqcup_{k\in\mathbb{N}} \Psi^k(\top)$ where $\top$ is the empty PER-object (see Section 2.2.4). Of course, given a PER-object $\Phi : \mathbf{PerC}^{n+1} \to \mathbf{PerC}$, the operation $\mathbf{PerC}^n \to \mathbf{PerC}$ which maps $(U_1, \dots, U_n)$ to $\mathsf{fix}(\Phi(U_1, \dots, U_n, \_))$ is a variable PER-object. We have already seen that the map $U \to U^\perp$ is a variable PER-object.

Lemma 10 *The operations $(U, V) \mapsto U \otimes V$, $U \mapsto U^\perp$, $U \mapsto U^I$ and $U \mapsto\, !U$ are variable PER-objects.*

Proof. We have already seen that $U \mapsto U^\perp$ is a variable PER-object.

We observe that $\otimes$ is monotone, in the sense that if $U \sqsubseteq U'$ and $V \sqsubseteq V'$, then $U \otimes V \sqsubseteq U' \otimes V'$. This results from the fact that $|U \otimes V| \subseteq |U' \otimes V'|$ and from the obvious equations $\eta_{|U\otimes V|,|U'\otimes V'|} = \eta_{|U|,|U'|} \otimes \eta_{|V|,|V'|}$ and $\rho_{|U\otimes V|,|U'\otimes V'|} = \rho_{|U|,|U'|} \otimes \rho_{|V|,|V'|}$. We check similarly that $!\_$ and $(\_)^I$ are monotone.

We show that $(U, V) \mapsto (U \multimap V)$ is a variable PER-object. It is monotone by the considerations above. Let $(U_\gamma)_{\gamma\in\Gamma}$ and $(V_\gamma)_{\gamma\in\Gamma}$ be directed families of PER-objects. We show that $U \multimap V = \bigsqcup_{\gamma\in\Gamma} (U_\gamma \multimap V_\gamma)$ where $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$ and $V = \bigsqcup_{\gamma\in\Gamma} V_\gamma$. Let $t_1, t_2 \subseteq |U \multimap V|$. Assume first that $t_1 \sim_{U\multimap V} t_2$; one has $t_1 \cap |U_\gamma \multimap V_\gamma| \sim_{U_\gamma\multimap V_\gamma} t_2 \cap |U_\gamma \multimap V_\gamma|$ because, if $x_1 \sim_{U_\gamma} x_2$, one has

$(t_i \cap |U_\gamma \multimap V_\gamma|)\, x_i = (t_i\, x_i) \cap |V_\gamma|$. Conversely, assume that $t_1 \sim_{\bigsqcup_{\gamma\in\Gamma}(U_\gamma \multimap V_\gamma)} t_2$ and let us show that $t_1 \sim_{U \multimap V} t_2$. So let $x_1, x_2 \subseteq |U|$ be such that $x_1 \sim_U x_2$, and let us show that $t_1\, x_1 \sim_V t_2\, x_2$. We have $t_i\, x_i = \bigcup_{\gamma\in\Gamma} (t_i \cap |U_\gamma \multimap V_\gamma|)\,(x_i \cap |U_\gamma|)$ and $(t_1 \cap |U_\gamma \multimap V_\gamma|)\,(x_1 \cap |U_\gamma|) \sim_{V_\gamma} (t_2 \cap |U_\gamma \multimap V_\gamma|)\,(x_2 \cap |U_\gamma|)$ for each $\gamma \in \Gamma$. We conclude applying Lemma 1 and using the fact that $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$. Since $U \otimes V = (U \multimap V^\perp)^\perp$, this shows that $(U, V) \mapsto U \otimes V$ is a variable PER-object.

One proves easily that $U \mapsto U^I$ is a variable PER-object.

To conclude, let us prove that $\Phi : U \mapsto (!U)^\perp$ is a variable PER-object. It is a monotone operation because $!\_$ is monotone as we have seen. So let $(U_\gamma)_{\gamma\in\Gamma}$ be a directed family of PER-objects and let us show that $\Phi(U) = \bigsqcup_{\gamma\in\Gamma} \Phi(U_\gamma)$, where $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$. Let $A'_1, A'_2 \subseteq \mathcal{M}_{\mathrm{fin}}(|!U|)$. Assume first that $A'_1 \sim_{\Phi(U)} A'_2$ and let $\gamma \in \Gamma$, we prove that $A'_1 \cap |\Phi(U_\gamma)| \sim_{\Phi(U_\gamma)} A'_2 \cap |\Phi(U_\gamma)|$. So let $x_1, x_2 \subseteq |U_\gamma|$ with $x_1 \sim_{U_\gamma} x_2$ and assume that $A'_1 \cap |\Phi(U_\gamma)| \cap x_1^! \neq \emptyset$. We have $x_1 \sim_U x_2$ and hence $A'_2 \cap x_2^! \neq \emptyset$, that is $A'_2 \cap |\Phi(U_\gamma)| \cap x_2^! \neq \emptyset$. Conversely, assume that $A'_1 \sim_{\bigsqcup_{\gamma\in\Gamma}\Phi(U_\gamma)} A'_2$ and let us prove that $A'_1 \sim_{\Phi(U)} A'_2$. So let $x_1, x_2 \subseteq |U|$ with $x_1 \sim_U x_2$ and assume that $A'_1 \cap x_1^! \neq \emptyset$; let m be an element of that intersection. Since Γ is directed and m is a *finite* multiset, one can find $\gamma \in \Gamma$ such that $m \in |\Phi(U_\gamma)|$. By assumption, we have $A'_1 \cap |\Phi(U_\gamma)| \sim_{\Phi(U_\gamma)} A'_2 \cap |\Phi(U_\gamma)|$ and $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$. We conclude using the fact that $(x_1 \cap |U_\gamma|)^! = x_1^! \cap |\Phi(U_\gamma)|$: we have $A'_1 \cap x_1^! \cap |\Phi(U_\gamma)| \neq \emptyset$, that is $(A'_1 \cap |\Phi(U_\gamma)|) \cap (x_1 \cap |U_\gamma|)^! \neq \emptyset$ and hence $(A'_2 \cap |\Phi(U_\gamma)|) \cap (x_2 \cap |U_\gamma|)^! \neq \emptyset$ which implies $A'_2 \cap x_2^! \neq \emptyset$. □

2.3.3 An extensional reflexive PER-object.

Consider the mapping of PER-object Φ_{e} defined by $\Phi_{\mathsf{e}}(U) = (!(U^{\mathbb{N}}))^\perp$. By Lemmas 7 and 10, Φ_{e} is a variable PER-object, and has therefore a least fixpoint, namely the PER-object $\mathcal{D}_{\mathsf{e}} = \bigsqcup_{k\in\mathbb{N}} \Phi_{\mathsf{e}}^k(\top)$. One has $\mathcal{D}_{\mathsf{e}} \Rightarrow \mathcal{D}_{\mathsf{e}} = (!\mathcal{D}_{\mathsf{e}})^\perp \parr \mathcal{D}_{\mathsf{e}} = (!\mathcal{D}_{\mathsf{e}})^\perp \parr \Phi_{\mathsf{e}}(\mathcal{D}_{\mathsf{e}}) = (!\mathcal{D}_{\mathsf{e}})^\perp \parr (!(\mathcal{D}_{\mathsf{e}}^{\mathbb{N}}))^\perp \simeq (!(\mathcal{D}_{\mathsf{e}} \,\&\, \mathcal{D}_{\mathsf{e}}^{\mathbb{N}}))^\perp$ by the Seely isomorphism of Section 2.2.6. We conclude since $\mathcal{D}_{\mathsf{e}} \,\&\, \mathcal{D}_{\mathsf{e}}^{\mathbb{N}} \simeq \mathcal{D}_{\mathsf{e}}^{\mathbb{N}}$ (by the strong isomorphism which maps $(1, a)$ to $(0, a)$ and $(2, (i, a))$ to $(i + 1, a)$). Therefore $\mathcal{D}_{\mathsf{e}}$ is an extensional model of the pure lambda-calculus in the Kleisli category $\mathbf{PerL}_!$.

The underlying set $|\mathcal{D}_{\mathsf{e}}|$ is the relational model of the pure lambda-calculus described in [BEM07]. We denote it as $\mathcal{D}_{\mathsf{r}}$. It is the least fixpoint (in the partially ordered class of sets) of the monotone and continuous operation $E \mapsto \mathcal{M}_{\mathrm{fin}}(\mathbb{N} \times E)$.

3 A linear Scott semantics

We describe now the linear Scott model that we want to connect with the relational semantics through an extensional collapse. We don't claim to introduce any novelty in this presentation: all the material of this section can be found in earlier work by Huth and al. [Hut93, HJK00] and Winskel [Win04]. More information and intuitions on this model can be found in these papers, in particular about the connections between the resource modalities $!\_$ and $?\_$ and various powerdomain constructions: our resource modalities are exactly the same as theirs, up to slight and irrelevant variations in the presentation (in particular, we insist on considering finite multisets instead of finite sets when defining $!S$, only for simplifying the description of the collapse). We won't mention further these properties here because they are not directly related with the result we aim at.

Given a preordered set $(S, \leq)$, we denote with S^{op} the opposite preorder. Given $x \subseteq S$, we denote with $\downarrow_S x$ (or simply $\downarrow x$ if the ambient preorder is clear from the context) the set $\{a \in S \mid \exists b \in x\ a \leq b\}$. And we set $\uparrow_S x = \downarrow_{S^{\mathrm{op}}} x$. We also define

$$\mathcal{I}(S) = \{x \subseteq S \mid \underset{S}{\downarrow} x = x\}$$

which, ordered by inclusion, is a prime-algebraic lattice.

3.1 $*$-autonomous structure

Let S and T be preorders. A function $f : \mathcal{I}(S) \to \mathcal{I}(T)$ is *linear* if it commutes with arbitrary lubs. In other words, for any family $(x_i)_{i \in I}$ of elements of $\mathcal{I}(S)$, we must have $f\left(\bigcup_{i \in I} x_i\right) = \bigcup_{i \in I} f(x_i)$. This implies in particular that f is monotone, and that $f(\emptyset) = \emptyset$ (of course, we do not necessarily have $f(S) = T$). We denote with **ScottL** the corresponding category.

We equip the hom-set $\mathbf{ScottL}(S, T)$ with the ordinary pointwise order: $f \leq g$ if $\forall x \in \mathcal{I}(X)\ f(x) \subseteq g(x)$. We define the *linear trace* of a linear map $f \in \mathbf{ScottL}(S, T)$ as

$$\mathsf{tr}^{\mathsf{S}}(f) = \{(a, b) \in S \times T \mid b \in f(\underset{S}{\downarrow} \{a\})\}\,.$$

This is similar to the usual definition of the trace of a stable linear map (see [Gir87, AC98]), the main difference being that there is no minimality requirement on a: such a requirement would not make sense in general because usually our preorders are not well-founded. Then it is easily checked that $\mathsf{tr}^{\mathsf{S}}(f) \in \mathcal{I}(S^{\mathrm{op}} \times T)$. Conversely, given any $t \in \mathcal{I}(S^{\mathrm{op}} \times T)$, we define a function

$$\begin{aligned} \mathsf{fun}^{\mathsf{S}}(t) : \mathcal{I}(S) &\to \mathcal{P}(T) \\ x &\mapsto t\,x \end{aligned}$$

and it is easy to check that $\mathsf{fun}^{\mathsf{S}}(t)$ takes its values in $\mathcal{I}(T)$ and is linear from $\mathcal{I}(S)$ to $\mathcal{I}(T)$.

Proposition 11 *The maps* tr^{S} *and* $\mathsf{fun}^{\mathsf{S}}$ *define an order isomorphism between the posets* $\mathbf{ScottL}(S, T)$ *and* $\mathcal{I}(S^{\mathrm{op}} \times T)$*. Moreover, these isomorphisms commute with composition (of maps and relations respectively).*

Therefore, we set $S \multimap T = S^{\mathrm{op}} \times T$. Thanks to the proposition above, we can consider the morphisms of the category **ScottL** as linear functions or as relations. For instance, as a function, the identity morphism $S \to S$ is of course the identity function $\mathcal{I}(S) \to \mathcal{I}(S)$, but as a relation, it is $\mathsf{Id}^{\mathsf{S}} = \{(a, b) \in S \times S \mid b \leq a\}$. In this paper, we prefer the relational viewpoint on morphisms for its similarity with morphisms in **Rel**.

The following easy lemma clarifies the connection between the two approaches, in a more general case where the relation is not assumed to be downward closed in $S \multimap T$.

Lemma 12 *Let* $t \subseteq S \times T$ *and let* $x \in \mathcal{I}(S)$*. One has* $\downarrow_T (t\,x) = (\downarrow_{S \multimap T} t)\,x$*.*

3.1.1 Isomorphisms.

An isomorphism (in the usual categorical sense) from S to T in **ScottL** is a relation $t \in \mathcal{I}(S \multimap T)$ such that $\mathsf{fun}^{\mathsf{S}}(t) : \mathcal{I}(S) \to \mathcal{I}(T)$ is an order isomorphism. As a relation, an isomorphism from S to T has no reason to be a bijection, not even a function. For instance, if $S = \{0\}$ and $T = \mathbb{N}$ (with the largest preorder, in which $n \leq m$ for all $n, m \in \mathbb{N}$), then the relation $\{(0, n) \mid n \in \mathbb{N}\}$ is an isomorphism from S to T (it is actually the only non-empty morphism from S to T).

We'll call *strong isomorphism* from S to T any function $\varphi : S \to T$ which is an isomorphism of preorders (that is, φ is bijective and $a \leq_S b$ iff $\varphi(a) \leq_T \varphi(b)$). Such a φ, considered as a set of pairs, is not an isomorphism in the categorical sense above in general, but $\downarrow_{S \multimap T} \varphi$ is. And we'll say that S and T are strongly isomorphic if there is a strong isomorphism from S to T.

3.1.2 Monoidal structure.

The *tensor product* of preorders is given by $S \otimes T = S \times T$. It is easily seen to be functorial. Indeed, let $s \in \mathcal{I}(S_1 \multimap S_2)$ and $t \in \mathcal{I}(T_1 \multimap T_2)$. Then, we set

$$s \otimes t = \{((a_1, b_1), (a_2, b_2)) \in (S_1 \otimes T_1) \multimap (S_2 \otimes T_2) \mid (a_1, a_2) \in s \text{ and } (b_1, b_2) \in t\}\,.$$

One can check that $s \otimes t \in \mathcal{I}((S_1 \otimes T_1) \multimap (S_2 \otimes T_2))$ and that $(s' \otimes t') \circ (s \otimes t) = (s' \circ s) \otimes (t' \circ t)$.

The neutral element of the tensor product is $1 = \{\star\}$ (actually, any non-empty preorder such that $a \leq b$ for all a, b is isomorphic to 1, and therefore is neutral for $\otimes$). The so defined symmetric monoidal category **ScottL** is monoidal closed, with linear evaluation morphism $\mathsf{ev}^{\mathsf{S}} \in \mathbf{ScottL}((S \multimap T) \otimes S, T)$ given by

$$\mathsf{ev}^{\mathsf{S}} = \{(((a, b), a'), b') \mid b' \leq_{|T|} b \text{ and } a \leq_{|S|} a'\}\,.$$

We use the same object 1 as dualizing object, but when used in that way, we denote it with $\perp$.

It is clear that $S \multimap \perp = S^{\mathrm{op}}$ (up to the identification of $a \in S$ with $(a, \star) \in S \multimap \perp$), and that the canonical map $S \to (S \multimap \perp) \multimap \perp$ coincides with the identity, so the monoidal category of preorders and linear maps is a $*$-autonomous category in the sense of Section 1.3.

3.2 Products and coproducts

Let $(S_i)_{i \in I}$ be a collection of preorders, the cartesian product of this family is denoted with $\&_{i \in I} S_i$ and is the disjoint union $\bigcup_{i \in I} (\{i\} \times S_i)$, endowed with the disjoint union of the preorder relations. One has $\mathcal{I}(\&_{i \in I}) = \prod_{i \in I} \mathcal{I}(S_i)$ up to a trivial and canonical isomorphism. The i-th projection $\pi_i^{\mathsf{S}} : \&_{i \in I} S_i \to S_i$ is given by

$$\pi_i^{\mathsf{S}} = \{((i, a), b) \mid a, b \in S_i \; b \leq a\}\,.$$

And given morphisms $t_i : T \to S_i$, the unique morphism $t = \langle t_i \rangle_{i \in I} : T \to \&_{i \in I} S_i$ characterized by $\forall i \, \pi_i^{\mathsf{S}} \, t = t_i$ is given by

$$t = \bigcup_{i \in I} \{(b, (i, a)) \mid (b, a) \in t_i)\}\,.$$

The sum $\oplus_{i \in I} S_i = (\&_{i \in I} {S_i}^{\mathrm{op}})^{\mathrm{op}}$ is the operation dual to this product, and coincides with it as easily checked.

If S is a preorder and I is a set, we use S^I for the product $\&_{i\in I} S_i$ where $S_i = S$ for each I. We use $\top$ for the product of the empty family of preorders: it is the terminal object, and, as a preorder, it is empty (so $\mathcal{I}(\top) = \{\emptyset\}$). It is obviously isomorphic to its dual, denoted with 0.

3.3 Exponentials

Given a preorder S, we define the preorder $!S$, whose elements are the finite multisets of elements of S, with the following preorder relation: given $p, q \in\, !S$, one has $p \leq_{!S} q$ if $\forall a \in \mathsf{supp}(p)\, \exists b \in \mathsf{supp}(q)\ a \leq_S b$. Of course we could have taken $!S = \mathcal{P}_{\mathrm{fin}}(S)$, with a similarly defined preorder, and the associated lattices of initial segments would have been trivially isomorphic. We choose multisets because our goal is to compare this preorder model with the relational model, where the exponentials are defined with finite multisets. This choice makes the study of the collapse much simpler.

Given $x \subseteq S$, we set $x^! = \mathcal{M}_{\mathrm{fin}}(x)$. The following is a straightforward but crucial observation.

Lemma 13 *Let $x \subseteq S$. We have $(\downarrow_{|X|} x)^! = \downarrow_{|!S|} (x^!)$.*

We'll use this remark quite often, tacitly. It implies that, if $x \in \mathcal{I}(S)$, then $x^! \in \mathcal{I}(!S)$. Given $t : S \to T$, we set

$$!t = \{(p, q) \in\, !S \times !T \mid \forall b \in q\, \exists a \in p\ (a, b) \in t\}\,.$$

Then one shows easily that $!t : !S \to !T$, and that this operation on morphisms is functorial. Moreover, it is quite useful to observe that

$$\forall x \in \mathcal{I}(S) \quad !t\, x^! = (t\, x)^!\,.$$

And this property actually characterizes the morphism $!t$.

3.3.1 Comonad structure of the exponential. As required by the definition of a Seely category (see Section 1.3), this functor $!\_$ has a structure of comonad, which is given by the natural morphism

$$\mathsf{d}^{\mathsf{S}}_S = \{(p, b) \in\, !S \times S \mid \exists a \in p\ b \leq a\} : !S \to S$$

usually called *dereliction* and

$$\mathsf{p}^{\mathsf{S}}_S = \{(p, [p_1, \dots, p_n]) \in\, !S \times !!S \mid p_1 + \cdots + p_n \leq_{!S} p\} : !S \to !!S$$

usually called *digging*. Observe that $\mathsf{d}^{\mathsf{S}}_S\, x^! = x$ and that $\mathsf{p}^{\mathsf{S}}_S\, x^! = (x^!)^!$, and that these equations characterize the morphisms $\mathsf{d}^{\mathsf{S}}_S$ and $\mathsf{p}^{\mathsf{S}}_S$. With these observations, it is trivial to check that these morphisms are natural (as announced) and provide the functor $!\_$ with a comonad structure.

3.3.2 Weakening and contraction. Given two preorders S_1 and S_2, there is a canonical and natural strong isomorphism between the preorders $!S_1 \otimes !S_2$ and $!(S_1 \,\&\, S_2)$, which is actually the preorder isomorphism

$$([a_1, \dots, a_n], [b_1, \dots, b_m]) \mapsto [(1, a_1), \dots, (1, a_n), (2, b_1), \dots, (2, b_m)]\,.$$

Similarly, there is a trivial isomorphism from 1 to $!\top$ (both are the one-point preorder): these are the Seely isomorphisms of the model. With all these structures, **ScottL** is a Seely category in the sense Section 1.3), it is the model discovered independently by Huth [Hut93] and Winskel [Win98].

Using these isomorphisms, and applying the $!\_$ functor to the diagonal map $\delta_S : S \to S \mathbin{\&} S$ (which, as easily checked, is the set $\{(a,(1,b)) \mid b \leq a\} \cup \{(a,(2,b)) \mid b \leq a\}$) and to the unique map $S \to \top$ (the empty map), we get the contraction and weakening maps:

$$\begin{aligned}\mathsf{contr}^{\mathsf{S}}_S &= \{(p,(q_1,q_2)) \mid q_1 + q_2 \leq_{!S} p)\} : !S \to !S \otimes !S \\ \mathsf{weak}^{\mathsf{S}}_S &= \{(p,\star) \mid p \in !S\} : !S \to 1\,.\end{aligned}$$

3.4 The Kleisli category

Remember that, in the associated Kleisli category $\mathbf{ScottL}_!$, a morphism from S to T is a linear morphism $t : !S \to T$:

$$\mathbf{ScottL}_!(S,T) = \mathbf{ScottL}(!S,T)\,.$$

Given such a morphism $t : !S \to T$, we can define a map

$$\begin{aligned}\mathsf{Fun}^{\mathsf{S}}(t) : \mathcal{I}(S) &\to \mathcal{I}(T) \\ x &\mapsto t\,x^!\end{aligned}$$

In other words, $\mathsf{Fun}^{\mathsf{S}}(t)(x) = \{b \in T \mid \exists p \in !S \ \mathsf{supp}(p) \subseteq x \text{ and } (p,b) \in t\}$

Observe that the function $S \to !S$ which maps x to $x^!$ is never linear (since it maps $\emptyset$ to $\{[]\}$; it is actually the "most non-linear" map from S to $S\ldots$), but is Scott continuous. Therefore, the map $\mathsf{Fun}^{\mathsf{S}}(t)$ is Scott-continuous as well.

Conversely, observe that $\mathcal{I}(S)$ is a Scott domain, whose compact elements are the finitely generated elements of $\mathcal{I}(S)$, that is, the elements x_0 of $\mathcal{I}(S)$ such that $x_0 = \downarrow_S u$ for some finite $u \subseteq S$. Given a Scott-continuous function $f : \mathcal{I}(S) \to \mathcal{I}(T)$, one defines the set

$$\mathsf{Tr}^{\mathsf{S}}(f) = \{(p,b) \in \mathcal{M}_{\mathrm{fin}}(S) \times T \mid b \in f(\underset{S}{\downarrow}(\mathsf{supp}(p)))\}\,.$$

that we call the *trace* of f.

Lemma 14 *Let S and T be preorders. The maps Tr^{S} and $\mathsf{Fun}^{\mathsf{S}}$ define an order isomorphism between $\mathcal{I}(!S \multimap T)$ and the set of Scott-continuous functions from $\mathcal{I}(S)$ to $\mathcal{I}(T)$, endowed with the pointwise order.*

Proof. Let $f, g : \mathcal{I}(S) \to \mathcal{I}(T)$ be Scott-continuous functions such that $f \leq g$ for the pointwise order. Let $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(f)$. Then $b \in f(\downarrow_S(\mathsf{supp}(p))) \subseteq g(\downarrow_S(\mathsf{supp}(p)))$, so $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(g)$ and hence the map Tr^{S} is monotone. Let $s, t \in \mathcal{I}(!S \multimap T)$ be such that $s \subseteq t$, let $x \in \mathcal{I}(S)$ and let $b \in \mathsf{Fun}^{\mathsf{S}}(s)(x)$. This means that there exists $p \in !S$ such that $(p,b) \in s$ and $\mathsf{supp}(p) \subseteq x$. Then $(p,b) \in t$ and hence we also have $b \in \mathsf{Fun}^{\mathsf{S}}(t)(x)$, and this shows that the map $\mathsf{Fun}^{\mathsf{S}}$ is monotone as well.

Let $f : \mathcal{I}(S) \to \mathcal{I}(T)$ be continuous, $f' = \mathsf{Fun}^{\mathsf{S}}(\mathsf{Tr}^{\mathsf{S}}(f))$ and let $x \in \mathcal{I}(S)$. Let $b \in f(x)$. Since f is continuous, there is a finite subset u of x such that

$b \in f(\downarrow_S(u))$. Let $p \in\, !S$ be such that $\mathsf{supp}(p) = u$. Then we have $(p, b) \in \mathsf{Tr}^{\mathsf{S}}(f)$ and hence $b \in f'(x)$. Conversely, if $b \in f'(x)$, let $p \in\, !S$ be such that $(p, b) \in \mathsf{Tr}^{\mathsf{S}}(f)$ and $\mathsf{supp}(p) \subseteq x$, then $b \in f(\downarrow_S(\mathsf{supp}(p))) \subseteq f(x)$ and we have shown that $f'(x) = f(x)$ for all $x \in \mathcal{I}(S)$, so $\mathsf{Fun}^{\mathsf{S}} \circ \mathsf{Tr}^{\mathsf{S}}$ is the identity map.

Conversely, let $t \in \mathcal{I}(!S \multimap T)$ and let $t' = \mathsf{Tr}^{\mathsf{S}}(\mathsf{Fun}^{\mathsf{S}}(t))$. Let $(p, b) \in t$, then $b \in \mathsf{Fun}(t)(\downarrow_S(\mathsf{supp}(p)))$, and hence $(p, b) \in t'$. Let $(p, b) \in t'$, then $b \in \mathsf{Fun}^{\mathsf{S}}(t)(\downarrow_S(\mathsf{supp}(p)))$ and hence there exists $q \in\, !S$ such that $(q, b) \in t$ and $\mathsf{supp}(q) \subseteq \downarrow_S(\mathsf{supp}(p))$, that is, $q \leq_{!S} p$. Since $(p, b) \leq_{!S \multimap T} (q, b) \in t$ and $t \in \mathcal{I}(!S \multimap T)$, we have $(p, b) \in t$, and this shows that $\mathsf{Tr}^{\mathsf{S}} \circ \mathsf{Fun}^{\mathsf{S}}$ is the identity map. $\Box$

3.4.1 The Kleisli category of preorders. This isomorphism is compatible with composition, as easily checked, so that we can consider $\mathbf{ScottL}_!$ as a full subcategory of the category of Scott domains and continuous functions. Moreover, it is easily checked that the cartesian products and function space constructions in both categories coincide: the cartesian product in $\mathbf{ScottL}_!$ of S and T is $S \mathbin{\&} T$, and we have seen that $\mathcal{I}(S \mathbin{\&} T) \simeq \mathcal{I}(S) \times \mathcal{I}(T)$ (with the product order) and their function space is $S \Rightarrow T = !S \multimap T$, and we have seen that $\mathcal{I}(!S \multimap T)$ is isomorphic (as a poset) to the space of continuous maps from $\mathcal{I}(S)$ to $\mathcal{I}(T)$, endowed with the pointwise order, which is precisely the function space of $\mathcal{I}(S)$ and $\mathcal{I}(T)$ in the category of Scott domains and continuous functions. The evaluation map $\mathsf{Ev}^{\mathsf{S}} \in \mathbf{ScottL}_!((S \Rightarrow T) \mathbin{\&} S, T) \simeq \mathbf{ScottL}(!(S \Rightarrow T) \otimes !S, T)$ satisfies

$$\mathsf{Ev}^{\mathsf{S}} = \{((r, p), b) \mid \exists (p', b') \in r \quad b \leq_T b' \text{ and } p' \leq_{!S} p\}$$

as easily checked from the general definition of this evaluation morphism in Section 1.3.3.

So $\mathbf{ScottL}_!$ is a full sub-CCC of the CCC of Scott domains and continuous functions.

3.5 The partially ordered class of preorders

We say that the preorder S is a substructure of the preorder T, and we write $S \sqsubseteq T$ if, for any $a_1, a_2 \in S$, one has $a_1 \leq_S a_2 \Leftrightarrow a_1 \leq_T a_2$. We denote with $\mathbf{ScottC}$ the corresponding partially ordered class. It is easy to check that $\mathbf{ScottC}$ is complete (any directed family $(S_\gamma)_{\gamma \in \Gamma}$ has a lub $\bigsqcup_{\gamma \in \Gamma} S_\gamma$), and that all the constructions we have introduced on preorders are *variable preorders*, that is, continuous class functions $\mathbf{ScottC}^n \to \mathbf{ScottC}$. Any variable preorder $\Phi : \mathbf{ScottC} \to \mathbf{ScottC}$ admits a least fixpoint. In particular, the operation $\Phi_{\mathsf{s}} : \mathbf{ScottC} \to \mathbf{ScottC}$ defined by $\Phi_{\mathsf{s}}(S) = (!(S^{\mathbb{N}}))^{\perp}$ is a variable preorder and therefore admits a least fixpoint $\mathcal{D}_{\mathsf{s}}$, which is an extensional model of the pure lambda-calculus (same computation as in Section 2.3.3).

4 The category of preorders with projections

We define a Seely category $\mathbf{PpL}$ whose objects are of a mixed nature. An object X of $\mathbf{PpL}$ is a pair $(|X|, \mathsf{D}(X))$ where $|X|$ is a preorder (object of the category $\mathbf{ScottL}$) which will also be considered as a simple set, that is, as an object of the category $\mathbf{Rel}$, by forgetting the preorder relation. The two aspects of these objects are related by the predicate $\mathsf{D}(X)$ on $\mathcal{P}(|X|)$ which satisfies a closure property defined as usual by a duality, whose definition involves the preorder relation on $|X|$. Using this predicate, we'll define a binary relation between subsets and downward closed subsets of $|X|$ and show that, in the Kleisli category associated with this Seely category, this binary relation behaves as a logical relation, proving that this Kleisli cartesian closed category is a sub-CCC of the heterogeneous category $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$ defined in Section 1.4.2, and which satisfies the conditions mentioned in that section (its objects are modest and the first projection functor is cartesian closed).

4.1 A duality on preorders

We introduce first the duality which will be essential for defining these objects. Let S be a preorder. Given $x, x' \subseteq S$, we say that x and x' are *in duality* (with respect to S) and write $x \perp_S x'$ if

$$x \cap x' = \emptyset \Rightarrow (\underset{S}{\downarrow} x) \cap x' = \emptyset\,.$$

Of course, the converse implication always holds so that, when it holds, the implication above is actually an equivalence. The intuition is clear: x and x' are in duality if x' cannot separate x from its downward closure.

This duality relation is symmetric in the following sense: since clearly $(\downarrow_S x) \cap x' = \emptyset \Leftrightarrow x \cap (\uparrow_S x') = \emptyset \Leftrightarrow (\downarrow_S x) \cap (\uparrow_S x') = \emptyset$, we have

$$\forall x, x' \subseteq S \quad x \perp_S x' \Leftrightarrow x' \perp_{S^{\mathrm{op}}} x\,.$$

If $D \subseteq \mathcal{P}(S)$, we set

$$D^{\perp_S} = \{x' \subseteq S \mid \forall x \in D \quad x \perp_S x'\}$$

With this definition, we have $D \subseteq D^{\perp_S \perp_{S^{\mathrm{op}}}}$. Indeed, let $x \in D$ and let $x' \in D^{\perp_S}$. We have $x \perp_S x'$, that is $x' \perp_{S^{\mathrm{op}}} x$, and since this holds for all $x' \in D^{\perp_S}$, we have $x \in D^{\perp_S \perp_{S^{\mathrm{op}}}}$. Moreover, if $D, E \subseteq \mathcal{P}(S)$, we have $D \subseteq E \Rightarrow E^{\perp_S} \subseteq D^{\perp_S}$. Therefore, one always has $D^{\perp_S \perp_{S^{\mathrm{op}}} \perp_S} = D^{\perp_S}$.

One can observe indeed that the operations $D \to D^{\perp_S}$ and $D \to D^{\perp_{S^{\mathrm{op}}}}$ define a Galois connection from $(\mathcal{P}(S), \subseteq)$ to $(\mathcal{P}(S), \supseteq)$ (this is usually the case with this kind of orthogonality construction in linear logic).

Let $D \subseteq \mathcal{P}(S)$ be such that $D = D^{\perp_S \perp_{S^{\mathrm{op}}}}$ (equivalently, $D = E^{\perp_{S^{\mathrm{op}}}}$ for some $E \subseteq \mathcal{P}(S)$). Then $\mathcal{I}(S) \subseteq D \subseteq \mathcal{P}(S)$. And one checks easily that $\mathcal{P}(S)^{\perp_S} = \mathcal{I}(S^{\mathrm{op}})$ and $\mathcal{I}(S)^{\perp_S} = \mathcal{P}(S^{\mathrm{op}})$. Let $(x_i)_{i \in I}$ be a family of elements of D. Then $\bigcup_{i \in I} x_i \in D$. Indeed, since $D = D^{\perp_S \perp_{S^{\mathrm{op}}}}$, it suffices to show that $\left(\bigcup_{i \in I} x_i\right) \perp_S x'$ for all $x' \in D^{\perp_S}$. So let $x' \in D^{\perp_S}$, and let us assume that $\left(\bigcup_{i \in I} x_i\right) \cap x' = \emptyset$. Then, for any $i \in I$, we have $x_i \cap x' = \emptyset$ and hence $\downarrow_S x_i \cap x' = \emptyset$ (since $x_i \in \mathsf{D}(X)$) and therefore $\left(\bigcup_{i \in I} \downarrow_S x_i\right) \cap x' = \emptyset$. We conclude because clearly $\left(\bigcup_{i \in I} \downarrow_S x_i\right) = \downarrow_S \left(\bigcup_{i \in I} x_i\right)$. So D, endowed with inclusion, is a complete lattice, whose least element is $\emptyset$, and largest element is S.

A *preorder with projection* (a PP for short; the reason for this terminology will appear later) is a pair $X = (|X|, \mathsf{D}(X))$ where $|X|$ is a preorder and $\mathsf{D}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathsf{D}(X) = \mathsf{D}(X)^{\perp_{|X|} \perp_{|X|^{\mathrm{op}}}}$. We define then

$$X^\perp = (|X|^{\mathrm{op}}, \mathsf{D}(X)^{\perp_{|X|}})\,.$$

By definition, we have $X^{\perp\perp} = X$. Remember that $\mathcal{I}(|X|) \subseteq \mathsf{D}(X) \subseteq \mathcal{P}(|X|)$.

Given two PPs X and Y, we define $X \otimes Y$ by setting $|X \otimes Y| = |X| \times |Y|$, endowed with the product order. Then $\mathsf{D}(X \otimes Y)$ is given by

$$\mathsf{D}(X \otimes Y) = \{x \times y \mid x \in \mathsf{D}(X) \text{ and } y \in \mathsf{D}(Y)\}^{\perp_{|X|\times|Y|}\perp_{|X|^{\mathrm{op}}\times|Y|^{\mathrm{op}}}}$$

We define accordingly $X \multimap Y = (X \otimes Y^{\perp_{|Y|}})^{\perp_{|X|\times|Y|^{\mathrm{op}}}}$, so that $|X \multimap Y| = |X|^{\mathrm{op}} \times |Y|$ and, for $t \subseteq |X \multimap Y|$, one has $t \in \mathsf{D}(X \multimap Y)$ iff, for all $x \in \mathsf{D}(X)$ and for all $y' \in \mathsf{D}(Y^{\perp})$, one has

$$t \cap (x \times y') = \emptyset \Rightarrow t \cap (\underset{|X|}{\downarrow} x \times \underset{|Y|}{\uparrow} y') = \emptyset\,.$$

Given $t \subseteq |X| \times |Y|$, remember that the transpose of t is $t^{\perp} = \{(b, a) \mid (a, b) \in t\} \subseteq |Y| \times |X|$. One checks easily that $t \in \mathsf{D}(X \multimap Y)$ iff $t^{\perp} \in \mathsf{D}(Y^{\perp} \multimap X^{\perp})$.

Fortunately, there is an easy functional characterization of the elements of $\mathsf{D}(X \multimap Y)$.

Proposition 15 *Let X and Y be PPs. Let $t \subseteq |X| \times |Y|$. One has $t \in \mathsf{D}(X \multimap Y)$ iff the two following conditions are satisfied.*

- *For all $x \in \mathsf{D}(X)$, one has $t\,x \in \mathsf{D}(Y)$*
- *and, for all $x \in \mathsf{D}(X)$, one has $\downarrow_{|Y|}(t\,x) = \downarrow_{|X \multimap Y|} t \downarrow_{|X|} x$.*

For any $t \subseteq |X| \times |Y|$, the second condition is equivalent to each of the following three statements

- $\forall x \in \mathsf{D}(X)\ \downarrow_{|X \multimap Y|} t \downarrow_{|X|} x \subseteq \downarrow_{|Y|}(t\,x)$
- $\forall x \in \mathsf{D}(X)\ \downarrow_{|Y|}(t \downarrow_{|X|} x) \subseteq \downarrow_{|Y|}(t\,x)$
- $\forall x \in \mathsf{D}(X)\ t \downarrow_{|X|} x \subseteq \downarrow_{|Y|}(t\,x)$.

Proof. The equivalences at the end of the proposition result from Lemma 12.

Assume first that $t \in \mathsf{D}(X \multimap Y)$. Let $x \in \mathsf{D}(X)$. We show first that $t\,x \in \mathsf{D}(Y) = \mathsf{D}(Y^{\perp})^{\perp_{|Y|^{\mathrm{op}}}}$, so let $y' \in \mathsf{D}(Y^{\perp})$ and let us assume that $(t\,x) \cap y' = \emptyset$. This is equivalent to $t \cap (x \times y') = \emptyset$, and since $t \in \mathsf{D}(X \multimap Y)$, we have $t \cap \uparrow_{X \multimap Y}(x \times y') = \emptyset$, that is $t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} y') = \emptyset$. But this implies $t \cap (x \times \uparrow_{|Y|} y') = \emptyset$, that is, $(t\,x) \cap \uparrow_{|Y|} y' = \emptyset$. Since this holds for all $y' \in \mathsf{D}(Y^{\perp})$, we have shown that $t\,x \in \mathsf{D}(Y)$.

We must show now that $t \downarrow_{|X|} x \subseteq \downarrow_{|Y|}(t\,x)$. So let $b \in t \downarrow_{|X|} x$, we have $\uparrow_{|Y|} b \in \mathsf{D}(Y^{\perp})$ and $t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} b) \neq \emptyset$, that is $t \cap \uparrow_{X \multimap Y}(x \times \{b\}) \neq \emptyset$. Since $t \in \mathsf{D}(X \multimap Y)$, this shows that $t \cap (x \times \uparrow_{|Y|} b) \neq \emptyset$, that is $(t\,x) \cap \uparrow_{|Y|} b \neq \emptyset$, that is $b \in \downarrow_{|Y|}(t\,x)$ as required.

Assume conversely that the two conditions of the statement are satisfied, and let us show that $t \in \mathsf{D}(X \multimap Y)$. So let $x \in \mathsf{D}(X)$ and $y' \in \mathsf{D}(Y^{\perp})$, and assume that $t \cap \uparrow_{X \multimap Y}(x \times y') \neq \emptyset$. Equivalently, we have $t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} y') \neq \emptyset$, that is $(t \downarrow_{|X|} x) \cap \uparrow_{|Y|} y' \neq \emptyset$. By our second assumption, we have therefore $\downarrow_{|Y|}(t\,x) \cap \uparrow_{|Y|} y' \neq \emptyset$, and hence $t \cap (x \times y') \neq \emptyset$ since $t\,x \in \mathsf{D}(Y)$ and $y' \in \mathsf{D}(Y^{\perp})$. $\square$

4.2 The linear category

Let **PpL** be the category whose objects are the PPs, and with $\mathbf{PpL}(X,Y) = \mathsf{D}(X \multimap Y)$, composition defined as the usual relational composition.

4.2.1 Identity and composition. By Proposition 15, the identity relation $\mathsf{Id} \subseteq |X| \times |X|$ belongs to $\mathsf{D}(X \multimap X)$.

As to composition, let $s \in \mathsf{D}(X \multimap Y)$ and $t \in \mathsf{D}(Y \multimap Z)$, then we show that the relational composition $u = t\,s$ of these morphisms belongs to $\mathsf{D}(Y \multimap Z)$, using Proposition 15. So let $x \in \mathsf{D}(X)$. First, we have $u\,x = t\,(s\,x) \in \mathsf{D}(Z)$ since $s\,x \in \mathsf{D}(Y)$. Next we must prove that $u \downarrow_{|X|} x \subseteq \downarrow_{|Z|} (u\,x)$. But $u \downarrow_{|X|} x = t\,(s \downarrow_{|X|} x) \subseteq t \downarrow_{|Y|} (s\,x)$ since $s \in \mathsf{D}(X \multimap Y)$ and $x \in \mathsf{D}(X)$. Since $s\,x \in \mathsf{D}(Y)$ and $t \in \mathsf{D}(Y \multimap Z)$, we have $t \downarrow_{|Y|} (s\,x) \subseteq \downarrow_{|Z|} (t\,(s\,x)) = \downarrow_{|Z|} (u\,x)$ as required.

Observe last that $\downarrow_{X \multimap Z} (t\,s) = (\downarrow_{Y \multimap Z} t)\,(\downarrow_{X \multimap Y} s)$. The "$\subseteq$" inclusion is straightforward, we check the converse. Let $(a,c) \in (\downarrow_{Y \multimap Z} t)\,(\downarrow_{X \multimap Y} s)$. Let $b \in |Y|$ be such that $(b,c) \in \downarrow_{Y \multimap Z} t$ and $(a,b) \in \downarrow_{X \multimap Y} s$. Let $(a',b') \in s$ be such that $a' \leq_{|X|} a$ and $b' \geq_{|Y|} b$, and let $(b'',c') \in t$ be such that $b'' \leq_{|Y|} b$ and $c' \geq_{|Z|} c$. We have $b'' \leq b'$ and hence (e.g.) $(b',c) \leq_{|Y \multimap Z|} (b'',c') \in t$ and $(a,b') \leq_{|X \multimap Y|} (a',b') \in s$ and we conclude.

4.2.2 Tensor product. Given two PPs X and Y, we have defined a PP $X \otimes Y$ in Section 4.1. We turn now this operation into a functor which will endow the category **PpL** with a monoidal structure. For this purpose, it is convenient to characterize first the "bilinear" morphisms: this is the purpose of the next lemma which is a binary version of Proposition 15.

Lemma 16 *Let X_1, X_2 and Y be PPs. Let $t \subseteq |X_1 \otimes X_2 \multimap Y|$. One has $t \in \mathbf{PpL}(X_1 \otimes X_2, Y)$ iff, for all $x_1 \in \mathsf{D}(X_1)$ and $x_2 \in \mathsf{D}(X_2)$, one has*

- $t\,(x_1 \otimes x_2) \in \mathsf{D}(Y)$
- *and* $\downarrow_{|Y|} (t\,(x_1 \otimes x_2)) = (\downarrow_{|X_1 \otimes X_2 \multimap Y|} t)\,(\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2)$.

The second condition is equivalent to $t\,(\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) \subseteq \downarrow_{|Y|} (t\,(x_1 \otimes x_2))$.

Proof. The conditions are necessary by Proposition 15. We prove that they are sufficient, so assume that they hold. We prove that $t^\perp \in \mathsf{D}(Y^\perp \multimap (X_1 \otimes X_2)^\perp)$, using Proposition 15, so let $y' \in \mathsf{D}(Y^\perp)$.

We show first that $t^\perp\,y' \in \mathsf{D}((X_1 \otimes X_2)^\perp)$. So let $x_1 \in \mathsf{D}(X_1)$ and $x_2 \in \mathsf{D}(X_2)$ and assume that $(t^\perp\,y') \cap (x_1 \otimes x_2) = \emptyset$, hence $(t\,(x_1 \otimes x_2)) \cap y' = \emptyset$. But we have $t\,(x_1 \otimes x_2) \in \mathsf{D}(Y)$, and hence $(t\,(x_1 \otimes x_2)) \cap \uparrow_{|Y|} y' = \emptyset$, and hence, by our second hypothesis, $(\downarrow_{|X_1 \otimes X_2 \multimap Y|} t)\,(\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) \cap \uparrow_{|Y|} y' = \emptyset$. Therefore $(\downarrow_{|X_1 \otimes X_2 \multimap Y|} t)^\perp \uparrow_{|Y|} y' \cap (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) = \emptyset$, which clearly implies that $t^\perp\,y' \cap (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) = \emptyset$, showing that $t^\perp\,y' \in \mathsf{D}((X_1 \otimes X_2)^\perp)$.

Last, we must show that $t^\perp \uparrow_{|Y|} y' \subseteq \uparrow_{|X_1 \otimes X_2|} (t^\perp\,y')$ so let $(a_1,a_2) \in t^\perp\,(\uparrow_Y y')$. We have that $\downarrow_{X_1 \otimes X_2} \{(a_1,a_2)\} \cap t^\perp\,(\uparrow_Y y') \neq \emptyset$, that is $(t \downarrow_{X_1 \otimes X_2} \{(a_1,a_2)\}) \cap \uparrow_{|Y|} y' \neq \emptyset$ and hence $(t \downarrow_{X_1 \otimes X_2} \{(a_1,a_2)\}) \cap y' \neq \emptyset$ since $t \downarrow_{X_1 \otimes X_2} \{(a_1,a_2)\} \in \mathsf{D}(Y)$ by our assumption on t. Therefore we have $\downarrow_{X_1 \otimes X_2} \{(a_1,a_2)\} \cap t^\perp\,y' \neq \emptyset$, that is $(a_1,a_2) \in \uparrow_{|X_1 \otimes X_2|} (t^\perp\,y')$. $\square$

We can now easily define the functorial action of the tensor product.

Let $t_i \in \mathbf{PpL}(X_i, Y_i)$ for $i = 1, 2$. Let $t_1 \otimes t_2 \subseteq |(X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2)|$ be defined as usual as $t_1 \otimes t_2 = \{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in t_i \text{ for } i = 1, 2\}$. Then we show that $t_1 \otimes t_2 \in \mathbf{PpL}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ using Lemma 16. So let $x_i \in \mathsf{D}(X_i)$ for $i = 1, 2$. We have $(t_1 \otimes t_2)\,(x_1 \otimes x_2) = (t_1\, x_1) \otimes (t_2\, x_2) \in \mathsf{D}(Y_1 \otimes Y_2)$ since we have $t_i\, x_i \in \mathsf{D}(Y_i)$ for $i = 1, 2$. Moreover, we have

$$\begin{aligned} t_1 \otimes t_2 \,(\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) &= (t_1\,(\downarrow_{|X_1|} x_1)) \otimes (t_2\,(\downarrow_{|X_2|} x_2)) \\ &\subseteq \downarrow_{|Y_1|} (t_1\, x_1) \otimes \downarrow_{|Y_2|} (t_2\, x_2) \\ &= \downarrow_{|Y_1 \otimes Y_2|} ((t_1 \otimes t_2)\,(x_1 \otimes x_2)) \end{aligned}$$

applying Proposition 15 to t_1 and t_2.

4.2.3 Strong isomorphisms.

Let X and Y be PPs. A *strong isomorphism* from X to Y is a preorder isomorphism $\theta : |X| \to |Y|$ such that, for any $x \subseteq |X|$, one has $x \in \mathsf{D}(X)$ iff $\theta(x) \in \mathsf{D}(Y)$. A strong isomorphism from X to Y is an isomorphism (in the categorical sense), as easily seen using Lemma 16. The converse is certainly true as well, but we don't need it.

4.2.4 Associativity and symmetry isomorphisms.

Let α be the obvious bijection $|(X_1 \otimes X_2) \otimes X_3| \to |X_1 \otimes (X_2 \otimes X_3)|$. Then α is a preorder isomorphism which is also a strong isomorphism of PPs (this results actually from the forthcoming Lemma 17). Similarly, the bijection $\sigma : |X_1 \otimes X_2| \to |X_2 \otimes X_1|$ is a strong isomorphism. This shows that the category **PpL**, equipped with the above defined tensor product, is a monoidal category (of course, the unit of this tensor product is the PP $1 = (\{*\}, \{\emptyset, \{*\}\})$.

4.2.5 Linear function space and monoidal closedness.

We have already defined $X \multimap Y = (X \otimes Y^\perp)^\perp$. We show that this object is the linear function space from X to Y. This results straightforwardly from the following strong isomorphism.

Lemma 17 *The obvious bijection $\lambda : |(Z \otimes X) \multimap Y| \to |Z \multimap (X \multimap Y)|$ is a strong isomorphisms from $(Z \otimes X) \multimap Y$ to $Z \multimap (X \multimap Y)$.*

Proof. We already know that λ is a preorder isomorphism.

Let $t \in \mathsf{D}((Z \otimes X) \multimap Y)$ and let us prove that $t' = \lambda(t) \in \mathsf{D}(Z \multimap (X \multimap Y))$, using Lemma 15. So let $z \in \mathsf{D}(Z)$, we show first that $t'\, z \in \mathsf{D}(X \multimap Y)$. Let $x \in \mathsf{D}(X)$, we have $(t'\, z)\, x = t\,(z \otimes x) \in \mathsf{D}(Y)$. Next, we have $(t'\, z) \downarrow_{|X|} x = t\,(z \otimes \downarrow_{|X|} x) \subseteq t\,(\downarrow_{|Z|} z \otimes \downarrow_{|X|} x) \subseteq \downarrow_{|Y|} (t\,(z \otimes x)) = \downarrow_{|Y|} ((t'\, z)\, x)$ by Lemma 15 applied to t, and hence, by the same lemma applied to $t'\, z$, we have $t'\, z \in \mathsf{D}(X \multimap Y)$. We must show now that $t' \downarrow_{|Z|} z \subseteq \downarrow_{|X \multimap Y|} (t'\, z)$, so let $(a, b) \in t' \downarrow_{|Z|} z$. We have $b \in (t' \downarrow_{|Z|} z) \downarrow_{|X|} a = t\,(\downarrow_{|Z|} z \otimes \downarrow_{|X|} a) \subseteq \downarrow_Y (t\,(z \otimes \downarrow_{|X|} a))$ so we can find $b' \in |Y|$ with $b' \geq b$, $c \in z$ and $a' \leq a$ such that $((c, a'), b') \in t$, that is $(c, (a', b')) \in t'$. Hence $(a', b') \in t'\, z$, and therefore $(a, b) \in \downarrow_{|X \multimap Y|} (t'\, z)$ as required. □

Since we have taken $\mathbf{PpL}(X, Y) = \mathsf{D}(X \multimap Y)$ it results easily from that lemma that the monoidal category **PpL** is monoidal closed, with $X \multimap Y$ as function space.

The category **PpL** is clearly $*$-autonomous (with $\bot = 1^\bot = 1$ as dualizing object), since $X \multimap \bot = (X \otimes 1)^\bot$ and this latter PP is isomorphic to $X^\bot$ by the strong PP isomorphism which maps $a \in |X|$ to $(a, *)$, see Section 1.3.1.

4.2.6 The "par" connective. The co-tensor product, or *par*, is defined as $X \mathbin{⅋} Y = (X^\bot \otimes Y^\bot)^\bot = X^\bot \multimap Y$ and has the same associativity and symmetry properties as the tensor product. Also, there is a *mix* morphism $\mathsf{mix} : X \otimes Y \to X \mathbin{⅋} Y$, which is the diagonal set $\mathsf{mix} = \{((a,b),(a,b)) \mid a \in |X| \text{ and } b \in |Y|\}$. As it is well known, this relation is a morphism because $1 = 1^\bot = \bot$. A natural question is whether this morphism is an isomorphism, as in both categories **ScottL** and **RelL** (these categories are *compact closed*), and we provide now a counter-example showing that this is not the case in general.

4.2.7 The morphism mix is not an isomorphism in general. Let X be the PP defined by $|X| = \mathbb{N}$ (the natural numbers, with the usual order) and $\mathsf{D}(X) = \mathcal{P}(\mathbb{N})$, and let $Y = X^\bot$. We check first that the "successor" relation $s = \{(n, n+1) \mid n \in \mathbb{N}\}$ belongs to $\mathsf{D}(Y \mathbin{⅋} X) = \mathsf{D}(X \multimap X)$. Let $x \in \mathsf{D}(X) = \mathcal{P}(\mathbb{N})$. Obviously $s\,x \in \mathsf{D}(X)$, and, if $b \in s{\downarrow_X}\,x$, then we have $b > 0$ and $b - 1 \in {\downarrow_X}\,x$. Let $c \in x$ such that $c \geq b-1$. We have $c+1 \in s\,x$ and hence $b \in {\downarrow_X}\,(s\,x)$.

On the other hand, we have $\mathsf{Id} \in \mathsf{D}(Y \multimap Y) = \mathsf{D}((Y \otimes X)^\bot)$ and, since $|Y|$ is $\mathbb{N}$ with the opposite order, we have $s \cap {\downarrow_{|Y \multimap Y|}}\,\mathsf{Id} \neq \emptyset$ (indeed $s \subseteq {\downarrow_{|Y \multimap Y|}}\,\mathsf{Id}$). But $s \cap \mathsf{Id} = \emptyset$, therefore $s = \mathsf{mix}^{-1}\,s \notin \mathsf{D}(Y \otimes X)$, which shows that $\mathsf{mix}^{-1} \notin \mathbf{PpL}(Y \mathbin{⅋} X, Y \otimes X)$.

This means that **PpL** is not compact closed, see [Day77].

4.3 The additives

Given a family $(X_i)_{i \in I}$ of PPs, we define their cartesian product $X = \&_{i\in I} X_i$ by setting $|X| = \bigcup_{i \in I}\{i\} \times |X_i|$ and saying that a set $x \subseteq |X|$ belongs to $\mathsf{D}(X)$ if, for all $i \in I$, one has $\pi_i\,x \in \mathsf{D}(X_i)$ (where $\pi_i \subseteq |X \multimap X_i|$ is $\pi_i = \{((i,a),a) \mid a \in |X_i|\}$, so that $\pi_i\,x = \{a \in |X_i| \mid (i,a) \in x\}$; we'll use the notation x_i for $\pi_i\,x$ in the sequel).

One must check that $\mathsf{D}(X) = \mathsf{D}(X)^{\bot_{|X|}\bot_{|X|^{\mathrm{op}}}}$. For this it will suffice to show that, for all $x' \subseteq |X|$, one has $x' \in \mathsf{D}(X)^{\bot_{|X|}}$ iff $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$ for all $i \in I$; this will show that X defined above is a PP, with $X^\bot = \&_{i \in I} X_i^\bot$. Assume first that $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$ for all $i \in I$ and assume that ${\downarrow_{|X|}}\,x \cap x' \neq \emptyset$ for some $x \in \mathsf{D}(X)$. There exists $i \in I$ such that ${\downarrow_{|X_i|}}\,x_i \cap x'_i \neq \emptyset$, and therefore $x_i \cap x'_i \neq \emptyset$, and hence $x \cap x' \neq \emptyset$. Conversely, assume that $x' \in \mathsf{D}(X)^{\bot_{|X|}}$ and let $i \in I$, we must show that $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$. So let $y \in \mathsf{D}(X_i)$ and assume that ${\downarrow_{|X_i|}}\,y \cap x'_i \neq \emptyset$. Let $x = \{i\} \times y \subseteq |X|$, we have $x \in \mathsf{D}(X)$ (remember the definition of $\mathsf{D}(X)$ and the fact that $\emptyset \in \mathsf{D}(Y)$ for any PP Y) and ${\downarrow_{|X|}}\,x \cap x' \neq \emptyset$. Therefore we have $x \cap x' \neq \emptyset$, that is $y \cap x'_i \neq \emptyset$.

It is straightforward to check that $\&_{i \in I} X_i$ is the cartesian product of the family $(X_i)_{i\in I}$, with the relations π_i as projections.

4.4 The exponentials

We have seen that **PpL** is a cartesian $*$-autonomous category. We equip it now with an exponential comonad which will give it the structure of a Seely category in the sense of Section 1.3.

Let X be a PP. We define $!X$ by setting first $|!X| = !|X|$; remember that this means that $|!X|$ is the set of all finite multisets of elements of $|X|$, with the preorder defined as follows: $p \le q$ iff $\forall a \in p\, \exists b \in q\ a \le_{|X|} b$. Given $x \subseteq |X|$, we set $x^! = \mathcal{M}_{\mathrm{fin}}(x)$, and remember that we have the following property:

$$\mathop{\downarrow}_{|!X|}(x^!) = (\mathop{\downarrow}_{|X|} x)^! \,. \tag{2}$$

We set then

$$\mathsf{D}(!X) = \{x^! \mid x \in \mathsf{D}(X)\}^{\perp_{|!X|}\perp_{|!X|^{\mathrm{op}}}} \,.$$

Just as in Section 4.2.2, the first thing to prove is an analogue of Proposition 15 adapted to relations whose domain is an exponential.

Lemma 18 *Let X and Y be PPs and let $t \subseteq |!X \multimap Y|$. We have $t \in \mathsf{D}(!X \multimap Y)$ iff, for all $x \in \mathsf{D}(X)$,*

- $t\,x^! \in \mathsf{D}(Y)$
- *and* $\downarrow_{|Y|}(t\,x^!) = (\downarrow_{|!X \multimap Y|} t)\,(\downarrow_{|X|} x)^!$

and the second condition is equivalent to $t\,(\downarrow_{|X|} x)^! \subseteq \downarrow_{|Y|}(t\,x^!)$.

The proof is similar to that of Lemma 16.

Let $t \in \mathbf{PpL}(X,Y)$, we define $!t \subseteq |!X \multimap !Y|$ by

$$!t = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid (a_i,b_i) \in t \text{ for all } i = 1,\dots,n\} \,.$$

Using Lemma 18, we prove that $!t \in \mathbf{PpL}(!X, !Y)$. So let $x \in \mathsf{D}(X)$. We have $!t\,x^! = (t\,x)^! \in \mathsf{D}(!Y)$ since $t\,x \in \mathsf{D}(Y)$. Next we have $!t\,(\downarrow_{|X|} x)^! = (t \downarrow_{|X|} x)^! \subseteq (\downarrow_{|Y|}(t\,x))^!$ by Proposition 15 applied to t, and we conclude because $(\downarrow_{|Y|}(t\,x))^! = \downarrow_{|!Y|}(t\,x)^! = \downarrow_{|!Y|}(!t\,x^!)$, using Equation (2).

We check that the usual comonad structure of the exponential in the relational model gives rise to a comonad structure for the $!\_$ functor we have just defined.

We define first d_X as $\mathsf{d}_X = \mathsf{d}_{|X|} = \{([a],a) \mid a \in |X|\} \subseteq |!X \multimap X|$. Given $x \in \mathsf{D}(X)$, we have $\mathsf{d}_X\,x^! = x$ and $\mathsf{d}_X\,(\downarrow_{|X|} x)^! = \downarrow_{|X|} x = \downarrow_{|X|}(\mathsf{d}_X\,x^!)$ and so $\mathsf{d}_X \in \mathbf{PpL}(!X, X)$ by Lemma 18. Similarly, we define p_X as $\mathsf{p}_X = \mathsf{p}_{|X|} = \{(m_1+\cdots+m_n,[m_1,\dots,m_n]) \mid m_1,\dots,m_n \in |!X|\} \subseteq |!X \multimap !!X|$ and we show that $\mathsf{p}_X \in \mathsf{D}(!X \multimap !!X)$, using Lemma 18 again. So let $x \in \mathsf{D}(X)$, we have $\mathsf{p}_X\,x^! = x^{!!} \in \mathsf{D}(!!X)$, since $x^! \in \mathsf{D}(!X)$. Next we have $\mathsf{p}_X\,(\downarrow_{|X|} x)^! = (\downarrow_{|X|} x)^{!!} = \downarrow_{|!!X|}(x^{!!}) = \downarrow_{|!!X|}(\mathsf{p}_X\,x^!)$ and this completes the proof that p_X is a morphism.

4.4.1 Seely isomorphism.

We show that the PPs $!(X \,\&\, Y)$ and $!X \otimes !Y$ are isomorphic, by the bijection $\theta : |!(X \,\&\, Y)| \to |!X \otimes !Y|$ which maps the multiset $[(1,a_1),\dots,(1,a_l),(2,b_1),\dots,(2,b_r)]$ (with $a_i \in |X|$ and $b_j \in |Y|$) to $([a_1,\dots,a_l],[b_1,\dots,b_r])$.

We show that θ is a morphism from $!(X \mathbin{\&} Y)$ to $!X \otimes !Y$. So let $x \in \mathsf{D}(X)$ and $y \in \mathsf{D}(Y)$. We have $\theta\,\langle x,y\rangle^! = x^! \otimes y^! \in \mathsf{D}(!X \otimes !Y)$ which shows by Lemma 18 that θ is a morphism, since it is a preorder isomorphism (so that the second condition of the lemma is trivially satisfied). Conversely, let $\rho = \theta^{-1}$ and let $\rho' \subseteq |!X| \times |(!Y \multimap !(X \mathbin{\&} Y))|$ be given by

$$\rho' = \{(p,(q,m)) \mid m = \theta(p,q)\}\,.$$

By monoidal closedness, it suffices to prove that ρ' is a morphism from $!X$ to $!Y \multimap !(X \mathbin{\&} Y)$, and for this, we apply twice Lemma 18 as follows. First, let $x \in \mathsf{D}(X)$, we must show that $\rho'\,x^! \in \mathsf{D}(!Y \multimap !(X \mathbin{\&} Y))$. For this, let $y \in \mathsf{D}(Y)$, we have $(\rho'\,x^!)\,y^! = \langle x,y\rangle^! \in \mathsf{D}(!(X \mathbin{\&} Y))$. Next, we have

$$(\rho'\,x^!)\,(\underset{|Y|}{\downarrow} y)^! = \langle x, \underset{|Y|}{\downarrow} y\rangle^!$$

on the one hand and

$$\underset{|!(X\&Y)|}{\downarrow}((\rho'\,x^!)\,y^!) = \underset{|!(X\&Y)|}{\downarrow}\langle x,y\rangle^! = (\underset{|X\&Y|}{\downarrow}\langle x,y\rangle)^!$$

on the other hand, from which it clearly results that

$$(\rho'\,x^!)\,(\underset{|Y|}{\downarrow} y)^! \subseteq \underset{|!(X\&Y)|}{\downarrow}((\rho'\,x^!)\,y^!)$$

and therefore $\rho'\,x^! \in \mathsf{D}(!Y \multimap !(X \mathbin{\&} Y))$ by Lemma 18. To finish the proof, we must show that $\rho'\,(\downarrow_{|X|} x)^! \subseteq \downarrow_{|!Y\multimap!(X\&Y)|}(\rho'\,x^!)$, so let $q \in |!Y|$ and $m \in |!(X \mathbin{\&} Y)|$ and assume that $(q,m) \in \rho'\,(\downarrow_{|X|} x)^!$. There exists $p \in |!X|$ such $p \in (\downarrow_{|X|} x)^!$ and $m = \theta(p,q)$. Since $p \in (\downarrow_{|X|} x)^!$, we can find $p' \in x^!$ such that $p \leq_{|!X|} p'$. Let $m' = \theta(p',q)$, we have $(q,m') \in \rho'\,x^!$ and hence $(q,m) \in \downarrow_{|!Y\multimap!(X\&Y)|}(\rho'\,x^!)$ since $m \leq_{|(X\&Y)^!|} m'$.

There is also an obvious isomorphism from $!\top$ to 1 (the "0-ary version" of the Seely isomorphism).

4.4.2 Cartesian closedness. Equipped with this structure (the comonad $(!\_,\mathsf{d},\mathsf{p})$, the Seely isomorphism), the cartesian star-autonomous category $\mathbf{PpL}$ is a model of linear logic in the sense of Section 1.3. It gives rise therefore to a cartesian closed category, which is the Kleisli category $\mathbf{PpL}_!$ of that comonad. The cartesian product of $(X_i)_{i\in I}$ in $\mathbf{PpL}_!$ is $X = \&_{i\in I} X_i$ with projections $\pi_i \circ \mathsf{d}_X$ (simply denoted as π_i). The object of morphisms from X to Y is $X \Rightarrow Y = !X \multimap Y$ with evaluation morphism Ev (defined in Section 2.1).

4.5 The partially ordered class of PPs

Let X and Y be two PPs. We say that X is a subobject of Y and we write $X \sqsubseteq Y$ if $|X| \sqsubseteq |Y|$ (in the sense of Section 3.5) and if $\eta_{|X|,|Y|} \in \mathbf{PpL}(X,Y)$ and $\rho_{|X|,|Y|} \in \mathbf{PpL}(Y,X)$. This means that the two following conditions must hold:

$$\begin{aligned}
&\forall x \subseteq |X| \quad x \in \mathsf{D}(X) \Rightarrow x \in \mathsf{D}(Y)\\
&\forall y \subseteq |Y| \quad y \in \mathsf{D}(Y) \Rightarrow (y \cap |X| \in \mathsf{D}(X) \text{ and } (\underset{|Y|}{\downarrow} y) \cap |X| \subseteq \underset{|X|}{\downarrow}(y \cap |X|))\,.
\end{aligned}$$

Observe that, in the second condition, the converse inclusion always holds because $|X| \sqsubseteq |Y|$.

It is clear that $\sqsubseteq$ is an order relation on the class of PPs; let us denote with **PpC** the corresponding partially ordered class.

As usual, the first thing to observe is that linear negation is covariant with respect to this notion.

Lemma 19 *If $X \sqsubseteq Y$ then $X^\perp \sqsubseteq Y^\perp$.*

Proof. Same proof as for Lemma 7. $\square$

4.5.1 Completeness. We prove now that this partially ordered class has all directed lubs, in order to be able to compute least fixpoints of variable types for defining a model of the pure lambda-calculus.

We first introduce the a natural candidate of lub for a directed family.

Lemma 20 *Let $(X_\gamma)_{\gamma\in\Gamma}$ a directed family of PPs. Let $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma$ be defined as follows: $|X| = \bigsqcup_{\gamma\in\Gamma} |X_\gamma|$ (in the partially ordered class* **ScottC***) and $\mathsf{D}(X) = \{x \subseteq |X| \mid \forall\gamma \in \Gamma\ x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)\}$. Then X is a PP.*

Proof. Observe first that, if $x \in \mathsf{D}(X_\gamma)$, then $x \in \mathsf{D}(X)$. Indeed, let $\delta \in \Gamma$, we must check that $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$. So let $\varepsilon \in \Gamma$ be such that $\gamma, \delta \leq \varepsilon$. Since $X_\gamma \sqsubseteq X_\varepsilon$, we have $x \in \mathsf{D}(X_\varepsilon)$, and since $X_\delta \sqsubseteq X_\varepsilon$, we have $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$.

For proving the lemma, we build $X' = \bigsqcup_{\gamma\in\Gamma} X_\gamma^\perp$ (this makes sense since the family $(X_\gamma^\perp)_{\gamma\in\Gamma}$ is directed by Lemma 19), and we show that $X = X'^\perp$. Since obviously $|X| = |X'^\perp|$ (as preorders), it remains to show that $\mathsf{D}(X) = \mathsf{D}(X')^\perp$.

First, let $x \in \mathsf{D}(X)$ and let us show that $x \in \mathsf{D}(X')^\perp$. So let $x' \in \mathsf{D}(X')$ and assume that $\downarrow_{|X|} x \cap x' \neq \emptyset$. Let $a \in x$ and let $a' \in x'$ be such that $a' \leq_{|X|} a$. Let $\gamma \in \Gamma$ be such that $a, a' \in |X_\gamma|$ (so that $a' \leq_{|X_\gamma|} a$). We have $x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$, $x' \cap |X_\gamma| \in \mathsf{D}(X_\gamma^\perp)$ and $a' \in \downarrow_{|X_\gamma|} (x \cap |X_\gamma|) \cap (x' \cap |X_\gamma|)$, and hence $x \cap x' \neq \emptyset$.

Conversely, let $x \in \mathsf{D}(X')^\perp$, and let us show that $x \in \mathsf{D}(X)$. So let $\gamma \in \Gamma$ and let us show that $x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$. Let $x' \in \mathsf{D}(X_\gamma^\perp)$ and assume that $\downarrow_{|X_\gamma|} x \cap x' \neq \emptyset$. By our initial observation, we have $x' \in \mathsf{D}(X')$. Since $\downarrow_{|X_\gamma|} x \cap x' \neq \emptyset$, we have $\downarrow_{|X|} x \cap x' \neq \emptyset$ and hence $x \cap x' \neq \emptyset$. $\square$

Next we show that the object introduced in Lemma 20 is actually the lub of the directed family of PPs under consideration.

Lemma 21 *$\bigsqcup_{\gamma\in\Gamma} X_\gamma$ is the least upper bound of the family $(X_\gamma)_{\gamma\in\Gamma}$ in the partially ordered class* **PpC***.*

Proof. Let $\delta \in \Gamma$, we check that $X_\delta \sqsubseteq \bigsqcup_{\gamma\in\Gamma} X_\gamma = X$. We have already seen that, if $x \in \mathsf{D}(X_\delta)$, then $x \in \mathsf{D}(X)$. So let $x \in \mathsf{D}(X)$. By definition, we have $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$. We have to check that $\downarrow_{|X|} x \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (x \cap |X_\delta|)$, so let $a' \in \downarrow_{|X|} x \cap |X_\delta|$ and let $a \in x$ such that $a' \leq_{|X|} a$. We can find $\varepsilon \geq \delta$ such that $a, a' \in |X_\varepsilon|$. Then $a' \in \downarrow_{|X_\varepsilon|} x \cap |X_\delta|$ and since $X_\delta \sqsubseteq X_\varepsilon$, we have $\downarrow_{|X_\varepsilon|} x \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (x \cap |X_\delta|)$ and hence $a' \in \downarrow_{|X_\delta|} (x \cap |X_\delta|)$ as required.

Let Y be a PP such that $X_\gamma \sqsubseteq Y$ for each $\gamma \in \Gamma$ and let us show that $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma \sqsubseteq Y$. We already know that $\bigsqcup_{\gamma\in\Gamma} |X_\gamma| \sqsubseteq |Y|$. First, let $x \in \mathsf{D}(X)$ and let us show that $x \in \mathsf{D}(Y)$. So let $y' \in \mathsf{D}(Y^\perp)$ and assume that $\downarrow_{|X|} x \cap y' \neq \emptyset$. Let $a' \in \downarrow_{|X|} x \cap y'$ and let $a \in x$ be such that $a' \leq_{|X|} a$. Let $\delta \in \Gamma$ be such that $a, a' \in |X_\delta|$, so that $a' \leq_{|X_\delta|} a$. We have $a' \in \downarrow_{|X_\delta|} (x \cap |X_\delta|) \cap (y' \cap |X_\delta|)$, $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$ (by definition of X) and $y' \cap |X_\delta| \in \mathsf{D}(X_\delta^\perp)$ (since $X_\delta \sqsubseteq Y$, and by Lemma 19). Hence $x \cap y' \neq \emptyset$, and this shows that $x \in \mathsf{D}(X)$.

Next, let $y \in \mathsf{D}(Y)$. We must show first that $y \cap |X| \in \mathsf{D}(X)$, but this results immediately from the definition of X and from the fact that $X_\delta \sqsubseteq Y$ for each $\delta \in \Gamma$. Last, we must show that $\downarrow_{|Y|} y \cap |X| \subseteq \downarrow_{|X|} (y \cap |X|)$. Let $a' \in \downarrow_{|Y|} y \cap |X|$. Let $\delta \in \Gamma$ be such that $a' \in |X_\delta|$. Since $X_\delta \sqsubseteq Y$, we have $\downarrow_{|Y|} y \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (y \cap |X_\delta|)$ and we conclude because $a' \in \downarrow_{|Y|} y \cap |X_\delta|$ and, obviously, $\downarrow_{|X_\delta|} (y \cap |X_\delta|) \subseteq \downarrow_{|X|} (y \cap |X|)$. □

4.5.2 Variable PPs and least fixpoints thereof.

A *variable PP* is a functor $\Phi : \mathbf{PpC}^n \to \mathbf{PpC}$ which commutes with the lubs of directed families of PPs (as usual we say then that Φ is *continuous*).

We first observe that the standard logical operations on PPs are variable PPs.

Lemma 22 *The operations $(X, Y) \mapsto X \otimes Y$, $X \to X^I$ and $X \mapsto\ !X$ are variable PPs.*

Proof. We observe first that these operations are monotone, as in the proof of Lemma 10.

So the operation $(X, Y) \mapsto (X \multimap Y)$ is monotone, we prove that it is continuous. Let $(X_\gamma)_{\gamma \in \Gamma}$ and $(Y_\gamma)_{\gamma \in \Gamma}$ be directed families of PPs, and let X and Y be their lubs. Then $(X_\gamma \multimap Y_\gamma)_{\gamma \in \Gamma}$ is a directed family of PPs (we have just seen that $\_ \multimap \_$ is monotonous wrt. $\sqsubseteq$), let Z be its lub. We must show that $Z = X \multimap Y$. We already know that $|Z| = |X \multimap Y|$ and that $Z \sqsubseteq X \multimap Y$, so it will be enough to show that $\mathsf{D}(X \multimap Y) \subseteq \mathsf{D}(Z)$. So let $t \in \mathsf{D}(X \multimap Y)$ and let $\gamma \in \Gamma$, we must prove that $t_\gamma = t \cap |X_\gamma \multimap Y_\gamma| \in \mathsf{D}(X_\gamma \multimap Y_\gamma)$. Let $x \in \mathsf{D}(X_\gamma)$, we have $x \in \mathsf{D}(X)$ and $t_\gamma\, x = (t\, x) \cap |Y_\gamma| \in \mathsf{D}(Y_\gamma)$. Moreover, $t_\gamma \downarrow_{|X_\gamma|} x = (t \downarrow_{|X_\gamma|} x) \cap |Y_\gamma| \subseteq (t \downarrow_{|X|} x) \cap |Y_\gamma| \subseteq \downarrow_{|Y|} (t\, x) \cap |Y_\gamma|$ since $t \in \mathsf{D}(X \multimap Y)$. Therefore, since $Y_\gamma \sqsubseteq Y$, we have $t_\gamma \downarrow_{|X_\gamma|} x \subseteq \downarrow_{|Y_\gamma|} ((t\, x) \cap |Y_\gamma|) = \downarrow_{|Y_\gamma|} (t_\gamma\, x)$ (remember that $x \in \mathsf{D}(X_\gamma)$) and this concludes the proof that $t_\gamma \in \mathsf{D}(X_\gamma \multimap Y_\gamma)$, and therefore also the proof that $\_ \multimap \_$ is a variable PP.

The operation $\Phi : X \mapsto (!X)^\perp$ is monotone, and we conclude by proving that it is continuous. Let $(X_\gamma)_{\gamma \in \Gamma}$ be a directed family, let X be its lub, and let Y be the lub of the directed family $(\Phi(X_\gamma))_{\gamma \in \Gamma}$. We have $Y \sqsubseteq \Phi(X)$ and $|Y| = |\Phi(X)|$, so it will be sufficient to prove that $\mathsf{D}(\Phi(X)) \subseteq \mathsf{D}(Y)$. Let $A' \in \mathsf{D}(\Phi(X))$ and let $\gamma \in \Gamma$, we must prove that $A' \cap |\Phi(X_\gamma)| \in \mathsf{D}(\Phi(X_\gamma))$. Let $x \in \mathsf{D}(X_\gamma)$ and assume that $A' \cap \downarrow_{|!X_\gamma|} (x^!) \neq \emptyset$. Then we have $A' \cap \downarrow_{|!X|} (x^!) \neq \emptyset$ and hence $A' \cap x^! \neq \emptyset$, since $x \in \mathsf{D}(X)$, that is $(A' \cap |\Phi(X_\gamma)|) \cap x^! \neq \emptyset$. □

Of course, any variable PP $\Phi : \mathbf{PpC} \to \mathbf{PpC}$ admits a least fixpoint, namely $\bigsqcup_{k \in \mathbb{N}} \Phi^k(\top)$ (remember that $\top = (\emptyset, \{\emptyset\})$, so that $\top$ is the least element of $\mathbf{PpC}$ for the preorder $\sqsubseteq$).

4.5.3 An extensional reflexive PP.

The operation $\Phi_{\mathsf{pp}} : \mathbf{PpC} \to \mathbf{PpC}$ defined by $\Phi_{\mathsf{pp}}(X) = (!(X^{\mathbb{N}}))^\perp$ is a variable PP and has therefore a least fixpoint that we denote with $\mathcal{D}_{\mathsf{pp}}$. One checks easily (as in Section 2.3.3) that $\mathcal{D}_{\mathsf{pp}}$ is an extensional reflexive object in the CCC $\mathbf{PpL}_!$.

4.6 PPs are heterogeneous logical relations

We know from Section 2.1.2 and Section 3.4.1 that $\mathbf{Rel}_!$ and $\mathbf{ScottL}_!$ are CCCs and that $\mathbf{ScottL}_!$ is well-pointed, so we can apply to these categories the constructions of Section 1.4.2. We'll see that, up to canonical isomorphisms, $\mathbf{PpL}_!$ is a sub-cartesian closed category of $\mathsf{e}_{\mathsf{mod}}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.

If E is a set considered as an object of $\mathbf{Rel}_!$, a point of E (that is an element of $\mathbf{Rel}_!(\top, E)$) is just a subset of E. And if S is a preordered set considered as an object of $\mathbf{ScottL}_!$, a point of S is an element of $\mathcal{I}(S)$.

4.6.1 Heterogeneous relation associated with a PP. Given a PP X, we define an object $\mathsf{h}(X)$ of the category $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$ by setting $\ulcorner \mathsf{h}(X) \urcorner = |X|$ (considered as a simple set), $\llcorner \mathsf{h}(X) \lrcorner = |X|$ (considered as a preordered set) and

$$x \Vdash_{\mathsf{h}(X)} u \quad \text{if} \quad x \in \mathsf{D}(X) \text{ and } u = \mathop{\downarrow}_{|X|} x\,.$$

Given a morphism $t \in \mathbf{PpL}_!(X, Y)$, we define a pair of morphisms $\mathsf{h}(t) = (\ulcorner \mathsf{h}(t) \urcorner, \llcorner \mathsf{h}(t) \lrcorner)$ with $\ulcorner \mathsf{h}(t) \urcorner = t \in \mathbf{Rel}_!(\ulcorner \mathsf{h}(X) \urcorner, \ulcorner \mathsf{h}(Y) \urcorner)$ and $\llcorner \mathsf{h}(t) \lrcorner = \downarrow_{|!X \multimap Y|} t$, which belongs to $\mathbf{ScottL}_!(\llcorner \mathsf{h}(X) \lrcorner, \llcorner \mathsf{h}(Y) \lrcorner)$.

The next result shows that this correspondence turns $\mathbf{PpL}_!$ into a sub-cartesian closed category of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.

Theorem 23 *The operation h defined above is a full and faithful cartesian closed functor from $\mathbf{PpL}_!$ to $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.*

Proof. Observe first that $\mathsf{h}(t) \in \mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)(\mathsf{h}(X), \mathsf{h}(Y))$ (with the notations above). Indeed, due to the definition of $\Vdash_{\mathsf{h}(X)}$ and of $\Vdash_{\mathsf{h}(Y)}$, this amounts to checking that, for any $x \in \mathsf{D}(X)$, one has $t\, x^! \in \mathsf{D}(Y)$ and $\downarrow_{|Y|} (t\, x^!) = \downarrow_{|!X \multimap Y|} t\, (\downarrow_{|X|} x)^!$. This holds by Lemma 18.

Let us check the functoriality of h, so let $s \in \mathbf{PpL}_!(X, Y)$ and $t \in \mathbf{PpL}_!(Y, Z)$. One has first $\ulcorner \mathsf{h}(t \circ s) \urcorner = t \circ s = \ulcorner \mathsf{h}(t) \urcorner \circ \ulcorner \mathsf{h}(s) \urcorner$. Next, we have $\llcorner \mathsf{h}(t \circ s) \lrcorner = \downarrow_{|!X \multimap Z|} (t \circ s)$. Let $x \in \mathsf{D}(X)$. We have, applying again Lemma 18,

$$\begin{aligned}
\llcorner \mathsf{h}(t \circ s) \lrcorner (\mathop{\downarrow}_{|X|} x)^! &= \mathop{\downarrow}_{|!X \multimap Z|} (t \circ s)\, (\mathop{\downarrow}_{|X|} x)^! \\
&= \mathop{\downarrow}_{|Z|} ((t \circ s)\, x^!) \\
&= \mathop{\downarrow}_{|Z|} (t\, ((s\, x^!)^!)) \\
&= \mathop{\downarrow}_{|!Y \multimap Z|} t\, (\mathop{\downarrow}_{|Y|} (s\, x^!))^! \\
&= \mathop{\downarrow}_{|!Y \multimap Z|} t\, (\mathop{\downarrow}_{|!X \multimap Y|} s\, (\mathop{\downarrow}_{|X|} x)^!)^! \\
&= (\mathop{\downarrow}_{|!Y \multimap Z|} t \circ \mathop{\downarrow}_{|!X \multimap Y|} s)\, (\mathop{\downarrow}_{|X|} x)^!
\end{aligned}$$

and hence $\llcorner \mathsf{h}(t \circ s) \lrcorner = \llcorner \mathsf{h}(t) \lrcorner \circ \llcorner \mathsf{h}(s) \lrcorner$ because the category $\mathbf{ScottL}_!$ is well-pointed, and because any element of $\mathcal{I}(|X|)$ can be written $\downarrow_{|X|} x$ for some

$x \in \mathsf{D}(X)$ (remember that $\mathcal{I}(|X|) \subseteq \mathsf{D}(X)$). One proves similarly that identities are preserved.

Fullness of h results again from Lemma 18 (used in the converse direction). It remains to prove that this functor is cartesian closed.

Let $(X_i)_{i\in I}$ be a finite family of PPs and let $X = \&_{i\in I} X_i$, so that $\ulcorner \mathsf{h}(X) \urcorner = \&_{i\in I} \ulcorner \mathsf{h}(X_i) \urcorner$ and $\llcorner \mathsf{h}(X) \lrcorner = \&_{i\in I} \llcorner \mathsf{h}(X_i) \lrcorner$. Moreover, $\ulcorner \mathsf{h}(\pi_i) \urcorner = \pi_i$ and $\llcorner \mathsf{h}(\pi_i) \lrcorner = \downarrow_{|!X_i \multimap X_i|} \pi_i = \pi_i^{\mathsf{S}}$. Last, given $x = \langle x_i \rangle_{i\in I} \in \mathcal{P}(|X|)$ and $u = \langle u_i \rangle_{i \in I} \in \mathcal{I}(|X|)$, we have $x \Vdash_{\mathsf{h}(X)} u$ iff $x \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = u$. The first of these two conditions is equivalent to $\forall i \in I\, x_i \in \mathsf{D}(X_i)$ and the second one is equivalent to $\forall i \in I \downarrow_{|X_i|} x_i = u_i$ and therefore $x \Vdash_{\mathsf{h}(X)} u \Leftrightarrow \forall i \in I\, x_i \Vdash_{X_i} u_i$ and this shows that h commutes with cartesian products.

It remains to show that h commutes with the function space construction, so let X and Y be PPs and let $Z = (X \Rightarrow Y) = (!X \multimap Y)$. We clearly have $\ulcorner \mathsf{h}(Z) \urcorner = \ulcorner \mathsf{h}(X) \urcorner \Rightarrow \ulcorner \mathsf{h}(Y) \urcorner$ and $\llcorner \mathsf{h}(Z) \lrcorner = \llcorner \mathsf{h}(X) \lrcorner \Rightarrow \llcorner \mathsf{h}(Y) \lrcorner$. Next we have $\ulcorner \mathsf{h}(\mathsf{Ev}) \urcorner = \mathsf{Ev}$ and $\llcorner \mathsf{h}(\mathsf{Ev}) \lrcorner = \downarrow_{|Z|} \mathsf{Ev} = \mathsf{Ev}^{\mathsf{S}}$ (see Section 3.4.1). Finally, let $t \in \mathcal{P}(|Z|)$ and let $w \in \mathcal{I}(|Z|)$. Assume first that $t \Vdash_{\mathsf{h}(Z)} w$, that is $t \in \mathsf{D}(Z)$ and $\downarrow_{|Z|} t = w$. We must prove that $t \Vdash_{\mathsf{h}(X) \Rightarrow \mathsf{h}(Y)} w$. So let $x \in \mathcal{P}(|X|)$ and $u \in \mathcal{I}(|X|)$ be such that $x \Vdash_X u$, that is $x \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = u$. By definition of $t(x)$ and $w(u)$ (see Section 1.1), we have $t(x) = t\, x^!$ and $w(u) = w\, u^! = (\downarrow_{|Z|} t)\, (\downarrow_{|X|} x)^! = \downarrow_{|Y|} (t(x))$ by Lemma 18. By the same lemma, we have $t(x) \in \mathsf{D}(Y)$, and hence $t(x) \Vdash_{\mathsf{h}(Y)} w(u)$ as required. Conversely, assume that $t \Vdash_{\mathsf{h}(X) \Rightarrow \mathsf{h}(Y)} w$; we must prove that $t \Vdash_{\mathsf{h}(Z)} w$. We apply again Lemma 18, so let $x \in \mathsf{D}(X)$. We have $x \Vdash_X \downarrow_{|X|} x$ and hence $t(x) \in \mathsf{D}(Y)$ (that is $t\, x^! \in \mathsf{D}(Y)$) and $\downarrow_{|Y|} (t\, x^!) = w\, (\downarrow_{|X|} x)^!$ (by definition of $\Vdash_{\mathsf{h}(X) \Rightarrow \mathsf{h}(Y)}$). We prove that $\downarrow_{|Z|} t = w$. Let $(m, b) \in |Z|$. We have $\downarrow_{|Y|} (t\, (\downarrow_{|X|} \mathsf{supp}(m))^!) = w\, (\downarrow_{|X|} \mathsf{supp}(m))^!$. Assume first that $(m, b) \in \downarrow_{|Z|} t$ and let $(m', b') \in t$ be such that $(m, b) \leq_{|Z|} (m', b')$. Then $m' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ and hence $b \in \downarrow_{|Y|} (t\, (\downarrow_{|X|} \mathsf{supp}(m))^!)$. So let $m'' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ be such that $(m'', b) \in w$. Since $w \in \mathcal{I}(|Z|)$, we have $(m, b) \in w$. Conversely, assume that $(m, b) \in w$. Since $m \in (\downarrow_{|X|} \mathsf{supp}(m))^!$, we have $b \in \downarrow_{|Y|} (t\, (\downarrow_{|X|} \mathsf{supp}(m))^!)$ so we can find $(m', b') \in t$ such that $m' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ and $b \leq b'$, that is $(m, b) \leq_{|Z|} (m', b')$, which show that $(m, b) \in \downarrow_{|Z|} t$. Therefore, x being an element of $\mathsf{D}(X)$, we have $\downarrow_{|Y|} (t\, x^!) = \downarrow_{|Z|} t\, (\downarrow_{|X|} x)^!$ and so $t \in \mathsf{D}(Z)$ by Lemma 18. This concludes the proof that $t \Vdash_Z w$, and therefore we have $\mathsf{h}(Z) = \mathsf{h}(X) \Rightarrow \mathsf{h}(Y)$. Therefore h is a CCC functor. $\square$

So we can consider $\mathbf{PpL}_!$ as a sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, and, considered as objects of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, all the objects of $\mathbf{PpL}_!$ are modest. In order to show that $\mathbf{ScottL}_!$ represents the extensional collapse of $\mathbf{Rel}_!$ in the sense of Section 1.4.2, we must show that the functor $\varepsilon : \mathbf{PpL}_! \to \mathbf{PerL}_!$ is a CCC functor since we have seen in Section 2.2.7 that $\mathbf{PerL}_!$ is a sub-CCC of $\mathsf{e}(\mathbf{Rel}_!)$ through an inclusion functor denoted as q.

4.7 A functor from PPs to PER-objects

We first define this functor on the linear category instead of defining only on the cartesian closed category. In order to avoid confusion and clarify the situation, we give a different name to this larger functor and call it ε_0.

Given a PP X, we obviously define a PER (denoted with B_X for the time being) on $\mathcal{P}(|X|)$ by saying that $x\ B_X\ y$ if $x, y \in \mathsf{D}(X)$ and $\downarrow_X x = \downarrow_X y$. Observe that $x\ B_X \downarrow_X x$ for any $x \in \mathsf{D}(X)$.

The first thing to prove is that this definition gives rise to a PER which satisfies the closure property of a PER-object.

Lemma 24 *For any PP X, one has $B_X^\perp = B_{X^\perp}$ and therefore $B_X^{\perp\perp} = B_X$.*

Proof. Let $x', y' \subseteq |X|$. Assume first that $x'\ B_X^\perp\ y'$ and let us show that $x'\ B_{X^\perp}\ y'$. We prove first that $x' \in \mathsf{D}(X)^\perp$, so let $x \in \mathsf{D}(X)$, and assume that $x' \cap \downarrow_{|X|} x \neq \emptyset$, we must show that $x' \cap x \neq \emptyset$. This results from the fact that $x\ B_X \downarrow_{|X|} x$. Similarly we get $y' \in \mathsf{D}(X)^\perp$. We must show now that $\uparrow_{|X|} x' = \uparrow_{|X|} y'$, so let $a \in \uparrow_{|X|} x'$. This means that $\downarrow_{|X|} a \cap x' \neq \emptyset$. Since $\downarrow_{|X|} a\ B_X \downarrow_{|X|} a$, we get $\downarrow_{|X|} a \cap y' \neq \emptyset$, that is $a \in \uparrow_{|X|} y'$.

Conversely, assume that $x'\ B_{X^\perp}\ y'$ and let us show that $x'\ B_X^\perp\ y'$. So let $x, y \subseteq |X|$ be such that $x\ B_X\ y$, and assume that $x \cap x' \neq \emptyset$; we must show that $y \cap y' \neq \emptyset$. We have *a fortiori* $\downarrow_{|X|} x \cap \uparrow_{|X|} x' \neq \emptyset$, that is $\downarrow_{|X|} y \cap \uparrow_{|X|} y' \neq \emptyset$. But then, since $y \in \mathsf{D}(X)$ and $y' \in \mathsf{D}(X)^\perp$, we get $y \cap y' \neq \emptyset$. □

We can rephrase this result as follows.

Lemma 25 *For any PP X, $\varepsilon_0(X) = (|X|, B_X)$ is a PER-object and we have $\varepsilon_0(X^\perp) = \varepsilon_0(X)^\perp$.*

The relation B_X can therefore also be denoted with $\sim_{\varepsilon_0(X)}$.

Next we show that, at linear function types, this PER admits a functional characterization: it is a "linear logical relation".

Lemma 26 *Let X and Y be PPs and let $s_1, s_2 \in \mathcal{P}(|X \multimap Y|)$. One has $s_1 \sim_{\varepsilon_0(X \multimap Y)} s_2$ iff for all $x_1, x_2 \in \mathcal{P}(|X|)$, if $x_1 \sim_{\varepsilon_0(X)} x_2$ then $s_1\, x_1 \sim_{\varepsilon_0(Y)} s_2\, x_2$. This means that $\varepsilon_0(X \multimap Y) = \varepsilon_0(X) \multimap \varepsilon_0(Y)$.*

Proof. Assume first that $s_1 \sim_{\varepsilon_0(X \multimap Y)} s_2$. Let $x_1, x_2 \subseteq |X|$ be such that $x_1 \sim_{\varepsilon_0(X)} x_2$, we want to show that $s_1\, x_1 \sim_{\varepsilon_0(Y)} s_2\, x_2$. Let $y'_1, y'_2 \subseteq |Y|$ be such that $y'_1 \sim_{\varepsilon_0(Y^\perp)} y'_2$. One has $(s_1\, x_1) \cap y'_1 \neq \emptyset$ iff $s_1 \cap (x_1 \times y'_1) \neq \emptyset$ and, since $x_1 \in \mathsf{D}(X)$ and $y'_1 \in \mathsf{D}(Y)^\perp$, this latter condition holds iff $s_1 \cap \downarrow_{|X \otimes Y^\perp|} (x_1 \times y'_1) \neq \emptyset$, which in turn is equivalent to $\downarrow_{|X \multimap Y|} s_1 \cap \downarrow_{|X \otimes Y^\perp|} (x_1 \times y'_1) \neq \emptyset$ since $s_1 \in \mathsf{D}(X \multimap Y)$. Since $\downarrow_{|X \multimap Y|} s_1 = \downarrow_{|X \multimap Y|} s_2$ (because $s_1 \sim_{\varepsilon_0(X \multimap Y)} s_2$) and $\downarrow_{|X \otimes Y^\perp|} (x_1 \times y'_1) = \downarrow_{|X \otimes Y^\perp|} (x_2 \times y'_2)$ (because $x_1 \sim_{\varepsilon_0(X)} x_2$ and $y'_1 \sim_{\varepsilon_0(Y^\perp)} y'_2$), we conclude that $(s_1\, x_1) \cap y'_1 \neq \emptyset \Leftrightarrow (s_1\, x_2) \cap y'_2 \neq \emptyset$, and this shows that $s_1\, x_1 \sim_{\varepsilon_0(Y)} s_2\, x_2$ by Lemma 24.

Conversely, assume that $s_1\, x_1 \sim_{\varepsilon_0(Y)} s_2\, x_2$ whenever $x_1 \sim_{\varepsilon_0(X)} x_2$, and let us show that $s_1 \sim_{\varepsilon_0(X \multimap Y)} s_2$. Observe that our assumption implies that $s_1\, x_1 \sim_{\varepsilon_0(Y)} s_1\, x_2$ (indeed, $x_2 \sim_{\varepsilon_0(X)} x_2$, hence $s_1\, x_2 \sim_{\varepsilon_0(Y)} s_2\, x_2$ and we can apply transitivity of the relation $\sim_{\varepsilon_0(Y)}$). We show first that $s_1 \in \mathsf{D}(X \multimap Y)$. So let $x \in \mathsf{D}(X)$. We have $x \sim_{\varepsilon_0(X)} x$ and hence $s_1\, x \sim_{\varepsilon_0(Y)} s_2\, x$, which implies $s_1\, x \in \mathsf{D}(X)$. Let $b \in s_1 \downarrow_{|X|} x$, we show that $b \in \downarrow_{|Y|} (s_1\, x)$. We have $x \sim_{\varepsilon_0(X)} \downarrow_{|X|} x$ and hence $s_1\, x \sim_{\varepsilon_0(Y)} s_1 \downarrow_{|X|} x$ which implies $\downarrow_{|Y|} (s_1\, x) = \downarrow_{|Y|} (s_1 \downarrow_{|X|} x)$ and we conclude since $b \in \downarrow_{|Y|} (s_1 \downarrow_{|X|} x)$. By Proposition 15, we have $s_1 \in \mathsf{D}(X \multimap Y)$, and of course the same holds for s_2 by symmetry. It remains to show that $\downarrow_{|X \multimap Y|} s_1 = \downarrow_{|X \multimap Y|} s_2$.

Let $(a,b) \in \downarrow_{|X \multimap Y|} s_1$. This means that $\downarrow_{|X \otimes Y^\perp|} (a,b) \cap s_1 \neq \emptyset$, that is $(s_1 \downarrow_{|X|} a) \cap \uparrow_{|Y|} b \neq \emptyset$. But $\downarrow_X a \sim_{\varepsilon_0(X)} \downarrow_X a$ and hence $s_1 \downarrow_{|X|} a \sim_{\varepsilon_0(Y)} s_2 \downarrow_{|X|} a$ and since $\uparrow_{|Y|} b \sim^\perp_{\varepsilon_0(Y)} \uparrow_{|Y|} b$, we have $(s_2 \downarrow_{|X|} a) \cap \uparrow_{|Y|} b \neq \emptyset$, that is $(a,b) \in \downarrow_{|X \multimap Y|} s_2$. □

In particular, for any PPs X and Y, one has $\mathbf{PpL}(X,Y) = \mathbf{PerL}(\varepsilon_0(X), \varepsilon_0(Y))$ and so the operation ε_0 is a full and faithful functor, which is the identity on morphisms. Indeed, composition of morphisms is defined in the same way in both categories, as the standard composition of relations.

Next we prove that the functor ε_0 commutes on the nose with all constructions of linear logic: ε_0 is an LL-functor in the sense of Section 1.3.4.

Lemma 27 *Let X and Y be PPs. We have $\varepsilon_0(X \otimes Y) = \varepsilon_0(X) \otimes \varepsilon_0(Y)$, that is, the functor ε_0 is strict monoidal.*

Proof. Apply the fact that $X \otimes Y = (X \multimap Y^\perp)^\perp$, Lemma 25 and Lemma 26. □

Lemma 28 *The functor ε_0 commutes with all cartesian products.*

The proof is a straightforward verification.

Lemma 29 *Let X be a PP, one has $\varepsilon_0(!X) = !\varepsilon_0(X)$.*

Proof. By Lemma 25, it suffices to show that $\varepsilon_0(!X)^\perp = (!\varepsilon_0(X))^\perp$. Let $A'_1, A'_2 \subseteq |!X|$.

On the one hand, $A'_1 \sim_{\varepsilon_0(!X)^\perp} A'_2$ means that $A'_1 \sim^\perp_{\varepsilon_0(!X)} A'_2$, that is

$$\forall A_1, A_2 \subseteq |!X| \; A_1 \sim_{\varepsilon_0(!X)} A_2 \Rightarrow (A_1 \cap A'_1 \neq \emptyset \Leftrightarrow A_2 \cap A'_2 \neq \emptyset)\,,$$

and remember that $A_1 \sim_{\varepsilon_0(!X)} A_2$ means that $A_1, A_2 \in \mathsf{D}(!X)$ and $\downarrow_{|!X|} A_1 = \downarrow_{|!X|} A_2$. By Lemma 25, $A'_1 \sim_{\varepsilon_0(!X)^\perp} A'_2$ is also equivalent to $A'_1 \sim_{\varepsilon_0((!X)^\perp)} A'_2$, that is

$$A'_1, A'_2 \in \mathsf{D}(!X)^\perp \quad \text{and} \quad \underset{|!X|}{\uparrow} A'_1 = \underset{|!X|}{\uparrow} A'_2\,. \tag{3}$$

On the other hand, $A'_1 \sim_{(!\varepsilon_0(X))^\perp} A'_2$ means that $A'_1 \sim^\perp_{!\varepsilon_0(X)} A'_2$, that is

$$\forall x_1, x_2 \subseteq |X| \; x_1 \sim_{\varepsilon_0(X)} x_2 \Rightarrow (x_1^! \cap A'_1 \neq \emptyset \Leftrightarrow x_2^! \cap A'_2 \neq \emptyset)$$

and remember that $x_1 \sim_{\varepsilon_0(X)} x_2$ means that $x_1, x_2 \in \mathsf{D}(X)$ and $\downarrow_{|X|} x_1 = \downarrow_{|X|} x_2$.

Hence $x_1 \sim_{\varepsilon_0(X)} x_2$ implies $x_1^!, x_2^! \in \mathsf{D}(!X)$ and $\downarrow_{|!X|} x_1^! = (\downarrow_{|X|} x_1)^! = (\downarrow_{|X|} x_2)^! = \downarrow_{|!X|} x_2^!$, that is $x_1^! \sim_{\varepsilon_0(!X)} x_2^!$ and hence $A'_1 \sim^\perp_{\varepsilon_0(!X)} A'_2 \Rightarrow A'_1 \sim^\perp_{!\varepsilon_0(X)} A'_2$.

Let us prove the converse implication, so assume that $A'_1 \sim^\perp_{!\varepsilon_0(X)} A'_2$ and let us prove that property (3) holds. We prove first that $A'_1 \in \mathsf{D}(!X)^\perp$. So let $x \in \mathsf{D}(X)$ and assume that $A'_1 \cap x^! = \emptyset$. Since $x \sim_{\varepsilon_0(X)} \downarrow_{|X|} x$, we have $x^! \sim_{!\varepsilon_0(X)} (\downarrow_{|X|} x)^! = \downarrow_{|!X|} (x^!)$, and hence $A'_1 \cap \downarrow_{|!X|} (x^!) = \emptyset$ since we have $A'_1 \sim^\perp_{!\varepsilon_0(X)} A'_1$. It remains to show that $\uparrow_{|!X|} A'_1 = \uparrow_{|!X|} A'_2$, we only prove the

"$\subseteq$" inclusion. So let $m \in |!X|$ and assume that $m \in \uparrow_{|!X|} A'_1$. This means that $A'_1 \cap \downarrow_{|!X|} m \neq \emptyset$, and since $\downarrow_{|!X|} m \sim_{!\varepsilon_0(X)} \downarrow_{|!X|} m$, we have $m \in \uparrow_{|!X|} A'_2$. $\square$

Theorem 30 *The functor ε_0 is an LL-functor.*

Proof. This results from Lemmas 26, 27, 28 and 29, from the fact that ε_0 acts trivially on morphisms and from the fact that the operations on morphisms are defined in the same way in both categories. $\square$

It follows that ε_0 is a cartesian closed functor from $\mathbf{PpL}_!$ to $\mathbf{PerL}_!$ which itself is a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!)$ through the inclusion functor q (see Section 2.2.7). Moreover, when considering $\mathbf{PpL}_!$ as a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, the functor ε and ε_0 coincide. This can be stated more precisely as follows (we recall that the definition of the functor h is given in Section 4.6.1).

Theorem 31 *We have $\varepsilon \circ \mathsf{h} = \mathsf{q} \circ \varepsilon_0 : \mathbf{PpL}_! \to \mathsf{e}(\mathbf{Rel}_!)$.*

The proof is a straightforward verification. It follows that, when restricted to the image $\mathcal{H}$ of the full and faithful CCC functor h, which is a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, the functor $\varepsilon : \mathcal{H} \to \mathsf{e}(\mathbf{Rel}_!)$ is cartesian closed.

Let U be a discrete object of $\mathsf{e}(\mathbf{Rel}_!)$; this means that $\sim_U$ is the equality on $\mathcal{P}(|U|)$. Let X be the PP defined by: $|X| = |U|$ with the discrete preorder relation, and $\mathsf{D}(X) = \mathcal{P}(|U|)$. Then one has $\varepsilon_0(X) = U$ and all the conditions of 1.4.2 are fulfilled. We can state the main result of the paper.

Theorem 32 $\mathbf{ScottL}_!$ *represents the extensional collapse of the category* $\mathbf{Rel}_!$.

Our purpose now is to extend this result to reflexive objects, according to Section 1.4.4. But before that we give more information about the forgetful functor $\mathbf{PpL} \to \mathbf{ScottL}$.

4.8 A functor from PPs to preorders

Remember from Section 1.4.2 that there is a second projection CCC-functor $\sigma : \mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!) \to \mathbf{ScottL}_!$ which induces a CCC-functor $\mathbf{PpL}_! \to \mathbf{ScottL}_!$ by composition with the full and faithful CCC-functor $\mathsf{h} : \mathbf{PpL}_! \to \mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$ of Section 4.6.1. We want to show that this functor is induced by an LL-functor σ_0 from $\mathbf{PpL}$ to $\mathbf{ScottL}$. This will complete the linear picture of the extensional collapse and will be useful for dealing with the extensional collapse of reflexive objects.

Given a PP X, we set $\sigma_0(X) = |X|$, which is a preorder. Given two PPs X and Y and $t \in \mathbf{PpL}(X, Y) = \mathsf{D}(X \multimap Y)$, we set

$$\sigma_0(t) = \mathop{\downarrow}_{|X \multimap Y|} t \in \mathcal{I}(|X \multimap Y|) \simeq \mathbf{ScottL}(|X|, |Y|)\,.$$

In other words, the linear map $\sigma_0(t) : \mathcal{I}(|X|) \to \mathcal{I}(|Y|)$ is given by $\sigma_0(t)(x) = \downarrow_{|Y|} (t\,x)$ (see Lemma 12).

Lemma 33 *The operation σ_0 on morphisms is a functor, that is $\sigma_0(\mathsf{Id}_X) = \mathsf{Id}^{\mathsf{S}}_X$ and, given $s \in \mathbf{PpL}(X, Y)$ and $t \in \mathbf{PpL}(Y, Z)$, one has $\sigma_0(t\,s) = \sigma_0(t)\,\sigma_0(s)$.*

Proof. See Section 4.2.1, where the proof is given. □

Theorem 34 *The functor σ_0 is an LL-functor.*

Proof. This is a routine verification.

As an example, let X and Y be PPs. We have $\sigma_0(!X) = |!X| = !|X| = !\sigma_0(X)$. Let $t \in \mathbf{PpL}(X, Y)$, we prove that $\sigma_0(!t) = !\sigma_0(t)$. Let $(p, q) \in |!X| \times |!Y|$. If $(p, q) \in \sigma_0(!t)$, we can find $(p', q') \in !t$ such that $p' \leq_{|!X|} p$ and $q \leq_{|!Y|} q'$; we show that $(p, q) \in !\sigma_0(t) = !(\downarrow_{|X\multimap Y|} t)$. Let $b \in q$, let $b' \in q'$ such that $b \leq_{|Y|} b'$. Let $a' \in p'$ be such that $(a', b') \in t$ (since $(p', q') \in !t$). Let $a \in p$ be such that $a' \leq_{|X|} a$ (since $p' \leq_{|!X|} p$). We have $(a', b') \in t$ and $(a, b) \leq_{|X\multimap Y|} (a', b')$, hence $(a, b) \in \sigma_0(t)$ and this shows that $(p, q) \in !\sigma_0(t)$.

Assume conversely that $(p, q) \in !\sigma_0(t)$ and let us show that $(p, q) \in \sigma_0(!t)$. Let us write $q = [b_1, \dots, b_n]$. For each $i \in \{1, \dots n\}$, let us choose $a_i \in p$ such that $(a_i, b_i) \in \sigma_0(t) = \downarrow_{|X\multimap Y|} t$ and let $(a'_i, b'_i) \in t$ be such that $a'_i \leq_{|X|} a_i$ and $b_i \leq_{|Y|} b'_i$. Let $p' = [a'_1, \dots, a'_n]$ and $q' = [b'_1, \dots, b'_n]$. We have $(p', q') \in !t$. Moreover, we have $q \leq_{|!Y|} q'$ and $p' \leq_{|!X|} p$ and hence $(p, q) \in \sigma_0(!t)$ as required.

Last, let us check that $\sigma_0(\mathsf{p}_X) = \mathsf{p}^{\mathsf{S}}_{\sigma_0(X)}$. Let $(p, P) \in !|X| \times !!|X|$, so that P can be written $P = [p_1, \dots, p_n]$ with $p_1, \dots, p_n \in |!X|$. Assume first that $(p, P) \in \sigma_0(\mathsf{p}_X) = \downarrow_{|!X\multimap !!X|} \mathsf{p}_X$ and let us show that $(p, P) \in \mathsf{p}^{\mathsf{S}}_{\sigma_0(X)}$, that is $p_1 + \cdots + p_n \leq_{!|X|} p$. So let $a \in p_1 + \cdots + p_n$, and let $i \in \{1, \dots, n\}$ be such that $a \in p_i$. Let $(p', P') \in \mathsf{p}_X$ be such that $p' \leq_{!|X|} p$ and $P \leq_{!!|X|} P'$, so that $P' = [p'_1, \dots, p'_k]$ with $p' = p'_1 + \cdots + p'_k$. Let $j \in \{1, \dots, k\}$ be such that $p_i \leq_{!|X|} p'_j$. Let $a' \in p'_j$ be such that $a \leq_{|X|} a'$ (remember that $a \in p_i$). Then we have $a' \in p'$ and hence we can find $a'' \in p$ such that $a' \leq_{|X|} a''$. This shows that $p_1 + \cdots + p_n \leq_{!|X|} p$ as required. Conversely, assume that $(p, P) \in \mathsf{p}^{\mathsf{S}}_{\sigma_0(X)}$ (that is $p_1 + \cdots + p_n \leq_{!|X|} p$) and let us show that $(p, P) \in \sigma_0(\mathsf{p}_X)$. We have $(p_1 + \cdots + p_n, P) \in \mathsf{p}_X$ by definition of p_X and we have $(p, P) \leq_{|!X\multimap !!X|} (p_1 + \cdots + p_n, P)$ since $p_1 + \cdots + p_n \leq_{!|X|} p$. Therefore $(p, P) \in \sigma_0(\mathsf{p}_X)$ as announced. □

It follows that σ_0 is a cartesian closed functor from $\mathbf{PpL}_!$ to $\mathbf{ScottL}_!$. This functor σ_0 is related with σ by the following property.

Proposition 35 *One has $\sigma_0 = \sigma \circ \mathsf{h} : \mathbf{PpL}_! \to \mathbf{ScottL}_!$.*

The proof is a straightforward verification.

4.9 Extensional collapse of the reflexive object

It is straightforward from the definition of $\mathbf{PpC}$ that σ_0 is a continuous class function from $\mathbf{PpC}$ to $\mathbf{ScottC}$.

Remember from Section 4.5.3 that we have defined a reflexive object $\mathcal{D}_{\mathsf{pp}}$ in $\mathbf{PpL}_!$ as the least fixpoint of a continuous class function $\Phi_{\mathsf{pp}} : \mathbf{PpC} \to \mathbf{PpC}$, in other words $\mathcal{D}_{\mathsf{pp}} = \bigsqcup_{n\in\mathbb{N}} \Phi^n_{\mathsf{pp}}(\top)$. By continuity of σ_0, we have $\sigma_0(\mathcal{D}_{\mathsf{pp}}) = \bigsqcup_{n\in\mathbb{N}} \sigma_0(\Phi^n_{\mathsf{pp}}(\top)) = \bigsqcup_{n\in\mathbb{N}} \Phi^n_{\mathsf{e}}(\top) = \mathcal{D}_{\mathsf{s}}$ (see Section 2.3.3) since σ_0 is an LL-functor from $\mathbf{PpL}$ to $\mathbf{ScottL}$.

Setting $\mathcal{D}_{\mathsf{h}} = \mathsf{h}(\mathcal{D}_{\mathsf{pp}})$, we define a reflexive object in $\mathsf{e}(\mathbf{PerL}_!, \mathbf{ScottL}_!)$ which satisfies $\sigma(\mathcal{D}_{\mathsf{h}}) = \mathcal{D}_{\mathsf{s}}$ by Proposition 35.

On the other hand, it is clear that the first component $\ulcorner\mathcal{D}_{\mathsf{h}}\urcorner$ of $\mathcal{D}_{\mathsf{h}} = (\ulcorner\mathcal{D}_{\mathsf{h}}\urcorner, \llcorner\mathcal{D}_{\mathsf{h}}\lrcorner, \Vdash_{\mathcal{D}_{\mathsf{h}}})$ (see Section 1.4.2) coincides with $\mathcal{D}_{\mathsf{r}}$, so that we have proved the following result.

Theorem 36 *The reflexive object $\mathcal{D}_{\mathsf{s}}$ is the extensional collapse of $\mathcal{D}_{\mathsf{r}}$ in the sense of Section 1.4.2.*

4.9.1 Continuity of ε_0. To conclude the paper, we show that the reflexive objects defined in $\mathbf{PpL}_!$ and $\mathbf{PerL}_!$ are related by the functor ε_0.

Let X and Y be PPs such that $X \sqsubseteq Y$. Since $\eta_{|X|,|Y|} \in \mathbf{PpL}(X, Y)$ and since ε_0 acts trivially on morphisms, we have $\eta_{|X|,|Y|} \in \mathbf{PerL}(\varepsilon_0(X), \varepsilon_0(Y))$. Similarly, we have $\rho_{|X|,|Y|} \in \mathbf{PerL}(\varepsilon_0(Y), \varepsilon_0(X))$. Therefore $\varepsilon_0(X) \sqsubseteq \varepsilon_0(Y)$, that is ε_0 is a monotone class function from $\mathbf{PpC}$ to $\mathbf{PerC}$.

Theorem 37 *The monotone class function $\varepsilon_0 : \mathbf{PpC} \to \mathbf{PerC}$ is continuous.*

Proof. Let $(X_\gamma)_{\gamma\in\Gamma}$ be a directed family of PPs and let $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma \in \mathbf{PpC}$. We already know that $|X| = \bigcup_{\gamma\in\Gamma} |X_\gamma|$ and so we have to prove that, given $x, y \subseteq |X|$, the two following conditions are equivalent:

1. $x, y \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = \downarrow_{|X|} y$
2. for all $\gamma \in \Gamma$, $x \cap |X_\gamma|$, one has $y \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$ and $\downarrow_{|X_\gamma|} (x \cap |X_\gamma|) = \downarrow_{|X_\gamma|} (y \cap |X_\gamma|)$.

That (1) implies (2) results from the monotonicity of ε_0 (for each $\gamma \in \Gamma$, we have $X_\gamma \sqsubseteq X$ and hence $\varepsilon_0(X_\gamma) \sqsubseteq \varepsilon_0(X)$), so let us prove the converse and assume that (2) holds. That $x, y \in \mathsf{D}(X)$ results directly from the definition of X (see Section 4.5.1). We conclude by checking that $\downarrow_{|X|} x \subseteq \downarrow_{|X|} y$. For this, it is sufficient to have $x \subseteq \downarrow_{|X|} y$, so let $a \in x$. Let $\gamma \in \Gamma$ be such that $a \in x \cap |X_\gamma|$. By assumption, $a \in \downarrow_{|X_\gamma|} (y \cap |X_\gamma|)$, so let $b \in y \cap |X_\gamma|$ be such that $a \leq_{|X_\gamma|} b$. Since $|X|$ is the lub of the $|X_\gamma|$s in the partially ordered class $\mathbf{ScottC}$, we have $a \leq_{|X|} b$ and this concludes the proof. □

As a consequence of this result, we have $\varepsilon_0(\mathcal{D}_{\mathsf{pp}}) = \mathcal{D}_{\mathsf{e}}$ (where $\mathcal{D}_{\mathsf{e}}$ is defined in Section 2.3.3).

Conclusion and further work

One of our motivations in this work was to understand more deeply the connection between the quantitative and the qualitative approach to the denotational semantics of linear logic. The quantitative model **Rel** provides very detailed information about the use of resources by programs, but the price to pay is that the interpretations in this model tend to be infinite even for very simple types and expressions. On the other hand, the qualitative semantics **ScottL** forgets a lot of information, keeping track only of the presence of elementary tokens in the interpretation of programs, and not of their quantities. Consider for instance the finite type hierarchy based on booleans. In both models, the basic type **Bool** has four points (the four subsets of $\{\mathbf{t}, \mathbf{f}\}$). In $\mathbf{ScottL}_!$, the interpretation of all types remain finite, whereas in $\mathbf{Rel}_!$, all types but **Bool** have an infinite interpretation. This means that the extensional collapse of $\mathbf{Rel}_!$ skims off a very small part of the quantitative model and we aimed at a more concrete grasp of this selection process: the present work is a first step in this direction, the main tool for understanding the situation being the concept of PP.

A remarkable difference between the two models is that **Rel** accommodates differential linear logic and differential interaction nets whereas **ScottL** doesn't.

The main novelty of differential linear logic [ER06] with respect to ordinary linear logic is the existence of a *codereliction* rule of type $A \multimap !A$. In the general categorical setting of Section 1.3.1, this rule must be interpreted as a natural transformation $\partial_X \in \mathcal{C}(X, !X)$. When trying to find such a codereliction morphism in the preorder model of Section 3, the only possibility is to define it as $\partial_S \in \mathcal{I}(S \multimap !S)$ given by $\partial_S = \{(a, m) \mid \forall b \in m\ b \leq a\}$ because we must have $\mathsf{d}_S \circ \partial_S = \mathsf{Id}_S$. The problem is that one does not define a natural transformation in that way. The reason for this phenomenon is that differential linear logic is a fundamentally *quantitative* logic which allows to count (by means of codereliction, precisely) how many times a piece of data is used by a function. This quantitative information is lost in the preorder model. In order to understand the collapse in a syntactic setting, we would like to endow the simple resource terms of [ER08] with a PP structure (as we did with a finiteness structure in [Ehr10]) in order to characterize the sets of resource terms which are "extensional": we know that those which arise in the Taylor expansion of lambda-terms have this property and we would like to know if there are more.

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