A Relational Semantics for Parallelism and Non-Determinism in a Functional Setting☆

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Abstract

We recently introduced an extensional model of the pure λ-calculus living in a canonical cartesian closed category of sets and relations [6]. In the present paper, we study the non-deterministic features of this model. Unlike most traditional approaches, our way of interpreting non-determinism does not require any additional powerdomain construction. We show that our model provides a straightforward semantics of *non-determinism* (*may* convergence) by means of *unions* of interpretations, as well as of *parallelism* (*must* convergence) by means of a binary, non-idempotent operation available on the model, which is related to the *mix rule* of Linear Logic. More precisely, we introduce a λ-calculus extended with non-deterministic choice and parallel composition, and we define its operational semantics (based on the *may* and *must* intuitions underlying our two additional operations). We describe the interpretation of this calculus in our model and show that this interpretation is 'sensible' with respect to our operational semantics: a term converges if, and only if, it has a non-empty interpretation.

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Contents

Introduction

Pure and typed λ-terms are specifications of sequential and deterministic processes. Several extensions of the λ-calculus with parallel and/or non-deterministic constructs have been proposed in the literature, either to increase the expressive power of the language, in the typed [21, 19, 16] and untyped [4, 5] settings, or to study the interplay between higher order features and parallel/non-deterministic features [18, 9, 10].

When introducing non-determinism in a functional setting, it is crucial to specify what notion of convergence is chosen. Two widely used notions are:

- the *must* convergence: a non-deterministic choice converges if all its components do. This characterizes the *demonic* non-determinism.
- the *may* convergence: a non-deterministic choice converges if at least one of its components does. This characterizes the *angelic* non-determinism.

The usual denotational models of functional calculi do not accommodate *may* non-determinism: let TRUE and FALSE be two convergent terms[2], whose denotations in standard models are distinct. What semantic value should take the non-deterministic term TRUE + FALSE, which *may* converges to TRUE and to FALSE? The value should be both TRUE and FALSE if we want the semantics to be invariant under reduction!

The typical way of interpreting "multi-valued" terms, like the one above, is to use models based on *powerdomains* [20], often defined as filter models with respect to suitable notions of intersection and union types [9, 10]. The semantics of TRUE + FALSE becomes some kind of join of both values, available in the powerdomain (similar techniques are also used for interpreting *must* non-determinism). In this framework, both kinds of non-determinism are modelled by some idempotent, commutative and associative operations.

In a recent paper [13], Faure and Miquel define a categorical counterpart of the syntactical notion of parallel execution: the *aggregation monad*. Powerdomains, sets with union and multisets with multi-union are all instances of aggregation monads (in categories of domains and of sets, respectively). In general, the notion of parallel composition modelled by an aggregation monad is neither idempotent, nor commutative, nor associative.

There are however models of the ordinary λ-calculus where aggregation, considered as parallel composition (that is, as *must* non-determinism), can be interpreted without introducing any additional structure, such as the above mentioned aggregation monads or powerdomain constructions.

This is the case in models of multiplicative exponential linear logic (MELL), where aggregation can be interpreted by the *mix rule*, if available. This rule allows to "put together" any two proofs whatsoever [8]. More precisely, parallel composition is obtained by combining the mix rule

[2] They could be the actual boolean constants in a typed λ-calculus with constants, or the projections $\lambda xy.x$, $\lambda xy.y$ as pure λ-terms.

with the contraction rule. Indeed, mix can be seen as a linear morphism $X \otimes Y \multimap X \parr Y$, so that there is a morphism $?A \otimes ?A \multimap ?A$, obtained by composing the mix morphism $?A \otimes ?A \multimap ?A \parr ?A$ with the contraction morphism $?A \parr ?A \multimap ?A$. This composite morphism defines a commutative algebra structure on $?A$, which is used to model the "parallel composition" of MELL proofs. Thus, to obtain a model of parallel λ-calculus, it is sufficient to solve the equation $\mathcal{D} \cong \mathcal{D} \Rightarrow \mathcal{D}$, with an object $\mathcal{D}$ of shape $?A$.

This is precisely what we did in [6], in a particularly simple model of linear logic: the model of sets and relations. Similar constructions are possible in other, richer models, such as the well known model of coherence spaces [14], or the model of hypercoherences [11]: the mix rule is available there, as well as in many other models. This shows that coherence (which prevents the above join of TRUE and FALSE) is not an obstacle to the interpretation of the *must* non-determinism in the pure λ-calculus[3]. Our model $\mathcal{D}$ of [6] satisfies the recursive equation $\mathcal{D} = ?(A)$ where $A = (\mathcal{D}^{\mathbb{N}})^{\perp}$, and therefore, $\mathcal{D}$ has the commutative algebra structure mentioned above. It is precisely this structure that we use for interpreting parallel composition, just as Danos and Krivine did in [8] for an extension of $\lambda\mu$-calculus with a parallel composition operation.

However, the category of sets and relations has another feature, which allows for a direct interpretation of the *may* non-determinism as well: morphisms are arbitrary relations between sets (interpreting types), and hence morphisms are closed under arbitrary unions. Thanks to this union operation on morphisms, *may* non-determinism can be interpreted directly, without introducing any additional powerdomain construction or aggregation monad. Of course, this operation is not available in the coherence or hypercoherence space models. Note that, if we consider $M + N \to M$ as a reduction rule of our calculus, then our semantics is not invariant under reduction, since the process of performing non-deterministic choices entails a non recoverable loss of information. But the situation is fundamentally similar with the powerdomain-based interpretations.

To summarize, in our model $\mathcal{D}$, the semantic counterparts of *may* and *must* non-determinism are at hand: they are simply the set-theoretic union and the mix-based algebraic operation. In this framework, parallel composition is no longer idempotent. This is quite natural if we consider each component of a parallel composition as the specification of a process whose

[3]In a typed language like PCF, this would be more problematic, since the object interpreting the type of booleans does not have the above mentioned structure.

execution requires the consumption of some kind of resources.

Contents. We introduce the $\lambda_{+\|}$-*calculus* which is a λ-calculus extended with parallel composition and non-deterministic choice. We define the operational semantics of $\lambda_{+\|}$-calculus using two different approaches. In the first one, which is rather canonical, we define a one-step head-reduction rule and we characterize the calculus as a term rewriting system. In the second one, we define the operational semantics by associating with each term a 'generalized' head normal form, which is a set of multisets of terms whose head subterms are variables. Roughly speaking, the operational value of a term is the collection of all possible outcomes of its head reductions. When the head subterm is $M + N$ (*may* non-deterministic choice), the head reduction goes on by choosing either M or N, and when the head subterm is $M\|N$ (*must* parallelism), the head reduction forks.

Thus, the operational values of the terms are characterized in two different ways: as sets of normal forms with respect to a canonical head-reduction rule, and as limits of an inductively defined sequence of sets. We prove the equivalence of the two approaches, and we use the latter to study the relationship between the operational and denotational semantics of this calculus.

We provide the denotational semantics of the $\lambda_{+\|}$-calculus in $\mathcal{D}$, considered as a λ-model, and endowed with two additional operations which turn it into a semiring. We prove the soundness with respect to β-reduction, and we show that the interpretations of the head normal forms of a term M are included in the interpretation of M. Next, we generalize Krivine's realizability technique to our extended calculus, showing that our denotational model is *sensible*: the operational value of a term is non-empty (i.e., a term is solvable) if, and only if, its denotation is non-empty.

Finally, we focus our attention on the contextual preorder on $\lambda_{+\|}$-terms, which is canonically associated to solvability. We show that the denotational interpretation is adequate with respect to this preorder. However, we also show that this model is not fully abstract. Intuitively, the lack of full abstraction is due to the fact that parallel composition is idempotent from the operational point of view, whilst it is not idempotent from the denotational one.

As usual, this mismatch can be fixed either by adding some resource sensitive operators that increase the expressivity of the language, or by decreasing the discriminating power of the model. We discuss these alternatives in the final section of this paper.

1. Preliminaries

To keep this article self-contained we summarize some definitions and results that will be used in the sequel. In particular, we present our semantic framework **MRel** and we recall the construction of a specific reflexive object $\mathcal{D}$ of **MRel**, which we have introduced in [6]. Our main reference for category theory is [1].

1.1. Multisets and Sequences

Let S be a set. We denote by $\mathcal{P}(S)$ the collection of all subsets of S. A *multiset* m over S can be defined as an unordered list $m = [a_1, a_2, \ldots]$ with repetitions such that $a_i \in S$ for all i. For each $a \in S$ the *multiplicity of* a *in* m is the number of occurrences of a in m. Given a multiset m over S, its *support* is the set of elements of S belonging to m.

A multiset m is called *finite* if it is a finite list, we denote by $[]$ the empty multiset. Given two multisets $m_1 = [a_1, a_2, \ldots]$ and $m_2 = [b_1, b_2, \ldots]$ the *multi-union* of m_1, m_2 is defined by $m_1 \uplus m_2 = [a_1, b_1, a_2, b_2, \ldots]$. We will write $\mathcal{M}_f(S)$ for the set of all finite multisets over S.

We denote by $\mathbb{N}$ the set of natural numbers. Given two $\mathbb{N}$-indexed sequences $\sigma = (\sigma_1, \sigma_2, \ldots), \tau = (\tau_1, \tau_2, \ldots)$ of multisets we define the *multi-union* of σ and τ componentwise as $\sigma \mathbin{\overline{\uplus}} \tau = (\sigma_1 \uplus \tau_1, \sigma_2 \uplus \tau_2, \ldots)$. An $\mathbb{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all, but a finite number of indices i. If S is a set, then we denote by $\mathcal{M}_f(S)^{(\omega)}$ the set of all quasi-finite $\mathbb{N}$-indexed sequences of multisets over S. We write $\star$ for the $\mathbb{N}$-indexed sequence of empty multisets, i.e., $\star$ is the only element of $\mathcal{M}_f(\emptyset)^{(\omega)}$.

1.2. Category Theory

In the following, $\mathbf{C}$ is a locally small[4] *cartesian closed category* (*ccc*, for short) and A, B, C are arbitrary objects of $\mathbf{C}$.

We denote by $A \& B$ the *categorical product*[5] of A and B, and by $\pi_1 \in \mathbf{C}(A \& B, A)$, $\pi_2 \in \mathbf{C}(A \& B, B)$ the associated *projections*. Given a pair of arrows $f \in \mathbf{C}(C, A)$ and $g \in \mathbf{C}(C, B)$, $\langle f, g\rangle \in \mathbf{C}(C, A \& B)$ is the unique arrow such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$.

[4] This means that $\mathbf{C}(A, B)$ is a set (called *homset*) for all objects A, B.

[5] We use the symbol & instead of $\times$ because, in the category we will be interested in, the categorical product is the disjoint union. The usual notation is kept to denote the set-theoretical product.

We will write $A \Rightarrow B$ for the *exponential object* and $\mathrm{eval}_{AB} \in \mathbf{C}(A \Rightarrow B \,\&\, A, B)$ for the *evaluation morphism* relative to A, B. Whenever A, B are clear from the context we will simply call it eval.

Moreover, for all objects A, B, C and arrow $f \in \mathbf{C}(C \,\&\, A, B)$ we denote by $\Lambda(f) \in \mathbf{C}(C, A \Rightarrow B)$ the unique morphism such that $\mathrm{eval}_{AB} \circ \langle \Lambda(f) \circ \pi_1, \pi_2 \rangle = f$. Finally, $\mathbb{1}$ denotes the *terminal object* and $!_A$ the only morphism in $\mathbf{C}(A, \mathbb{1})$.

We recall that in every ccc the following equalities hold:

$$
\begin{array}{llrr}
(\mathsf{pair}) & \langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle & \Lambda(f) \circ g = \Lambda(f \circ (g \times \mathrm{Id})) & (\mathsf{nat} - \Lambda) \\
(\beta_{\mathsf{cat}}) & \mathrm{eval} \circ \langle \Lambda(f), g \rangle = f \circ \langle \mathrm{Id}, g \rangle & \Lambda(\mathrm{eval}) = \mathrm{Id} & (\mathrm{Id}_{\mathsf{cat}})
\end{array}
$$

where $f_1 \times f_2$ is the *product map* defined by $\langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$.

Given a set I and a family $(A_i)_{i \in I}$ of objects of $\mathbf{C}$, we denote the I-*indexed product* of $(A_i)_{i \in I}$ by $\Pi_{i \in I} A_i$. If the object $\Pi_{i \in I} A_i$ exists in the category $\mathbf{C}$ for all families $(A_i)_{i \in I}$ such that the cardinality of I is less than or equal to $\aleph_0$, then we say that $\mathbf{C}$ *has countable products*.

Let us fix now an object A. For all sets I, we write A^I for the I-indexed product of an adequate number of copies of A, $\pi_i^I \in \mathbf{C}(A^I, A)$ for the projection on the i-th component, and Π_J^I, where $J \subseteq I$, for $\langle \pi_i^I \rangle_{i \in J} \in \mathbf{C}(A^I, A^J)$.

We say that the ccc $\mathbf{C}$ *has enough points* if, for all objects A, B and morphisms $f, g \in \mathbf{C}(A, B)$, whenever $f \neq g$, there exists a morphism $h \in \mathbf{C}(\mathbb{1}, A)$ such that $f \circ h \neq g \circ h$. Similarly, an object A *has enough points* if the above property holds for all $f, g \in \mathbf{C}(A, A)$.

1.3. **MRel***: a Cartesian Closed Category of Sets and Relations*

We now present the category **MRel**, which is the Kleisli category of the functor $\mathcal{M}_f(-)$ over the $\star$-autonomous category **Rel** of sets and relations. We provide here a direct definition, since in the sequel we will not use explicitly the monoidal structure of **Rel**.

- The objects of **MRel** are all the sets.
- A morphism from S to T is a relation from $\mathcal{M}_f(S)$ to T, in other words, $\mathbf{MRel}(S, T) = \mathcal{P}(\mathcal{M}_f(S) \times T)$.
- The identity of S is the relation $\mathrm{Id}_S = \{([a], a) \mid a \in S\} \in \mathbf{MRel}(S, S)$.
- The composition of $s \in \mathbf{MRel}(S, T)$ and $t \in \mathbf{MRel}(T, U)$ is defined by:

$$
t \circ s = \{(m, c) \mid \quad \begin{array}{l} \exists (m_1, b_1), \ldots, (m_k, b_k) \in s \text{ such that} \\ m = m_1 \uplus \ldots \uplus m_k \text{ and } ([b_1, \ldots, b_k], c) \in t\}. \end{array}
$$

We now provide an overview of the proof of cartesian closedness, and we show that **MRel** has countable products.

Theorem 1.1. *The category* **MRel** *is cartesian closed and has countable products.*

Proof. The terminal object $\mathbb{1}$ is the empty set $\emptyset$, and the unique element $!_S$ of $\mathbf{MRel}(S, \emptyset)$ is the empty relation.

Given two sets S_1 and S_2, their categorical product $S_1 \& S_2$ in **MRel** is their disjoint union:

$$S_1 \& S_2 = (\{1\} \times S_1) \cup (\{2\} \times S_2)$$

and the projections π_1, π_2 are given by:

$$\pi_i = \{([(i, a)], a) \mid a \in S_i\} \in \mathbf{MRel}(S_1 \& S_2, S_i), \text{ for } i = 1, 2.$$

It is easy to check that this is actually the categorical product of S_1 and S_2 in **MRel**; given $s \in \mathbf{MRel}(U, S_1)$ and $t \in \mathbf{MRel}(U, S_2)$, the corresponding morphism $\langle s, t \rangle \in \mathbf{MRel}(U, S_1 \& S_2)$ is given by:

$$\langle s, t \rangle = \{(m, (1, a)) \mid (m, a) \in s\} \cup \{(m, (2, b)) \mid (m, b) \in t\}.$$

This definition extends to arbitrary I-indexed families $(S_i)_{i \in I}$ of sets in the obvious way:

$$\&_{i \in I} S_i = \cup_{i \in I}(\{i\} \times S_i),$$
$$\pi_i = \{([(i, a)], a) \mid a \in S_i\} \in \mathbf{MRel}(\&_{i \in I} S_i, S_i), \text{ for } i \in I.$$

In particular, **MRel** has countable products.

Notice now that there exists a canonical bijection between $\mathcal{M}_f(S_1) \times \mathcal{M}_f(S_2)$ and $\mathcal{M}_f(S_1 \& S_2)$ which maps the pair $([a_1, \dots, a_p], [b_1, \dots, b_q])$ to the multiset $[(1, a_1), \dots, (1, a_p), (2, b_1), \dots, (2, b_q)]$. We will confuse this bijection with an equality, hence we will still denote by (m_1, m_2) the corresponding element of $\mathcal{M}_f(S_1 \& S_2)$.

Given two objects S and T, the exponential object $S \Rightarrow T$ is $\mathcal{M}_f(S) \times T$ and the evaluation morphism is given by:

$$\mathrm{eval}_{ST} = \{(([(m, b)], m), b) \mid m \in \mathcal{M}_f(S) \text{ and } b \in T\} \in \mathbf{MRel}(S \Rightarrow T \& S, T).$$

Again, it is easy to check that in this way we defined an exponentiation. Indeed, given any set U and any morphism $s \in \mathbf{MRel}(U \& S, T)$, there is exactly one morphism $\Lambda(s) \in \mathbf{MRel}(U, S \Rightarrow T)$ such that:

$$\mathrm{eval}_{ST} \circ (\Lambda(s) \times \mathrm{Id}S) = s,$$

namely, $\Lambda(s) = \{(p,(m,b)) \mid ((p,m),b) \in s\}$. $\Box$

Here, the points of an object S, i.e. the elements of $\mathbf{MRel}(\mathbb{1}, S)$, are the relations between $\mathcal{M}_f(\emptyset)$ and S, and hence, up to isomorphism, the subsets of S.

In the next subsection we will present a reflexive object in **MRel** which is extensional, although **MRel** is "strongly" non-extensional in the sense expressed by the following theorem.

Theorem 1.2. *No object $S \neq \mathbb{1}$ of* **MRel** *has enough points.*

Proof. We can always find $t_1, t_2 \in \mathbf{MRel}(S,S)$ such that $t_1 \neq t_2$ and, for all $s \in \mathbf{MRel}(\mathbb{1}, S)$, $t_1 \circ s = t_2 \circ s$. Recall that, by definition of composition, $t_1 \circ s = \{([], b) \mid \exists a_1, \ldots, a_n \in S \; ([], a_i) \in s \quad ([a_1, \ldots, a_n], b) \in t_1\} \in \mathbf{MRel}(\mathbb{1}, S)$, and similarly for $t_2 \circ s$. Hence, it is sufficient to choose $t_1 = \{(m_1, b)\}$ and $t_2 = \{(m_2, b)\}$ such that m_1, m_2 are different multisets with the same support. $\Box$

Corollary 1.3. **MRel** *does not have enough points.*

1.4. An Extensional Reflexive Object in **MRel**

From the category-theoretic point of view, a model of λ-calculus is a reflexive object of a cartesian closed category [2, Sec. 5.5].

Definition 1.4. *A* reflexive object *of a ccc* $\mathbf{C}$ *is a triple* $\mathcal{U} = (U, \mathcal{A}, \lambda)$ *such that U is an object of* $\mathbf{C}$*, and* $\lambda \in \mathbf{C}(U \Rightarrow U, U)$ *and* $\mathcal{A} \in \mathbf{C}(U, U \Rightarrow U)$ *satisfy* $\mathcal{A} \circ \lambda = \mathrm{Id}_{U \Rightarrow U}$. *$\mathcal{U}$ is called* extensional *if, moreover,* $\lambda \circ \mathcal{A} = \mathrm{Id}_U$*; in this case we have that* $U \cong U \Rightarrow U$.

We define a reflexive object $\mathcal{D}$ in **MRel**, which is extensional by construction.

Definition 1.5. *We let $(D_n)_{n \in \mathbb{N}}$ be the increasing family of sets defined by:*

- $D_0 = \emptyset$,
- $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)}$.

Finally, we set $D = \bigcup_{n \in \mathbb{N}} D_n$.

So we have $D_0 = \emptyset$ and $D_1 = \{\star\} = \{([], [], \dots)\}$. The elements of D_2 are quasi-finite sequences of multisets over a singleton, i.e., quasi-finite sequences of natural numbers, and so on.

More generally, an element of D can be represented as a finite tree which alternates two kinds of layers:

- ordered nodes (the quasi-finite sequences), where immediate subtrees are indexed by a possibly empty finite set of natural numbers,
- unordered nodes where subtrees are organised in a *non-empty* multiset.

Definition 1.6. *We say that* $\sigma \in D$ has rank n *if* $n \in \mathbb{N}$ *is minimum such that* $\sigma \in D_n$*.*

In order to define an isomorphism in **MRel** between D and $D \Rightarrow D = \mathcal{M}_f(D) \times D$ just notice that every element $\sigma = (\sigma_1, \sigma_2, \dots) \in D$ stands for the pair $(\sigma_1, (\sigma_2, \dots))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_f(D)$, we write $m :: \sigma$ for the element $\tau = (\tau_1, \tau_2, \dots) \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_f(D) \times D$ and D, and hence an isomorphism in **MRel** as follows:

Proposition 1.7. *(Bucciarelli, et al. [6]) The triple* $\mathcal{D} = (D, \mathcal{A}, \lambda)$ *where:*

- $\mathcal{A} = \{([m :: \sigma], (m, \sigma)) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D, D \Rightarrow D)$*,*
- $\lambda = \{([(m, \sigma)], m :: \sigma) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D \Rightarrow D, D)$*,*

is an extensional reflexive object of **MRel***.*

Of course, by Theorem 1.2, the object D does not have enough points.

2. The $\boldsymbol{\lambda}_{+\|}$-calculus: a Parallel and Non-Deterministic $\boldsymbol{\lambda}$-calculus

In this section we introduce the syntax and the operational semantics of a parallel and non-deterministic extension of λ-calculus that we call $\lambda_{+\|}$-*calculus*.

2.1. Syntax of $\lambda_{+\|}$-calculus

To begin with, we define the set $\Lambda_{+\|}$ of λ-terms enriched with two binary operators $+$ and $\|$, that is the set of terms generated by the following grammar (where x ranges over a countable set Var of variables):

$$M, N \ ::= \ x \mid \lambda x.M \mid MN \mid M + N \mid M \| N \ .$$

The elements of $\Lambda_{+\|}$ are called $\lambda_{+\|}$*-terms* and will be denoted by $M, N, P, \ldots$ Intuitively, $M + N$ denotes the *non-deterministic choice* between M and N, and $M\|N$ stands for their *parallel composition.*

As usual, we suppose that application associates to the left and λ-abstraction to the right. Moreover, to lighten the notation, we assume that application and λ-abstraction take precedence over $+$ and $\|$. The notions of *free* and *bound* variables of a term are defined in the obvious way.

Concerning specific $\lambda_{+\|}$-terms, that will be used in the following to build examples, we set:

$$\mathbf{I} \equiv \lambda x.x; \quad \Delta \equiv \lambda x.xx; \quad \Omega \equiv (\lambda x.xx)(\lambda x.xx),$$
$$\underline{n} \equiv \lambda sz.s^n(z) \text{ for each } n \in \mathbb{N}; \quad \mathbf{s} \equiv \lambda nxy.nx(xy),$$

where $\equiv$ denotes syntactical equality modulo α-conversion. Notice that $\underline{n}$ is the n-th Church numeral [2, Def 6.4.4] and $\mathbf{s}$ implements the successor function.

Notation 2.1.

- *We will write $\vec{P}$ for a (possibly empty) finite sequence of $\lambda_{+\|}$-terms $P_1 \ldots P_k$ and $\ell(\vec{P})$ for the length of $\vec{P}$.*
- *Given a sequence $\vec{P} \equiv P_1 \ldots P_k \in \Lambda_{+\|}$ with $k \geq 1$, we will denote by $\vec{P}_{\geq 2}$ the (possibly empty) sequence $P_2 \ldots P_k$.*

It is easy to check that every $\lambda_{+\|}$-term M has the form $\lambda\vec{x}.N\vec{P}$ where N, which is called *the head subterm of* M, is either a variable, a non-deterministic choice, a parallel composition or a λ-abstraction. Notice that, in this last case, we must have $\ell(\vec{P}) > 0$.

Definition 2.2. *A* substitution *is a finite set $s = \{(x_1, N_1), \ldots, (x_k, N_k)\}$ such that $x_i \neq x_j$ for all $1 \leq i < j \leq k$.*

Given a $\lambda_{+\|}$-term M and a substitution s as above, we denote by Ms the term obtained by substituting *simultaneously* the term N_j for all free occurrences of x_j (for $1 \leq j \leq k$) in M, subject to the usual proviso about renaming bound variables in M to avoid capture of free variables in the N_j's. If $s = \{(x, N)\}$ then we write $M[N/x]$ for Ms.

Remark 2.3. *In general, $M\{(x_1, N_1), \ldots, (x_k, N_k)\} \neq M[N_1/x_1] \cdots [N_k/x_k]$. For instance, $x\{(x, y), (y, z)\} = y$, whereas $x[y/x][z/y] = z$.*

Actually, k-ary substitutions will be only used in Section 5 in the proof of Lemma 5.9.

In this framework, contexts are $\lambda_{+\|}$-terms with some occurrences of a 'hole', denoted by $\langle\rangle$, inside.

Definition 2.4. *A* context *is inductively defined as follows:* $\langle\rangle$ *is a context;* x *is a context, for every variable* x*; if* C *is a context, then* $\lambda x.C$ *is a context for each variable* x*; if* C_1 *and* C_2 *are contexts then so are* $C_1 C_2$*,* $C_1 + C_2$ *and* $C_1 \| C_2$*.*

If M is a $\lambda_{+\|}$-term, we will write $C\langle M\rangle$ for the context $C\langle -\rangle$ where all the occurrences of $\langle\rangle$ have been simultaneously and syntactically replaced by M. Notice that this substitution can generate *capture* of free variables of M. Consider, for instance, the context $C\langle -\rangle \equiv \lambda x.(x + \langle\rangle)$ and the term $M \equiv \lambda y.yx$; in this case x, which occurs free in M, becomes bound in $C\langle N\rangle \equiv \lambda x.(x + \lambda y.yx)$.

2.2. Operational Semantics: One-Step Head Reduction

In this section we give the operational semantics of $\lambda_{+\|}$-calculus by defining a one-step head reduction rule. How should the head-reduction proceed when a sum or a parallel composition comes in head position?

- When the head subterm is of the shape $M + N$, a non-deterministic choice is performed, and the head reduction goes on by picking either M or N as new head subterm;
- when the head subterm is of the shape $M\|N$ two threads, having M and N as head subterms, are executed in parallel.

The following definition captures this intuitive idea.

Definition 2.5. *The* one-step head reduction *of* $\lambda_{+\|}$*-terms is the smallest binary relation* $\to_h \subseteq \Lambda_{+\|} \times \Lambda_{+\|}$ *such that, for all* $\lambda_{+\|}$*-terms* M, M', N, N', P *and variables* x*, we have:*

$$\frac{}{(\lambda x.M)N \to_h M[N/x]}\ (\beta) \qquad \frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'}\ (\eta)$$

$$\frac{}{M+N \to_h N}\ (+_c^\ell) \qquad \frac{}{M+N \to_h M}\ (+_c^r)$$

$$\frac{N \to_h N'}{N\|M \to_h N'\|M}\ (\|_a^\ell) \qquad \frac{M \to_h M'}{N\|M \to_h N\|M'}\ (\|_a^r)$$

$$\frac{}{(M\|N)P \to_h MP\|NP}\ (\|_{app}) \qquad \frac{M \to_h M' \quad M \not\equiv \lambda x.Q,\ Q_1\|Q_2}{MN \to_h M'N}\ (\nu)$$

We denote by $\to_h^*$ *the transitive and reflexive closure of* $\to_h$.

This reduction is similar to the $\to_{pn}^h$ reduction of [9], except for the fact that in our framework the head reduction of parallel composition is asynchronous; the relation between $\to_h$ and $\to_{pn}^h$ is discussed in Section 7.

Head reduction is clearly not Church-Rosser because of the $(+_c)$ rules. The set of head normal forms of a given $\lambda_{+\|}$-term M is defined as usual.

Definition 2.6. *Given a* $\lambda_{+\|}$*-term* M*, we define the* set $\mathsf{HNF}(M)$ of head normal forms of M *by* $\mathsf{HNF}(M) = \{N \mid M \to_h^* N$ *and* $N \not\to_h\}$.

In order to endow non-deterministic choice and parallel composition with may and must semantics, respectively, we say that a $\lambda_{+\|}$-term M is solvable if at least one head reduction starting from M terminates.

Definition 2.7. *A* $\lambda_{+\|}$*-term* M *is* solvable *if* $\mathsf{HNF}(M) \neq \emptyset$.

It is easy to see that a parallel composition is solvable if and only if both its components are solvable, and that a non-deterministic choice is solvable if and only if at least one of its components is.

Head normal forms, i.e., $\lambda_{+\|}$-terms N such that $N \not\to_h$, have the shape $\lambda\vec{x}.N'\vec{P}$, where N' is either a variable, as in the case of ordinary λ-calculus, or a parallel composition whose components are again head normal forms. In the latter case, $\vec{P}$ must be empty, because of the $(\|_{app})$ and (η) rules.

2.3. Operational Semantics: an Alternative Characterization

We introduce now an alternative characterization of $\mathsf{HNF}(M)$ that, instead of relying explicitely on a term rewriting system, is based on an inductive definition. (This can be seen as a first step from the operational to the denotational semantics of $\lambda_{+\|}$-calculus.)

The intuitive idea underlying the notion of "value" (formalized below) is the following:

- when a term of the form $N_1 + N_2$ gets in head position, either of the alternatives may be chosen to pursue the head reduction, and the final value is the union of the values obtained by each choice. In particular, if one of the choices produces a non-empty value, then the global value is non-empty.

- when a term of the form $N_1 \| N_2$ gets in head position, the head reduction forks, and the final value is obtained by "mixing" the values eventually obtained. In particular, if the value of one of the subprocesses is empty, then also the global value is.

Specifically, we use union (resp. multi-union) to get the value of $M_1 + M_2$ (resp. $M_1 \| M_2$) out of the values of M_1 and M_2.

As showed in Proposition 2.16, following this approach, we still associate with each $M \in \Lambda_{+\|}$ the value eventually obtained by head reducing M.

Definition 2.8. *A* basic head normal form *is a $\lambda_{+\|}$-term of the form $\lambda\vec{x}.y\vec{P}$. A* multiple head normal form *is a finite multiset of basic head normal forms. A* value *is a set of multiple head normal forms.*

To help the reader to get familiar with these notions, we first provide some simple examples of values:

- the value of $\mathbf{I} + \Delta$ is $\{[\mathbf{I}], [\Delta]\}$. In other words, the term $\mathbf{I} + \Delta$ has two different multiple head normal forms, which are singleton multisets;

- the value of $\mathbf{I} \| \Delta$ is $\{[\mathbf{I}, \Delta]\}$, then $\mathbf{I} \| \Delta$ has just one multiple head normal form;

- the values of $\mathbf{I} + \Omega$ and $\mathbf{I} \| \Omega$ are $\{[\mathbf{I}]\}$ and $\emptyset$, respectively. This is a consequence of the fact that the value of Ω is the empty-set.

In general, the value $H(M)$ of a $\lambda_{+\|}$-term M can be characterized as the limit of an increasing sequence $(H_n(M))_{n \in \mathbb{N}}$ of "partial" values, which are defined by induction on $n \in \mathbb{N}$ and by cases on the form of the head subterm of M.

Definition 2.9. *Let $M \equiv \lambda\vec{x}.N\vec{P}$ be a $\lambda_{+\|}$-term.*

- $H_0(M) = \emptyset$*;*

- $H_{n+1}(M) = \begin{cases} \{[M]\} & \text{if } N \equiv y, \\ H_n(\lambda\vec{x}.Q[P_1/y]P_2 \cdots P_{\ell(\vec{P})}) & \text{if } N \equiv \lambda y.Q, \\ H_n(\lambda\vec{x}.N_1\vec{P}) \cup H_n(\lambda\vec{x}.N_2\vec{P}) & \text{if } N \equiv N_1 + N_2, \\ \{m_1 \uplus m_2 \mid \exists m_i \in H_n(\lambda\vec{x}.N_i\vec{P}) \text{ for } i = 1, 2\} & \text{if } N \equiv N_1 \| N_2. \end{cases}$

Notice that, for all $M \in \Lambda_{+\|}$ and $n \in \mathbb{N}$, the value $H_n(M)$ is a finite set of multiple head normal forms. Since the sequence $(H_n(M))_{n\in\mathbb{N}}$ is increasing, we can define the (final) value of M as its limit.

Definition 2.10. *The* value of a $\lambda_{+\|}$-term M *is defined by* $H(M) = \bigcup_{n\in\mathbb{N}} H_n(M)$.

Of course, $H(M)$ may be infinite as shown in the example below.

Example 2.11. *Consider the $\lambda_{+\|}$-term*[6] $M \equiv \lambda n.\underline{0} + \mathbf{s}n$. *Let now* $N \equiv \mathbf{Y}M$, *where* $\mathbf{Y}$ *is some fixpoint combinator. To have simpler calculations, we suppose that* $\mathbf{Y}M$ *reduces to* $M(\mathbf{Y}M)$ *in just one step of head* β*-reduction. Then, we get:*

- $H_0(N) = \emptyset$,
- $H_1(N) = H_0(MN) = \emptyset$,
- $H_2(N) = H_1(MN) = H_0(\underline{0} + \mathbf{s}N) = \emptyset$,
- $H_3(N) = H_2(MN) = H_1(\underline{0} + \mathbf{s}N) = \{[\underline{0}]\} \cup H_0(\mathbf{s}N) = \{[\underline{0}]\}$.

Pursuing the calculation a little further, one gets $H_9(N) = \{[\underline{0}], [\underline{1}]\}$ *and, eventually,* $H(N) = \{[\underline{n}] \mid n \in \mathbb{N}\}$.

2.4. Equivalence Between the Two Approaches

We now prove that $H(M)$ and $\mathsf{HNF}(M)$ are essentially the same set: multi-unions in the former replace parallel compositions in head position in the latter. In the rest of the paper, we will use the characterization of solvability expressed in terms of $H(M)$ (see Corollary 2.17).

We start by defining an operator ι mapping head normal forms into multiple head normal forms. The idea is simply that, for instance, the head normal form $\mathbf{I}\|\mathbf{I}$ will be associated with the multiset $[\mathbf{I}, \mathbf{I}]$.

[6] We recall that $\underline{0}$ denotes the 0-th Church numeral and $\mathbf{s}$ the successor function.

Definition 2.12. *Let $Q \equiv \lambda\vec{x}.N\vec{P}$ be a head normal form. The multiset $\iota(Q)$ is defined by cases on N and by structural induction on Q as follows:*

$$\iota(Q) = \begin{cases} [Q] & \text{if } N \equiv x, \\ \iota(\lambda\vec{x}.Q_1) \uplus \iota(\lambda\vec{x}.Q_2) & \text{if } N \equiv Q_1 \| Q_2. \end{cases}$$

Remark that, in the definition above, if N is not a variable then $\ell(\vec{P}) = 0$ since Q is a head normal form.

The following simple lemmata, whose proofs are omitted, are useful for relating $\mathsf{HNF}(M)$ and $H(M)$.

Lemma 2.13. *Let P be a $\lambda_{+\|}$-term and $\vec{x}$ be a sequence of variables. If $\lambda\vec{x}.P \to_h^* Q$ then $Q \equiv \lambda\vec{x}.Q'$ and $P \to_h^* Q'$.*

The proof is a simple analysis of $\to_h$ rules. The relevant one is the (η) rule: in this case head abstractions persist and reductions take place in the body of the term.

Lemma 2.14. *Let $M \in \Lambda_{+\|}$. If $M \to_h^* N$ then:*

(a) $H(N) \subseteq H(M)$

(b) $\mathsf{HNF}(N) \subseteq \mathsf{HNF}(M)$.

Notice that head reductions do not preserve the values of terms, due to the non deterministic choice $(+_c)$. Nevertheless, it is easy to check that both the value and the set of head normal forms of a term can only decrease by head reducing it.

Lemma 2.15. *If $Q \in \Lambda_{+\|}$ is a head normal form, then $H(Q) = \{\iota(Q)\}$.*

In the following proposition we give the precise relationship between $\mathsf{HNF}(M)$ and $H(M)$.

Proposition 2.16. *Let $M \in \Lambda_{+\|}$, then we have that*

$$H(M) = \{\iota(Q) \mid Q \in \mathsf{HNF}(M)\}.$$

Proof. We start by proving $H(M) \subseteq \{\iota(Q) \mid Q \in \mathsf{HNF}(M)\}$. It is enough to show that, for all natural numbers n and for all multiple head normal forms m, if $m \in H_n(M)$ then there exists $Q \in \mathsf{HNF}(M)$ such that $m = \iota(Q)$. This is proven by induction on $n \in \mathbb{N}$.

- case $n = 0$. This case holds trivially, since $H_0(M)$ is empty.
- case $n+1$. Let $m \in H_{n+1}(M)$, and let us inspect the four possible cases for the head term N of $M \equiv \lambda\vec{x}.N\vec{P}$, as in Definition 2.9:
  1. $N \equiv y$: this case is trivial.
  2. $N \equiv \lambda y.Q$: let $M' \equiv \lambda\vec{x}.Q[P_1/y]P_2 \cdots P_{\ell(\vec{P})}$. By definition $m \in H_n(M')$, and by the inductive hypothesis there exists a $\lambda_{+\|}$-term $Q \in \mathsf{HNF}(M')$ such that $m = \iota(Q)$. We conclude by Lemma 2.14(b).
  3. $N \equiv N_1 + N_2$: let $M' \equiv \lambda\vec{x}.N_1\vec{P}$, and let us suppose, without loss of generality, that $m \in H_n(\lambda\vec{x}.N_1\vec{P})$. We conclude by using the inductive hypothesis and Lemma 2.14(b).
  4. $N \equiv N_1\|N_2$: then $m = m_1 \uplus m_2$ and $m_i \in H_n(\lambda\vec{x}.N_i\vec{P})$. By the inductive hypothesis, there exists $Q_i \in \mathsf{HNF}(\lambda\vec{x}.N_i\vec{P})$ such that $m_i = \iota(Q_i)$ for $i = 1, 2$. By Lemma 2.13, there exists a head reduction starting from M and ending with $Q' \equiv \lambda\vec{x}.(Q'_1\|Q'_2)$ such that $Q_i \equiv \lambda\vec{x}.Q'_i$. We conclude since $\iota(Q') = m_1 \uplus m_2$ by definition.

The proof of the other inclusion is given by a simple induction on the length of the derivation $M \to_h^* Q$. One uses Lemma 2.15 for the base case and Lemma 2.14(a) for the inductive step. $\square$

Corollary 2.17. *Let M be a $\lambda_{+\|}$-term. Then $\mathsf{HNF}(M) = \emptyset$ if, and only if, $H(M) = \emptyset$.*

As a matter of fact, given $M, N \in \Lambda_{+\|}$ it is possible that $H(M) = H(N)$ and $\mathsf{HNF}(M) \neq \mathsf{HNF}(N)$. Consider for instance $M \equiv (\mathbf{I}\|\mathbf{I})\|\mathbf{I}$ and $N \equiv \mathbf{I}\|(\mathbf{I}\|\mathbf{I})$. These terms can of course be consistently equated, and actually they will get the same denotational interpretation in our model. However, we can notice that in the general framework of aggregation monads, where $\|$ may be interpreted by a non associative operation, more discriminating notions of value could be considered.

From now on, we will focus on the value $H(M)$ of a given term M, forgetting about $\to_h$. Notice that the two points of view are equivalent, as showed in Proposition 2.16.

3. Some Remarkable Sets of Sovable Terms

In Definition 2.7 we say that a term is solvable if its set of head normal forms is non-empty; however, Corollary 2.17 allows us to shift our point of view, keeping the basic intuition of head normalization. Indeed, a $\lambda_{+\|}$-term M is solvable if, and only if, $H(M) \neq \emptyset$.

Notation 3.1. *We will write $\mathcal{N}$ for the set of solvable $\lambda_{+\|}$-terms.*

Among solvable terms, we single out the set $\mathcal{N}_0$ of head normal forms starting with a variable, and the set $\mathcal{N}_1$ of solvable terms having a multiple head normal form whose head variables are free.

Definition 3.2. *We set:*

- $\mathcal{N}_0 = \{x\vec{P} \mid x \in \mathrm{Var}$ *and* $\vec{P} \in \Lambda_{+\|}\}$, *and*
- $\mathcal{N}_1 = \{M \in \Lambda_{+\|} \mid \exists[\lambda\vec{x}_1.y_1\vec{P}_1, \dots, \lambda\vec{x}_k.y_k\vec{P}_k] \in H(M) \wedge (\forall j = 1..k)\ y_j \notin \vec{x}_j\}$.

The rest of this section is devoted to the proof the following proposition, which is the main technical tool used in the realizability argument of Section 5.

Proposition 3.3. *Let $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, then we have that:*

(i) if $Mx \in \mathcal{N}$ then $M \in \mathcal{N}$,

(ii) if $M\Omega \in \mathcal{N}_1$ then $M \in \mathcal{N}_1$,

(iii) if $M \in \mathcal{N}_1$ then $MN \in \mathcal{N}_1$ for all $N \in \Lambda_{+\|}$.

Notice that in the case of the pure λ-calculus the analogous properties are trivially true.

In order to prove the above proposition, we need to introduce some additional definitions and results.

Definition 3.4. *A multiple head normal form m is* head-free *if none of the head normal forms contained in m binds its head variable.*

The following definition extends the notion of application of λ-calculus to multiple head normal forms.

Definition 3.5. *Let m be a multiple head normal form and $N \in \Lambda_{+\|}$, then we set $mN = [MN \mid M \in m \cap \mathcal{N}_0] \uplus [P[N/x] \mid \lambda x.P \in m]$.*

Proposition 3.6. *Given a multiple head normal form m, we have that:*

- *mx is a multiple head normal form, for all $x \in \mathrm{Var}$;*

- *if m is head-free, then mN is a head-free multiple head normal form, for all $N \in \Lambda_{+\|}$.*

Proof. Straightforward. $\square$

We provide now three technical lemmata which will be used respectively for proving the three items of Proposition 3.3.

Lemma 3.7. *For all $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$ we have that for all $n \in \mathbb{N}$:*

$$m \in H_n(Mx) \Rightarrow \exists k \le n, \exists m' \in H_k(M) \text{ such that } m = m'x.$$

Proof. By induction on $n \in \mathbb{N}$.
If $n = 0$ then the implication follows trivially, since $H_0(Mx) = \emptyset$.
Suppose now that $n > 0$, then the proof is by cases on the shape of $M \equiv \lambda\vec{z}.M'\vec{P}$.

- If $M' \equiv y$ and $\ell(\vec{z}) = 0$, then $H_n(y\vec{P}x) = \{[y\vec{P}x]\}$. Hence, the only $m \in H_n(Mx)$ is $[y\vec{P}x]$ and the result follows taking $k = n$ and $m' = [y\vec{P}]$.

- If $M' \equiv y$ and $\ell(\vec{z}) > 0$, then $H_n((\lambda\vec{z}.y\vec{P})x) = H_{n-1}(\lambda\vec{z}_{\ge 2}.y[x/z_1]\vec{P}[x/z_1]) = \{[\lambda\vec{z}_{\ge 2}.y[x/z_1]\vec{P}[x/z_1]]\} = \{[\lambda\vec{z}.y\vec{P}]x\}$. Hence, if $m \in H_n((\lambda\vec{z}.y\vec{P})x)$, then $m = [\lambda\vec{z}.y\vec{P}]x$ and the result follows for $k = n$ and $m' = [\lambda\vec{z}.y\vec{P}]$.

- If $M' \equiv (\lambda y.Q)$ and $\ell(\vec{z}) = 0$, then $H_n((\lambda y.Q)\vec{P}x) = H_{n-1}(Q[P_1/y]\vec{P}_{\ge 2}x)$. Now, if $m \in H_n(Mx)$, then m also belongs to $H_{n-1}(Q[P_1/y]\vec{P}_{\ge 2}x)$ and, by the inductive hypothesis, there exist $k' \le n-1$ and $m' \in H_{k'}(Q[P_1/y]\vec{P}_{\ge 2})$ such that $m = m'x$. We can then conclude since $H_{k'}(Q[P_1/y]\vec{P}_{\ge 2}) = H_{k'+1}((\lambda y.Q)\vec{P})$ and $k = k'+1 \le n$.

- If $M' \equiv (\lambda y.Q)$, $\ell(\vec{z}) > 0$ and $H_n(Mx) \neq \emptyset$, then we have that $n > 2$ and

$$\begin{array}{rcl} H_n((\lambda\vec{z}.(\lambda y.Q)\vec{P})x) & = & H_{n-1}(\lambda\vec{z}_{\ge 2}.(\lambda y.Q[x/z_1])\vec{P}[x/z_1]) \\ & = & H_{n-2}(\lambda\vec{z}_{\ge 2}.Q[x/z_1][P_1[x/z_1]/y]\vec{P}_{\ge 2}[x/z_1]) \\ & = & H_{n-2}(\lambda\vec{z}_{\ge 2}.Q[P_1/y][x/z_1]\vec{P}_{\ge 2}[x/z_1]) \\ & = & H_{n-1}((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\ge 2})x). \end{array}$$

  Now, if $m \in H_n(Mx)$, then m also belongs to $H_{n-1}((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\ge 2})x)$ and, by the inductive hypothesis, there exist $k' \le n-1$ and $m' \in H_{k'}(\lambda\vec{z}.Q[P_1/y]\vec{P}_{\ge 2})$ such that $m = m'x$. We can conclude since $H_{k'}(\lambda\vec{z}.Q[P_1/y]\vec{P}) = H_{k'+1}(\lambda\vec{z}.(\lambda y.Q)\vec{P})$ and $k = k'+1 \le n$.

- If $M' \equiv M_1 + M_2$ and $\ell(\vec{z}) = 0$, then $H_n((M_1 + M_2)\vec{P}x) = \cup_{i=1,2} H_{n-1}(M_i\vec{P}x)$. If $m \in H_n(Mx)$ then m belongs to, say, $H_{n-1}(M_1\vec{P}x)$ and by the inductive hypothesis there exist $k' \leq n-1$ and $m' \in H_{k'}(M_1\vec{P})$ such that $m = m'x$. Thus, we conclude since $m' \in H_{k'+1}((M_1 + M_2)\vec{P})$ and $k = k' + 1 \leq n$.

- If $M' \equiv M_1 + M_2$, $\ell(\vec{z}) > 0$ and $H_n(Mx) \neq \emptyset$, then we have $n > 2$ and

$$\begin{array}{rcl} H_n((\lambda\vec{z}.(M_1 + M_2)\vec{P})x) & = & H_{n-1}(\lambda\vec{z}_{\geq 2}.(M_1[x/z_1] + M_2[x/z_1])\vec{P}[x/z_1]) \\ & = & \cup_{i=1,2} H_{n-2}(\lambda\vec{z}_{\geq 2}.M_i[x/z_1]\vec{P}[x/z_1]). \end{array}$$

  Thus if $m \in H_n(Mx)$ then m belongs to, say, $H_{n-2}(\lambda\vec{z}_{\geq 2}M_1[x/z_1]\vec{P}) = H_{n-1}((\lambda\vec{z}.M_1\vec{P})x)$ and, by the inductive hypothesis, there exist $k' \leq n-1$ and $m' \in H_{k'}(\lambda\vec{z}.M_1\vec{P})$ such that $m = m'x$. Hence, we conclude since $m' \in H_{k'+1}(\lambda\vec{z}.(M_1 + M_2)\vec{P})$ and $k = k' + 1 \leq n$.

- If $M' \equiv M_1 \| M_2$ and $\ell(\vec{z}) = 0$ then $m \in H_n((M_1\|M_2)\vec{P}x)$ implies that there exists $m_i \in H_{n-1}(M_i\vec{P}x)$ (for $i = 1,2$) such that $m = m_1 \uplus m_2$. By the inductive hypothesis there exist $k_1, k_2 \leq n-1$ and $m_i' \in H_{k_i}(M_i\vec{P})$ such that $m_i = m_i'x$ (for $i = 1,2$). Hence $m_1x \uplus m_2x \in H_{\max(k_1,k_2)+1}((M_1\|M_2)\vec{P})$ and we conclude since $m_1x \uplus m_2x = (m_1 \uplus m_2)x$ and $k = \max(k_1,k_2) + 1 \leq n$.

- If $M' \equiv M_1\|M_2$, $\ell(\vec{z}) > 0$ and $H(M) \neq \emptyset$, then we have $n > 2$ and $H_n((\lambda\vec{z}.(M_1\|M_2)\vec{P})x) = H_{n-1}(\lambda\vec{z}_{\geq 2}.(M_1[x/z_1]\|M_2[x/z_1])\vec{P}[x/z_1])$. Hence, if $m \in H_n(Mx)$ then there exists a multiple head normal form $m_i \in H_{n-2}(\lambda\vec{z}_{\geq 2}.M_i[x/z_1]\vec{P}[x/z_1]) = H_{n-1}((\lambda\vec{z}.M_i\vec{P})x)$ (for $i = 1,2$) such that $m = m_1 \uplus m_2$. By the inductive hypothesis there exist $k_1, k_2 \leq n-1$ and $m_i' \in H_{k_i}(\lambda\vec{z}.M_i\vec{P})$ such that $m_i = m_i'x$ (for $i = 1,2$). Hence $m_1x \uplus m_2x \in H_{\max(k_1,k_2)+1}(\lambda\vec{z}.(M_1\|M_2)\vec{P})$ and we conclude since $m_1x \uplus m_2x = (m_1 \uplus m_2)x$ and $k = \max(k_1,k_2) + 1 \leq n$.

$\square$

Lemma 3.8. *For all $M \in \Lambda_{+\|}$ we have that for all $n \in \mathbb{N}$:*

$m \in H_n(M\Omega)$ *head-free* $\Rightarrow \exists k \leq n, \exists m' \in H_k(M)$ *head-free, such that* $m = m'\Omega$.

Proof. By induction on n.
If $n = 0$ then the implication follows trivially, since $H_0(M\Omega) = \emptyset$.
Suppose now that $n > 0$, then the proof is by cases on the shape of $M \equiv \lambda\vec{z}.M'\vec{P}$.

- If $M' \equiv y$ and $\ell(\vec{z}) = 0$, then $H_n(y\vec{P}\Omega) = \{[y\vec{P}\Omega]\}$. Hence, the only $m \in H_n(M\Omega)$ is $[y\vec{P}\Omega]$ which is head-free and the result follows taking $k = n$ and $m' = [y\vec{P}]$.

- If $M' \equiv y$ and $\ell(\vec{z}) > 0$, then we can suppose $y \notin \vec{z}$, since otherwise it is easy to check that $H_n(M\Omega)$ contains no head-free multiple head normal form. In this case, we have: $H_{n-1}(\lambda\vec{z}_{\geq 2}.y\vec{P}[\Omega/z_1]) = \{[\lambda\vec{z}_{\geq 2}.y\vec{P}[\Omega/z_1]]\} = \{[\lambda\vec{z}.y\vec{P}]\Omega\}$. Hence, the only head-free multiple head normal form in $H_n(M\Omega)$ is $m = [\lambda\vec{z}.y\vec{P}]\Omega$ and we conclude since $H_{n-1}(\lambda\vec{z}.y\vec{P}) = \{[\lambda\vec{z}.y\vec{P}]\}$ and $m = [\lambda\vec{z}.y\vec{P}]$ is head-free.

- If $M' \equiv (\lambda y.Q)$ and $\ell(\vec{z}) = 0$, then $H_n((\lambda y.Q)\vec{P}\Omega) = H_{n-1}(Q[P_1/y]\vec{P}_{\geq 2}\Omega)$. Now, if there is a head-free multiple head normal form $m \in H_n(M\Omega)$, then m also belongs to $H_{n-1}(Q[P_1/y]\vec{P}_{\geq 2}\Omega)$. By the inductive hypothesis there exist $k' \leq n-1$ and $m' \in H_{k'}(Q[P_1/y]\vec{P}_{\geq 2})$ head-free such that $m = m'\Omega$. We conclude since $H_{k'}(Q[P_1/y]\vec{P}_{\geq 2}) = H_{k'+1}((\lambda y.Q)\vec{P})$ and $k = k'+1 \leq n$.

- If $M' \equiv (\lambda y.Q)$, $\ell(\vec{z}) > 0$ and $H_n(M\Omega) \neq \emptyset$, then we have that $n > 2$ and

$$\begin{array}{rcl} H_n((\lambda\vec{z}.(\lambda y.Q)\vec{P})\Omega) & = & H_{n-1}(\lambda\vec{z}_{\geq 2}.(\lambda y.Q[\Omega/z_1])\vec{P}[\Omega/z_1]) \\ & = & H_{n-2}(\lambda\vec{z}_{\geq 2}.Q[\Omega/z_1][P_1[\Omega/z_1]/y]\vec{P}_{\geq 2}[\Omega/z_1]) \\ & = & H_{n-2}(\lambda\vec{z}_{\geq 2}.Q[P_1/y][\Omega/z_1]\vec{P}_{\geq 2}[\Omega/z_1]) \\ & = & H_{n-1}((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})\Omega). \end{array}$$

  Now, if there is a head-free multiple head normal form $m \in H_n(M\Omega)$, then m also belongs to $H_{n-1}((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})\Omega)$ and, by the inductive hypothesis, there exist $k' \leq n-1$ and $m' \in H_{k'}(\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})$ head-free such that $m = m'\Omega$. Then we conclude since $H_{k'}(\lambda\vec{z}.Q[P_1/y]\vec{P}) = H_{k'+1}(\lambda\vec{z}.(\lambda y.Q)\vec{P})$ and $k = k'+1 \leq n$.

- If $M' \equiv M_1 + M_2$ and $\ell(\vec{z}) = 0$, then $H_n((M_1+M_2)\vec{P}\Omega) = \cup_{i=1,2} H_{n-1}(M_i\vec{P}\Omega)$. If there is a head-free multiple head normal form $m \in H_n(M\Omega)$ then m belongs to, say, $H_{n-1}(M_1\vec{P}\Omega)$ and, by the inductive hypothesis, there exist $k' \leq n-1$ and $m' \in H_{k'}(M_1\vec{P})$ head-free such that $m = m'\Omega$. Thus, we conclude since $m' \in H_{k'+1}((M_1+M_2)\vec{P})$ and $k = k'+1 \leq n$.

- If $M' \equiv M_1 + M_2$, $\ell(\vec{z}) > 0$ and $H_n(M\Omega) \neq \emptyset$, then we have that $n > 2$ and

$$\begin{array}{rcl} H_n((\lambda\vec{z}.(M_1+M_2)\vec{P})\Omega) & = & H_{n-1}(\lambda\vec{z}_{\geq 2}.(M_1[\Omega/z_1] + M_2[\Omega/z_1])\vec{P}[\Omega/z_1]) \\ & = & \cup_{i=1,2} H_{n-2}(\lambda\vec{z}_{\geq 2}.M_i[\Omega/z_1]\vec{P}[\Omega/z_1]). \end{array}$$

Thus if there is a head-free multiple head normal form $m \in H_n(M\Omega)$ then m belongs to, say, $H_{n-2}(\lambda\vec{z}_{\geq 2}.M_1[\Omega/z_1]\vec{P}) = H_{n-1}((\lambda\vec{z}.M_1\vec{P})\Omega)$ and by the inductive hypothesis there exist $k' \leq n-1$ and $m' \in H_{k'}(\lambda\vec{z}.M_1\vec{P})$ head-free such that $m = m'\Omega$. Hence, we conclude since $m' \in H_{k'+1}(\lambda\vec{z}.(M_1 + M_2)\vec{P})$ and $k = k' + 1 \leq n$.

- If $M' \equiv M_1 \| M_2$ and $\ell(\vec{z}) = 0$ then $m \in H_n((M_1\|M_2)\vec{P}\Omega)$, implies that there exists $m_i \in H_{n-1}(M_i\vec{P}\Omega)$ (for $i = 1,2$) such that $m = m_1 \uplus m_2$. Of course, if m is head-free then also m_1, m_2 are. Thus, by the inductive hypothesis, there exist $k_1, k_2 \leq n-1$ and $m'_i \in H_{k_i}(M_i\vec{P})$ head-free such that $m_i = m'_i\Omega$ (for $i = 1,2$). Hence $m_1\Omega \uplus m_2\Omega \in H_{\max(k_1,k_2)+1}((M_1\|M_2)\vec{P})$ and we conclude since $m_1\Omega \uplus m_2\Omega = (m_1 \uplus m_2)\Omega$ and $k = \max(k_1,k_2) + 1 \leq n$.

- If $M' \equiv M_1 \| M_2$ and $\ell(\vec{z}) > 0$, then we have $H_n((\lambda\vec{z}.(M_1\|M_2)\vec{P})\Omega) = H_{n-1}(\lambda\vec{z}_{\geq 2}.(M_1[\Omega/z_1]\|M_2[\Omega/z_1])\vec{P}[\Omega/z_1])$. Now, if $m \in H_n(M\Omega)$ then $n > 2$ and there exists $m_i \in H_{n-2}(\lambda\vec{z}_{\geq 2}.M_i[\Omega/z_1]\vec{P}[\Omega/z_1]) = H_{n-1}((\lambda\vec{z}.M_i\vec{P})\Omega)$ (for $i = 1,2$) such that $m = m_1 \uplus m_2$. Of course, if m is head-free then also m_1, m_2 are. By the inductive hypothesis there exist $k_1, k_2 \leq n-1$ and $m'_i \in H_{k_i}(\lambda\vec{z}.M_i\vec{P})$ head-free such that $m_i = m'_i\Omega$ (for $i = 1,2$). Hence $m_1\Omega \uplus m_2\Omega \in H_{\max(k_1,k_2)+1}(\lambda\vec{z}.(M_1\|M_2)\vec{P})$ and we conclude since $m_1\Omega \uplus m_2\Omega = (m_1 \uplus m_2)\Omega$ and $k = \max(k_1,k_2) + 1 \leq n$.

$\square$

Lemma 3.9. *For all $M, N \in \Lambda_{+\|}$ and for all $n \in \mathbb{N}$ if $m \in H_n(M)$ is head-free, then $mN \in H_{n+1}(MN)$.*

Proof. The proof is done by induction on $n \in \mathbb{N}$.
If $n = 0$ then there is no $m \in H_0(M)$ and the implication is trivially satisfied.
If $n > 0$ then the proof is done by cases on the shape of $M \equiv \lambda\vec{z}.M'\vec{P}$.

- If $M' \equiv y$ and $\ell(\vec{z}) = 0$, then $H_n(M) = \{[y\vec{P}]\}$. Since $[y\vec{P}]$ is head-free we have to check that $[y\vec{P}]N \in H_{n+1}(MN)$, and this follows since $H_{n+1}(MN) = \{[y\vec{P}N]\}$, by definition.

- If $M' \equiv y$ and $\ell(\vec{z}) > 0$, then $H_n(M) = \{[\lambda\vec{z}.y\vec{P}]\}$. If $y \in \vec{z}$, then $H_n(M)$ does not contain any head-free multiple head normal form and the implication trivially holds. Otherwise, if $y \notin \vec{z}$, then $[\lambda\vec{z}.y\vec{P}]$ is head-free and we have to check that $[\lambda\vec{z}.y\vec{P}]N \in H_{n+1}(MN)$. This follows since $[\lambda\vec{z}.y\vec{P}]N = [\lambda\vec{z}_{\geq 2}.y\vec{P}[N/z_1]]$ and $H_{n+1}((\lambda\vec{z}.y\vec{P})N) = H_n(\lambda\vec{z}_{\geq 2}.y\vec{P}[N/z_1]) = \{[\lambda\vec{z}_{\geq 2}.y\vec{P}[N/z_1]]\}$.

- If $M' \equiv \lambda y.Q$ and there exists a head-free multiple head normal form $m \in H_n(M) = H_{n-1}(\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})$ then, by the inductive hypothesis, we have $mN \in H_n((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})N)$. If $\ell(\vec{z}) = 0$ we conclude since $H_{n+1}(((\lambda y.Q)\vec{P})N) = H_n(Q[P_1/y]\vec{P}_{\geq 2}N)$. Otherwise, when $\ell(\vec{z}) > 0$, we have:

$$\begin{array}{rcl} H_{n+1}((\lambda\vec{z}.(\lambda y.Q)\vec{P})N) & = & H_n(\lambda\vec{z}_{\geq 2}.(\lambda y.Q[N/z_1])\vec{P}[N/z_1]) \\ & = & H_{n-1}(\lambda\vec{z}_{\geq 2}.Q[N/z_1][P_1[N/z_1]/y])\vec{P}_{\geq 2}[N/z_1]) \\ & = & H_{n-1}(\lambda\vec{z}_{\geq 2}.Q[P_1/y][N/z_1])\vec{P}_{\geq 2}[N/z_1]) \\ & = & H_n((\lambda\vec{z}.Q[P_1/y]\vec{P}_{\geq 2})N). \end{array}$$

- If $M' \equiv M_1 + M_2$, then $H_n(M) = H_{n-1}(\lambda\vec{z}.M_1\vec{P}) \cup H_{n-1}(\lambda\vec{z}.M_2\vec{P})$. Thus, if there is a head-free $m \in H_n(M)$ then m belongs to, say, $H_{n-1}(\lambda\vec{z}.M_1\vec{P})$ and by the inductive hypothesis we get $mN \in H_n((\lambda\vec{z}.M_1\vec{P})N)$. If $\ell(\vec{z}) = 0$ we conclude since $H_{n+1}((M_1 + M_2)\vec{P}N) = H_n(M_1\vec{P}N) \cup H_n(M_2\vec{P}N)$. Suppose now $\ell(\vec{z}) > 0$. We conclude since

$$\begin{array}{rcl} H_{n+1}(MN) & = & H_n(\lambda\vec{z}_{\geq 2}.(M_1[N/z_1] + M_2[N/z_1])\vec{P}[N/z_1]) \\ & = & \cup_{i=1,2} H_{n-1}(\lambda\vec{z}_{\geq 2}.M_i[N/z_1]\vec{P}[N/z_1]) \\ & = & \cup_{i=1,2} H_n((\lambda\vec{z}.M_i\vec{P})N). \end{array}$$

- If $M' \equiv M_1 \| M_2$ and $m \in H_n(M)$, then there is $m_i \in H_{n-1}(\lambda\vec{z}.M_i\vec{P})$ (for $i = 1, 2$) such that $m = m_1 \uplus m_2$. Of course, if m is head-free then also m_1, m_2 are. By the inductive hypothesis we have $m_iN \in H_n(\lambda\vec{z}.M_i\vec{P}N)$ (for $i = 1, 2$). Now, if $\ell(\vec{z}) = 0$, then it is straightforward to check that $(m_1 \uplus m_2)N \in H_{n+1}(MN)$ once noticed that $m_1N \uplus m_2N = (m_1 \uplus m_2)N$. If $\ell(\vec{z}) > 0$, we conclude since

$$\begin{array}{rcl} H_{n+1}(MN) & = & H_{n+1}((\lambda\vec{z}.(M_1\|M_2)\vec{P})N) \\ & = & H_n(\lambda\vec{z}_{\geq 2}.(M_1[N/z_1]\|M_2[N/z_1])\vec{P}[N/z_1]) \\ & = & \{m_1 \uplus m_2 \mid m_i \in H_{n-1}(\lambda\vec{z}_{\geq 2}.M_i[N/z_1]\vec{P}[N/z_1]) \text{ for } i = 1, 2\} \\ & = & \{m_1 \uplus m_2 \mid m_i \in H_n(((\lambda\vec{z}.M_i)\vec{P})N) \text{ for } i = 1, 2\}. \end{array}$$

□

We are now able to provide the complete proof of Proposition 3.3 (previously announced at page 18).

Proposition 3.3. *Let $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, then we have that:*

(i) *if* $Mx \in \mathcal{N}$ *then* $M \in \mathcal{N}$*,*

(ii) *if* $M\Omega \in \mathcal{N}_1$ *then* $M \in \mathcal{N}_1$*,*

(iii) *if* $M \in \mathcal{N}_1$ *then* $MN \in \mathcal{N}_1$ *for all* $N \in \Lambda_{+\|}$*.*

Proof. (i) If $Mx \in \mathcal{N}$ then there exists a multiset $m \in H(Mx)$. By definition of $H(-)$ we have that $m \in H_n(Mx)$ for some $n \in \mathbb{N}$. By Lemma 3.7 we know that there exists $m' \in H_k(M)$ for some $k \leq n$ and hence that $H(M)$ is non-empty. We conclude that $M \in \mathcal{N}$.
(ii) If $M\Omega \in \mathcal{N}_1$ then there is $m \in H(M)$ head-free. By definition of $H(-)$ we have that $m \in H_n(M\Omega)$ for some n. Then by Lemma 3.8 there exists m' head-free such that $m' \in H_k(M)$ for some $k \leq n$. We conclude that $M \in \mathcal{N}_1$.
(iii) If $M \in \mathcal{N}_1$ then there exists $m \in H(M)$ head-free. By definition of $H(-)$ we have that $m \in H_n(M)$ for some n. From Lemma 3.9 we have that $mN \in H_{n+1}(MN)$ for all $N \in \Lambda_{+\|}$ and hence that $mN \in H(MN)$. We conclude since, if m is head-free, then also mN is. □

4. A Relational Model of $\boldsymbol{\lambda}_{+\|}$-calculus

Exploiting the existence of countable products in **MRel** we have shown in [6] that the reflexive object $\mathcal{D} = (D, \mathcal{A}, \lambda)$ built in Section 1.4 can be turned into a λ-*model* [2, Def. 5.2.1]. This was not clear before, since the object D does not have enough points (see [2, Prop. 5.5.7(ii)]). The underlying set of the λ-model associated with D by our construction is the set of "finitary" morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$, where D^{Var} is the Var-indexed categorical product of countably many copies of D.

4.1. Finitary Morphisms in MRel

The morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$ are sets of pairs whose first projection is a finite multiset of elements in D^{Var}, and whose second projection is an element of D. Since categorical products in **MRel** are disjoint unions, a typical such pair is of the form:

$$([(x_1, \sigma_1^1), \ldots, (x_1, \sigma_1^{n_1}), \ldots, (x_k, \sigma_k^1), \ldots, (x_k, \sigma_k^{n_k})], \sigma)$$

where $k, n_1, \ldots, n_k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathrm{Var}$ and $\sigma_1^1, \ldots, \sigma_k^{n_k}, \sigma \in D$.

Notation 4.1. *Given* $m \in \mathcal{M}_f(D^{\mathrm{Var}})$ *and* $x \in \mathrm{Var}$*, we set* $m_x = [\sigma \mid (x, \sigma) \in m] \in \mathcal{M}_f(D)$ *and* $m_{-x} = [(y, \sigma) \in m \mid y \neq x] \in \mathcal{M}_f(D^{\mathrm{Var}})$*.*

In general, given an object U of a ccc $\mathbf{C}$, we say that a morphism $f \in \mathbf{C}(U^{\mathrm{Var}}, U)$ is "finitary" if there exist a finite subset $I \subseteq \mathrm{Var}$ and a morphism $f_I \in \mathbf{C}(U^I, U)$ such that $f = f_I \circ \Pi_I^{\mathrm{Var}}$ (see [6, Sec. 3.1]). Intuitively, a morphism f is finitary if it only depends on a finite number of arguments. Working in $\mathbf{MRel}$ it is more convenient to take the following equivalent definition.

Definition 4.2. *A morphism $r \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$ is* finitary *if there exists a finite set I of variables such that for all $(m, \sigma) \in r$ and $x \in \mathrm{Var}$ we have that $m_x \neq []$ entails $x \in I$.*

We denote by $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ the set of all finitary morphisms.

4.2. The Model

From [6, Thm. 1] we know that $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \bullet)$, where $\bullet$ is defined as usual by $r_1 \bullet r_2 = \mathrm{eval} \circ \langle \mathcal{A} \circ r_1, r_2 \rangle$, can be endowed with a structure of λ-model.

In order to interpret $\lambda_{+\|}$-terms as finitary morphisms of $\mathbf{MRel}$ we are going to define on $\mathbf{MRel}(D^{\mathrm{Var}}, D)$ two binary operations of *sum* and *aggregation* for modelling non-deterministic choice and parallel composition, respectively, and to prove that $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ is closed under these operations.

Definition 4.3. *Let $r_1, r_2 \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$, then:*

- *the* sum *of r_1 and r_2 is defined by $r_1 \oplus r_2 = r_1 \cup r_2$.*
- *the* aggregation *of r_1 and r_2 is defined by*

$$r_1 \odot r_2 = \{(m_1 \uplus m_2, \sigma_1 \bar{\uplus} \sigma_2) \mid \exists (m_i, \sigma_i) \in r_i, \ \text{for } i = 1, 2\}.$$

Proposition 4.4. *The set $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ is closed under sum and aggregation.*

Proof. Straightforward. In both cases, the union of the finite sets of variables I_1 and I_2 given by the finiteness of the arguments of the operation, is a witness of the finiteness of the result. $\square$

Composition is right-distributive over sum and aggregation.

Proposition 4.5. *Let $r, s \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$ and $t \in \mathbf{MRel}(D^{\mathrm{Var}}, D^{\mathrm{Var}})$, then:*

- $(r \oplus s) \circ t = (r \circ t) \oplus (s \circ t)$,

- $(r \odot s) \circ t = (r \circ t) \odot (s \circ t)$.

Proof. Straightforward. $\square$

The units of the operations $\oplus$ and $\odot$ are $0 = \emptyset$ and $1 = \{([], \star)\}$, respectively; $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \oplus, 0)$ and $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \odot, 1)$ are commutative monoids. Moreover, 0 annihilates $\odot$ and aggregation distributes over sum. Summing up, the following proposition gives an overview of the algebraic properties of $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ equipped with application, sum and aggregation.

Proposition 4.6.

- $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \oplus, \odot, 0, 1)$ *is a commutative semiring.*

- $\bullet$ *is right-distributive over* $\oplus$ *and* $\odot$*.*

- $\oplus$ *is idempotent (whereas* $\odot$ *is not).*

Proof. Straightforward. $\square$

4.3. The Absolute Interpretation

Before going through the formal definition of the interpretation of $\lambda_{+\|}$-terms, we present a short digression on the nature of such an interpretation.

In our framework, the $\lambda_{+\|}$-terms will be interpreted as morphisms in $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$, i.e., as subsets of $\mathcal{M}_f(D^{\mathrm{Var}}) \times D$. The occurrence of a particular pair

$$([(x_1, \sigma_1^1), \ldots, (x_1, \sigma_1^{n_1}), \ldots, (x_k, \sigma_k^1), \ldots, (x_k, \sigma_k^{n_k})], \sigma)$$

in the interpretation of a term M may be read as "in an environment ρ such that $\rho(x_i) = [\sigma_i^1, \ldots, \sigma_i^{n_i}]$ (for all $i = 1, \ldots, k$) the interpretation $[\![M]\!]_\rho$ contains σ".

Hence, here there is no need of providing explicitly an environment to the interpretation function as classically done for λ-models [2, Def. 5.2.1(ii)] because the whole information is coded inside the elements of the λ-model itself.

On the other hand, the categorical interpretation of a term M is usually defined with respect to a finite list of variables, containing the free variables of M [2, Def. 5.5.3(vii)]. Intuitively, our interpretation is defined with respect to the list of *all* variables, encompassing then all categorical interpretations.

These considerations lead us to the definition of an interpretation function

$$[\![-]\!] : \Lambda_{+\|} \to \mathbf{MRel}_f(D^{\mathrm{Var}}, D)$$

that we call the *absolute interpretation*[7] of $\lambda_{+\|}$-terms:

- $[\![x]\!] = \pi_x^{\mathrm{Var}}$, for $x \in \mathrm{Var}$,
- $[\![M_1 M_2]\!] = \mathrm{eval} \circ \langle \mathcal{A} \circ [\![M_1]\!], [\![M_2]\!] \rangle$,
- $[\![\lambda x.M]\!] = \lambda \circ \Lambda([\![M]\!] \circ \eta_x)$,
- $[\![M_1 + M_2]\!] = [\![M_1]\!] \oplus [\![M_2]\!]$,
- $[\![M_1 \| M_2]\!] = [\![M_1]\!] \odot [\![M_2]\!]$,

where $\eta_x \in \mathbf{MRel}(D^{\mathrm{Var}} \,\&\, D, D^{\mathrm{Var}})$ is defined componentwise, for $y \in \mathrm{Var}$, by:

$$\pi_y^{\mathrm{Var}} \circ \eta_x = \begin{cases} \pi_2 & \text{if } x \equiv y, \\ \pi_y^{\mathrm{Var}} \circ \pi_1 & \text{if } x \not\equiv y. \end{cases}$$

In what follows, we will use the inductive characterization of the interpretation of (some) $\lambda_{+\|}$-terms provided by the proposition below:

Proposition 4.7.

(i) $[\![x]\!] = \{([(x,\sigma)], \sigma) \mid \sigma \in D\}$,

(ii) $[\![MN]\!] = \{(m_0 \uplus m_1 \uplus \ldots \uplus m_k, \sigma) \mid \exists k \geq 0, (m_0, [\tau_1, \ldots, \tau_k] :: \sigma) \in [\![M]\!],$ $(m_i, \tau_i) \in [\![N]\!] \textit{ for } 1 \leq i \leq k\}$,

(iii) $[\![\lambda x.M]\!] = \{(m_{-x}, m_x :: \sigma) \mid (m, \sigma) \in [\![M]\!]\}$.

Proof. Simple calculations based on the definitions of Section 1. □

We show now the soundness of the interpretation with respect to β-conversion, which relies on the following lemma.

[7]See [17, Sec. 2.3.2] (and cf. [22]) for more details on the relations among the absolute, algebraic and categorical interpretations, and on how the former allows to recover the others.

Lemma 4.8. *If $M, N \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, then $[\![M[N/x]]\!] = [\![M]\!] \circ \eta_x \circ \langle \mathrm{Id}, [\![N]\!] \rangle$.*

Proof. By structural induction on M. The cases $M \equiv M_1 + M_2$ and $M \equiv M_1 \| M_2$ are settled by using Proposition 4.5. For the other cases, one can use Proposition 4.7 and the following characterization of $\eta_x \circ \langle \mathrm{Id}, [\![N]\!] \rangle \in \mathbf{MRel}(D^{\mathrm{Var}}, D^{\mathrm{Var}})$:

$$\begin{aligned} \eta_x \circ \langle \mathrm{Id}, [\![N]\!] \rangle = \ & \{([(y,\sigma)], (y,\sigma)) \mid \sigma \in D, y \not\equiv x\} \cup \\ & \{(m, (x,\sigma)) \mid (m,\sigma) \in [\![N]\!]\}. \end{aligned}$$

□

Lemma 4.9. *(Soundness) For all $M, N \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, we have $[\![(\lambda x.M)N]\!] = [\![M[N/x]]\!]$.*

Proof.

$$\begin{aligned} [\![(\lambda x.M)N]\!] &= \mathrm{eval} \circ \langle \mathcal{A} \circ \lambda \circ \Lambda([\![M]\!] \circ \eta_x), [\![N]\!] \rangle && \text{by def.} \\ &= \mathrm{eval} \circ \langle \Lambda([\![M]\!] \circ \eta_x), [\![N]\!] \rangle && \text{by } \mathcal{A} \circ \lambda = \mathrm{Id} \\ &= [\![M]\!] \circ \eta_x \circ \langle \mathrm{Id}, [\![N]\!] \rangle && \text{by } (\beta_{\mathsf{cat}}) \\ &= [\![M[N/x]]\!] && \text{by Lemma 4.8} \end{aligned}$$

.

□

We aim to prove that our model is *sensible* w.r.t. the operational semantics: a $\lambda_{+\|}$-term M has a non-empty interpretation if, and only if, M is solvable.

We start showing that the interpretation of every solvable term is non-empty (for the converse we will adapt Krivine's realizability method [15], see Section 5). This is an immediate corollary of the following propositions stating that the interpretation of a $\lambda_{+\|}$-term includes the union of the interpretations of its multiple head normal forms and that the interpretation of any head normal form is non-empty.

Proposition 4.10. *For all $M \in \Lambda_{+\|}$, we have $(\bigoplus_{m \in H(M)} (\bigodot_{N \in m} [\![N]\!])) \subseteq [\![M]\!]$.*

Proof. It is enough to show that $(\bigoplus_{m \in H_n(M)} (\bigodot_{N \in m} [\![N]\!])) \subseteq [\![M]\!]$ holds for all $n \in \mathbb{N}$; we prove it by induction on n. The case $n = 0$ is trivial. The proof of the inductive step goes by case analysis on the head subterm M' of $M \equiv \lambda \vec{z}.M'\vec{P}$.

- The case $M' \equiv x$ is trivial, and the case $M' \equiv \lambda y.Q$ is settled by Lemma 4.9.
- If $M' \equiv Q_1 \| Q_2$, we start by observing that $[\![M]\!] = [\![\lambda\vec{z}.Q_1\vec{P}]\!] \odot [\![\lambda\vec{z}.Q_2\vec{P}]\!]$. This is an easy consequence of the right distributivity of $\bullet$ over $\odot$ (Proposition 4.6) and of the fact that, by Proposition 4.7(iii), we have $[\![\lambda\vec{x}.(R_1\|R_2)]\!] = [\![\lambda\vec{x}.R_1]\!] \odot [\![\lambda\vec{x}.R_2]\!]$, for all $\vec{x} \in \mathrm{Var}$ and $R_1, R_2 \in \Lambda_{+\|}$. Then, we can conclude by the inductive hypothesis.
- The case $M' \equiv Q_1 + Q_2$ is similar, and simpler, once noted that $[\![M]\!] = [\![\lambda\vec{z}.Q_1\vec{P}]\!] \oplus [\![\lambda\vec{z}.Q_2\vec{P}]\!]$ (again, by Proposition 4.6 and Proposition 4.7(iii)).

□

We now show that every basic head normal form has a non-empty interpretation.

Proposition 4.11. *For all $x, \vec{y} \in \mathrm{Var}$ and $\vec{Q} \in \Lambda_{+\|}$ we have $[\![\lambda\vec{y}.x\vec{Q}]\!] \neq \emptyset$.*

Proof. By Proposition 4.7(iii), it is sufficient to prove that, for all $x \in \mathrm{Var}$ and $\vec{Q} \in \Lambda_{+\|}$, we have $[\![x\vec{Q}]\!] \neq \emptyset$. To conclude, it is easy to show by induction on k that $([(x,\star)],\star) \in [\![xQ_1 \dots Q_k]\!]$. □

Theorem 4.12. *For all $M \in \Lambda_{+\|}$, if $H(M) \neq \emptyset$ then $[\![M]\!] \neq \emptyset$.*

Proof. Let $[N_1, \dots, N_k] \in H(M)$. By Proposition 4.10, $\bigodot_{1\leq i\leq k} [\![N_i]\!] \subseteq [\![M]\!]$, and by Proposition 4.11 $[\![N_i]\!] \neq \emptyset$ for $1 \leq i \leq k$. We conclude that $\emptyset \neq \bigodot_{1\leq i\leq k} [\![N_i]\!] \subseteq [\![M]\!]$. □

5. Saturated Sets and the Realizability Argument

In this section, we generalize Krivine's realizability technique [15] to $\lambda_{+\|}$-calculus and we use it for proving that $\lambda_{+\|}$-terms having a non-empty interpretation are all solvable. For notations and terminology, we mainly follow [3].

The saturation of a set S of terms expresses the fact that S is closed under weak head expansions. For the pure λ-calculus, this amounts to the well known condition of being closed under weak head β-expansion. For the extension of the λ-calculus we are dealing with, three cases of weak head expansions, corresponding to the possible shapes of the head term, must be considered.

Definition 5.1. *A set* $S \subseteq \Lambda_{+\|}$ *is* saturated *if the following conditions hold:*

- *if* $M[N/x]\vec{P} \in S$ *then* $(\lambda x.M)N\vec{P} \in S$,
- *if* $(MQ\|NQ)\vec{P} \in S$ *then* $(M\|N)Q\vec{P} \in S$,
- *if* $M\vec{P} \in S$ *and* $N \in \Lambda_{+\|}$ *then* $(M+N)\vec{P} \in S$.

We recall that the sets $\mathcal{N}_0, \mathcal{N}_1$ and $\mathcal{N}$ have been defined in Section 3. It is easy to check that $\mathcal{N}$ is saturated, whilst $\mathcal{N}_0$ is not. In the realizability argument, only saturated sets included within $\mathcal{N}_0$ and $\mathcal{N}$ will be considered.

Definition 5.2. *The set* Sat_h *of* small saturated subsets of $\Lambda_{+\|}$ *is defined by:*

$$\mathrm{Sat}_h = \{S \subseteq \Lambda_{+\|} \mid \ S \textit{ is saturated and } \mathcal{N}_0 \subseteq S \subseteq \mathcal{N}\}.$$

Given $A, B \subseteq \Lambda_{+\|}$, we define $A \to B = \{M \in \Lambda_{+\|} \mid (\forall N \in A)\ MN \in B\}$. The operator $\to$ is contravariant in its first argument and covariant in its second one, in other words, $A \to B \subseteq A' \to B'$ for all $A' \subseteq A$ and $B \subseteq B'$.

Lemma 5.3. $\mathcal{N}_0 \subseteq \Lambda_{+\|} \to \mathcal{N}_0 \subseteq \mathcal{N}_0 \to \mathcal{N} \subseteq \mathcal{N}$.

Proof. The first inclusion follows by definition, the second one is a consequence of the contravariance/covariance of the arrow. For the third one, it is enough to prove that, for all $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, $H(Mx) \neq \emptyset$ entails $H(M) \neq \emptyset$; this holds by Proposition 3.3(i). □

The set Sat_h enjoys the following closure properties.

Lemma 5.4. *The set* Sat_h *is closed under the arrow operator, finite unions, finite intersections, and under the map* $\mathcal{F} : S \mapsto (\Lambda_{+\|} \to S)$.

Proof. Given two sets $S_1, S_2 \in \mathrm{Sat}_h$, it is straightforward to check that $S_1 \cap S_2$, $S_1 \cup S_2 \in \mathrm{Sat}_h$ and that $S_1 \to S_2$ and $\Lambda_{+\|} \to S_2$ are saturated. The inclusions $\mathcal{N}_0 \subseteq S_1 \to S_2 \subseteq \mathcal{N}$ and $\mathcal{N}_0 \subseteq \Lambda_{+\|} \to S_2 \subseteq \mathcal{N}$ follow easily from Lemma 5.3 and contravariance/covariance of the arrow. □

We are going to define a function $(-)^\bullet : D \to \mathrm{Sat}_h$, satisfying $(m :: \sigma)^\bullet = m^\bullet \to \sigma^\bullet$, where, for a multiset m of elements of D, $m^\bullet = \bigcap_{\alpha \in m} \alpha^\bullet$ and, in particular, $[]^\bullet = \Lambda_{+\|}$. Since $\star = [] :: \star$, the set $\star^\bullet$ must be a fixpoint of the function $\mathcal{F} : S \mapsto (\Lambda_{+\|} \to S)$. We now show that $\mathcal{N}_1$ is one of such fixpoints.

Proposition 5.5. *$\mathcal{N}_1 \in \mathrm{Sat}_h$ and $\mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$.*

Proof. The saturation of $\mathcal{N}_1$ and the fact that $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are both trivial. We now prove that $\mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$. Let $M \in \Lambda_{+\|} \to \mathcal{N}_1$. Since $M\Omega \in \mathcal{N}_1$, we get by Proposition 3.3(*ii*) that $M \in \mathcal{N}_1$. Conversely, let $M \in \mathcal{N}_1$ and $N \in \Lambda_{+\|}$. We conclude since, by Proposition 3.3(*iii*), we get $MN \in \mathcal{N}_1$. $\square$

Observe that any element $\sigma \in D$ may be written in a unique way as $\sigma = \sigma_1 :: \cdots :: \sigma_n :: \star$, with $n \geq 0$ and $\sigma_n \neq []$ (and of course $\sigma_1, \ldots, \sigma_n$ have ranks strictly smaller than that of σ). This is called the *standard decomposition* of σ.

Definition 5.6. *Given $\sigma \in D$, we define $(\sigma)^\bullet \in \mathrm{Sat}_h$ by induction on the rank k of σ. If $k = 0$, then $\sigma^\bullet = \star^\bullet = \mathcal{N}_1$. If $k > 0$ then $\sigma^\bullet = \sigma_1^\bullet \to \cdots \to \sigma_n^\bullet \to \mathcal{N}_1$, where $\sigma_1 :: \cdots :: \sigma_n :: \star$ is the standard decomposition of σ.*

Note that if $m \neq []$ or $\sigma \neq \star$, then the standard decomposition of $m :: \sigma$ is $m :: \sigma_1 :: \cdots :: \sigma_n :: \star$, where $\sigma_1 :: \cdots :: \sigma_n :: \star$ is the standard decomposition of σ. Hence, $(m :: \sigma)^\bullet = m^\bullet \to \sigma^\bullet$ holds in general, since $([] :: \star)^\bullet = \star^\bullet = \mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$.

We show now that the definition of $(-)^\bullet$ fits well with parallel composition.

Lemma 5.7. *Let $M, N \in \Lambda_{+\|}$, $\sigma = (\sigma_1, \sigma_2, \ldots), \tau = (\tau_1, \tau_2, \ldots) \in D$ and $\rho = \sigma \bar{\uplus} \tau$. If $M \in \sigma^\bullet$ and $N \in \tau^\bullet$, then $M\|N \in \rho^\bullet$.*

Proof. Let $\rho_n :: \cdots :: \rho_1 :: \star$ be the standard decomposition of ρ. We have to show that $M\|N \in \rho_n^\bullet \to \cdots \to \rho_1^\bullet \to \mathcal{N}_1$. We prove it by induction on n.

If $n = 0$, then $\sigma = \tau = \rho = \star$. Hence, we conclude since $\star^\bullet = \mathcal{N}_1$ and $\mathcal{N}_1$ is closed under parallel composition.

If $n > 0$, then we have to show that, for all $Q \in \rho_n^\bullet$, $(M\|N)Q \in (\rho')^\bullet$ where $\rho' = \rho_{n-1} :: \cdots :: \rho_1 :: \star$. Since $M \in \sigma_1^\bullet$ and $N \in \tau_1^\bullet$, we have that $MQ \in (\sigma')^\bullet$ and $NQ \in (\tau')^\bullet$, where $\sigma' = (\sigma_2, \sigma_3, \ldots)$ and $\tau' = (\tau_2, \tau_3, \ldots)^\bullet$. Moreover, $\rho' = \sigma' \bar{\uplus} \tau'$ and the standard decomposition of ρ' is strictly shorter than that of ρ. By the inductive hypothesis, we get $MQ\|NQ \in (\rho')^\bullet$. By saturation of $(\rho')^\bullet$, we conclude that $(M\|N)Q \in (\rho')^\bullet$, and hence $M\|N \in \rho^\bullet$. $\square$

We are now able to prove the main lemma, which constitutes the key tool in the realizability argument.

Definition 5.8. *A substitution* $s = \{(x_1, N_1), \ldots, (x_k, N_k)\}$ *is* adequate *for a multiset* $m \in \mathcal{M}_f(D^{\mathrm{Var}})$ *if:*

- $m_x \neq []$ *implies* $x \in \{x_1, \ldots, x_k\}$*, for all* $x \in \mathrm{Var}$*,*
- $N_i \in m_{x_i}^{\bullet}$ *for all* $1 \leq i \leq k$*.*

Observe that, if a substitution is adequate for some multiset $m \in \mathcal{M}_f(D^{\mathrm{Var}})$, then it is adequate for all submultisets of m.

Lemma 5.9. *Let* $M \in \Lambda_{+\|}$*,* $(m, \sigma) \in [\![M]\!]$ *and* s *be a substitution. If* s *is adequate for* m*, then* $Ms \in \sigma^{\bullet}$*.*

Proof. By structural induction on M.

- If $M \equiv x$, then $m = [(x, \sigma)]$ by Proposition 4.7(i). If s is adequate for m, then $(x, N) \in s$ for some $N \in [\sigma]^{\bullet}$. Hence, we have that $Ms = N \in [\sigma]^{\bullet} = \sigma^{\bullet}$.
- If $M \equiv PQ$, then by Proposition 4.7(ii), we have $m = m_0 \uplus m_1 \uplus \ldots \uplus m_k$ for some $k \geq 0$, and $\tau_1, \ldots, \tau_k \in D$ such that $(m_0, [\tau_1, \ldots, \tau_k] :: \sigma) \in [\![P]\!]$ and $(m_i, \tau_i) \in [\![Q]\!]$ *for* $1 \leq i \leq k$. Observe now that, if s is adequate for m then it is also adequate for $m_0, m_1, \ldots, m_k$, since they are all multisubsets of m. By the inductive hypothesis we have that:

  - $Ps \in ([\tau_1, \ldots, \tau_k] :: \sigma)^{\bullet} = [\tau_1, \ldots, \tau_k]^{\bullet} \to \sigma^{\bullet}$,
  - $Qs \in \tau_1^{\bullet}, \ldots, Qs \in \tau_k^{\bullet}$, which implies that $Qs \in [\tau_1, \ldots, \tau_k]^{\bullet}$.

  Hence, we can conclude that $(PQ)s \in \sigma^{\bullet}$.
- If $M \equiv \lambda x.P$, then by Proposition 4.7(iii), we have that $m = m'_{-x}$ and $\sigma = m'_x :: \sigma'$ for some $(m', \sigma') \in [\![P]\!]$. Let s be an adequate substitution for m'_{-x} and $Q \in (m'_x)^{\bullet}$. Since M is considered up to α-conversion, we can suppose without loss of generality that x does not occur in s. It is clear that $s' = s \cup \{(x, Q)\}$ is adequate for m' and hence, by the inductive hypothesis, we get $Ps' \in (\sigma')^{\bullet}$. Now we have that $Ps' = (Ps)[Q/x] \in (\sigma')^{\bullet}$ because x does not appear in s. Since $(\sigma')^{\bullet}$ is saturated and $(\lambda x.Ps) = (\lambda x.P)s$ we have that $(\lambda x.P)sQ \in (\sigma')^{\bullet}$. From the arbitrariness of $Q \in (m'_x)^{\bullet}$ we conclude that $(\lambda x.P)s \in (m'_x)^{\bullet} \to (\sigma')^{\bullet} = (m'_x :: \sigma')^{\bullet}$.
- If $M \equiv P + Q$, then (m, σ) belongs to, say, $[\![P]\!]$. Now, if s is adequate for m, then we get by the inductive hypothesis that $Ps \in \sigma^{\bullet}$ and we conclude, by saturation of $\sigma^{\bullet}$, that $(P + Q)s \in \sigma^{\bullet}$.

- If $M \equiv P\|Q$, then $m = m_1 \uplus m_2$ and $\sigma = \sigma_1 \overline{\uplus} \sigma_2$ with $(m_1, \sigma_1) \in [\![P]\!]$ and $(m_2, \sigma_2) \in [\![Q]\!]$. If s is adequate for m then it is also adequate for m_1, m_2 and, from the inductive hypothesis and Lemma 5.7, we conclude that $(P\|Q)s \in (\sigma_1 \overline{\uplus} \sigma_2)^\bullet$.

$\square$

Theorem 5.10. *For all $M \in \Lambda_{+\|}$, if $[\![M]\!] \neq \emptyset$ then $M \in \mathcal{N}$.*

Proof. Let $(m, \sigma) \in [\![M]\!]$. The substitution $s_{\mathrm{Id}} = \{(x, x) \mid m_x \neq []\}$ is adequate for m (note that $\mathrm{Var} \subset \mathcal{N}_0$), and $M s_{\mathrm{Id}} = M$. Hence, by Lemma 5.9, we conclude that $M \in \sigma^\bullet \subseteq \mathcal{N}$. $\square$

By Theorem 4.12 and Theorem 5.10 we finally get our main result.

Theorem 5.11. *For all $M \in \Lambda_{+\|}$, $H(M) \neq \emptyset \Leftrightarrow [\![M]\!] \neq \emptyset$.*

6. Adequacy and Full Abstraction

Results like Theorem 5.11 are often called "adequacy theorems". This can be a bit misleading if one consider that the notions of adequacy, and full abstraction, are relative to a given operational preorder on terms. So far, we have proved that the interpretation of a term is non-empty if and only if the term is solvable. Now, given a notion of solvability, we address the issue of the adequacy of the denotational interpretation with respect to the canonical contextual preorder, defined below.

Definition 6.1. *(Contextual preorder) Given $M, N \in \Lambda_{+\|}$, we write $M \sqsubseteq_o N$ if for all contexts $C\langle - \rangle$, $H(C\langle M \rangle]) \neq \emptyset$ entails $H(C\langle N \rangle) \neq \emptyset$.*

Then, adequacy, as expressed in Corollary 6.2, is an easy consequence of Theorem 5.11 and of monotonicity of the denotational interpretation with respect to the operation consisting in putting a term in a context.

Corollary 6.2. *For all $M, N \in \Lambda_{+\|}$, if $[\![M]\!] \subseteq [\![N]\!]$ then $M \sqsubseteq_o N$.*

The converse, namely the implication $M \sqsubseteq_o N \Rightarrow [\![M]\!] \subseteq [\![N]\!]$, is the other half of full abstraction, and does not hold here. Consider for instance the terms $\mathbf{I}$ and $\mathbf{I}\|\mathbf{I}$, and let us start showing that $[\![\mathbf{I}]\!] \not\subseteq [\![\mathbf{I}\|\mathbf{I}]\!]$. By definition, we have that $[\![\mathbf{I}]\!] = \{([], [\sigma] :: \sigma \mid \sigma \in D\}$ and $[\![\mathbf{I}\|\mathbf{I}]\!] = [\![\mathbf{I}]\!] \odot [\![\mathbf{I}]\!] = \{([], [\sigma, \sigma] :: (\sigma \overline{\uplus} \sigma)) \mid \sigma \in D\}$. Clearly, $[\![\mathbf{I}]\!] \not\subseteq [\![\mathbf{I}\|\mathbf{I}]\!]$.

It remains to show that $\mathbf{I} \sqsubseteq_o \mathbf{I} \| \mathbf{I}$. Instead of trying a syntactical "*tour de force*" for proving this quite intuitive statement, we can advocate the existence (in the folklore) of adequate models of $\lambda_{+\|}$-calculus where the aggregation monad interpreting the parallel operator $\|$ is idempotent. This is the case, for instance, of the model presented in [9], but this argument is weakeaned by the fact that operational semantics, and hence the operational preorders, are not exactly the same in the two frameworks. On the other hand, getting an idempotent version of $\|$ out of the relational framework is not easy: the replacement of multisets by sets and multi-unions by unions simply does not provide a model.

In a forthcoming paper [12], an adequate interpretation of $\lambda_{+\|}$-calculus endowed with an idempotent operator $\|$ will be provided. Moreover, the model presented there can be seen as the "extensional collapse" of our model.

Hence, $\mathbf{I} \sqsubseteq_o \mathbf{I} \| \mathbf{I}$ since in that particular adequate model $[\![\mathbf{I}]\!] = [\![\mathbf{I} \| \mathbf{I}]\!]$.

7. Related Works

The extension of λ-calculus with parallel and non-deterministic features has been the subject of a wealth of research works, some of which are cited in the introduction of the present one. Among those works, the papers [9] and [10], by Dezani-Ciancaglini, De Liguoro and Piperno have to be mentioned here, since they deal with exactly the language $\lambda_{+\|}$-calculus, focussing on the relation between its operational and denotational semantics.

Nevertheless, in [10], the $\lambda_{+\|}$-calculus is endowed with a *lazy* operational semantics. This means that the corresponding operational preorder is incomparable with ours. For instance, in their semantics, the term Ω is strictly smaller than $\lambda x.\Omega$, which is a normal form.

On the other hand, several notions of solvability have been examined in [9]; one of them, arising from the head rewriting relation $\to_{pn}^{h}$, is similar to ours.

To be precise, the roles of $+$ and $\|$ are switched in their framework with respect to ours: parallel composition behaves like a disjunction, and non-deterministic choice as a conjunction. The reduction rules are the same, except for the one concerning parallel composition which is synchronous ($\Omega \| \mathbf{I}$ is a normal form). A term is solvable if *all* its head reductions terminate. Altogether, a term M turns out to be solvable in our framework if, and only if, the term obtained by switching $+$ and $\|$ in M is solvable in the sense of [9].

The issue of full abstraction with respect to the canonical contextual preorder associated with solvability is left open in [9].

8. Conclusions and Further Work

We have defined a relational model $\mathcal{D}$ of a fairly standard parallel and non-deterministic extension of the pure untyped λ-calculus, equipped with a notion of observation given by a generalized form of head-normalization.

We have proved that the model $\mathcal{D}$ is adequate for the canonical contextual preorder.

Nevertheless, we have also shown that the full abstraction fails since, for instance, $\mathbf{I}$ and $\mathbf{I}\|\mathbf{I}$ are not separable, but their interpretations are different.

As suggested by the counterexample, the next step towards full abstraction should be to enrich the syntax of the language by some "resource sensitive" operators, to increase the discriminating power of contexts.

An alternative approach to obtain a full abstraction result would consist in keeping the language and its operational semantics unchanged, and providing a model with less discriminating power. To begin with, parallel compositon should be interpreted by an idempotent operation. The already mentioned model presented in [12] is actually a good candidate for providing a fully abstract interpretation of $\lambda_{+\|}$-calculus endowed with the observational preorder $\sqsubseteq_o$.

Finally, we already know from [17, Sec. 3.3] that the theory induced on the pure untyped λ-calculus by our model $\mathcal{D}$ is $\mathcal{H}^*$ (just as the theory induced by Scott's $\mathcal{D}_\infty$); it would be interesting to generalize such a result to the extended setting, as a step in the study and classification of $\lambda_{+\|}$-theories, and models.

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