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About the extensional collapse of **Rel**

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Qualitative vs. quantitative semantics

Traditionally, denotational semantics is based on domains (continuous, stable maps) and is *qualitative*.

This means that the interpretation of a term does not take into account the number of times arguments are used to produce a result.

Example: stable semantics makes no difference between the terms

- $M_1 = \lambda x^{\mathbf{Bool}}\,\mathtt{if}\,x\,\mathtt{then}\,(\mathtt{if}\,x\,\mathtt{then}\,\mathbf{t}\,\mathtt{else}\,\mathbf{f})\,\mathtt{else}\,\mathbf{f}$
- and $M_2 = \lambda x^{\mathbf{Bool}}\,x$

both are of type $\mathbf{Bool} \to \mathbf{Bool}$ and are interpreted as $\{(\{\mathbf{t}\},\mathbf{t}),(\{\mathbf{f}\},\mathbf{f})\}$ in the usual model of coherence spaces.

Quantitative semantics

In the simplest quantitative semantics, one replaces finite sets by finite multisets and one can remove coherence.

$M_1 = \lambda x^{\textbf{Bool}}\, \texttt{if}\, x\, \texttt{then}\, (\texttt{if}\, x\, \texttt{then}\, \mathbf{t}\, \texttt{else}\, \mathbf{f})\, \texttt{else}\, \mathbf{f}$ is interpreted as $\{([\mathbf{t},\mathbf{t}],\mathbf{t}),([\mathbf{t},\mathbf{f}],\mathbf{f}),([\mathbf{f}],\mathbf{f})\}$.

$M_2 = \lambda x^{\textbf{Bool}}\, x$ is interpreted as $\{([\mathbf{t}],\mathbf{t}),([\mathbf{f}],\mathbf{f})\}$.
This semantics is more informative in two ways:

- quantitative informations
- it takes into account non-deterministic behaviours of arguments (the point $([\mathbf{t},\mathbf{f}],\mathbf{f})$).

De Carvalho & Tortora de Falco: relational semantics is injective on normal proof-nets (MELL).

The linear logic viewpoint

The main difference between the two approaches is the interpretation of the LL exponential $!X$:

- in the qualitative setting (coherence spaces), $!X$ uses finite sets (actually finite cliques) of elements of X
- in the quantitative setting, $!X$ uses finite multisets of elements of X (coherence is not needed).

The other constructions (additives and multiplicatives) are very similar in both settinge.

It is possible to get rid of coherence in the qualitative setting as well and to replace finite cliques by finite sets in the construction of $!X$.

Scott semantics

But *this requires to move to Scott semantics*:

- In the quantitative setting, we can simply interpret a formula A as a set X with no additional structure. A proof of A is interpreted as a subset of X.
- But this does not work in the qualitative setting. In other words: if we interpret all formulae as in the quantitative setting, but for $!X$ for which we use $\mathcal{P}_{\mathrm{fin}}(X)$ instead of $\mathcal{M}_{\mathrm{fin}}(X)$, we don't get a model.

Failure of coherence-free qualitative semantics

Take $!X = \mathcal{P}_{\mathrm{fin}}(X)$ and

- if $s \subseteq X \times Y$ then define $!s \subseteq !X \times !Y$ by $!s = \{(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}) \mid n \in \mathbb{N} \text{ and } \forall i\, (a_i, b_i) \in s\}$
- $\mathrm{d}_X = \{(\{a\}, a) \mid a \in X\}$.

Then if $X = Y = \mathbf{Bool} = \{\mathbf{t}, \mathbf{f}\}$, $s = \{(\mathbf{t}, \mathbf{t}), (\mathbf{f}, \mathbf{t})\}$, we have

- $(\{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{t}\} = \{\mathbf{t}\}) \in !s$ and so $(\{\mathbf{t}, \mathbf{f}\}, \mathbf{t}) \in \mathrm{d}_{\mathbf{Bool}} \circ !s$
- but $(\{\mathbf{t}, \mathbf{f}\}, \mathbf{t}) \notin s \circ \mathrm{d}_{\mathbf{Bool}}$.

- In coherence spaces, since $s = \{(\mathbf{t}, \mathbf{t}), (\mathbf{f}, \mathbf{t})\}$ must be a clique in $\mathbf{Bool} \multimap \mathbf{Bool}$, we must have $\mathbf{t} \smile_{\mathbf{Bool}} \mathbf{f}$ and hence $\{\mathbf{t}, \mathbf{f}\} \notin |!\mathbf{Bool}|$ so $(\{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}) \notin !s$.
- In quantitative semantics $[\mathbf{t}, \mathbf{t}] \neq [\mathbf{t}]$ so $([\mathbf{t}, \mathbf{f}], [\mathbf{t}]) \notin !s$.

One has to endow X (et least) with a "Scott structure", that is, a preorder relation. Then proofs are interpreted as downwards closed sets.

The quantitative model

Category **Rel** of sets and relations $\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$.

- $E^\perp = E$
- $E \otimes F = E \parr F = E \times F$
- $\&_{i\in I} E_i = \bigoplus_{i\in I} E_i = \bigcup_{i\in I} \{i\} \times E_i$
- $!E = ?X = \mathcal{M}_{\mathrm{fin}}(E)$

If $t \in \mathbf{Rel}(E, F)$ then
$!t = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N} \text{ and } \forall i\ (a_i, b_i) \in t\}$.

- $\mathrm{d}_E = \{([a], a) \mid a \in E\} \in \mathbf{Rel}(!E, E)$ (dereliction)
- $\mathrm{p}_E = \{(m_1 + \cdots + m_n, [m_1, \dots, m_n]) \mid n \in \mathbb{N} \text{ and } \forall i\ m_i \in \mathcal{M}_{\mathrm{fin}}(E)\}$ (digging)
- $\mathrm{w}_E = \{([], *)\} \in \mathbf{Rel}(!E, 1)$ (weakening)
- $\mathrm{c}_E = \{(l + r, (l, r)) \mid l, r \in \mathcal{M}_{\mathrm{fin}}(E)\} \in \mathbf{Rel}(!E, !E \otimes !E)$ (contraction)

Moreover, **Rel** is a model of differential linear logic:

- $\overline{\mathsf{d}}_E = \{(a, [a]) \mid a \in E\} \in \mathbf{Rel}(E, !E)$ (codereliction), with $\mathsf{d}_E \circ \overline{\mathsf{d}}_E = \mathsf{Id}_E$.
- $\overline{\mathsf{w}}_E = \{(*, [])\} \in \mathbf{Rel}(1, !E)$ (coweakening)
- $\overline{\mathsf{c}}_E = \{(l, r, l + r) \mid l, r \in !E\} \in \mathbf{Rel}(!E \otimes !E, !E)$ (cocontraction)

$(!E, \overline{\mathsf{w}}_E, \overline{\mathsf{c}}_E, \mathsf{w}_E, \mathsf{c}_E)$ is a commutative bialgebra and moreover

$$\begin{tikzcd}
E \arrow[rr, "\overline{\mathsf{d}}_E"] \arrow[d, "0"'] & & !E \arrow[dll, "\mathsf{w}_E"] \\
1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, "\overline{\mathsf{d}}_E"] \arrow[d, "\overline{\mathsf{d}}_E \otimes \overline{\mathsf{w}}_E + \overline{\mathsf{w}}_E \otimes \overline{\mathsf{d}}_E"'] & & !E \arrow[dll, "\mathsf{c}_E"] \\
!E \otimes !E & &
\end{tikzcd}
\qquad \text{(Leibniz)}$$

$$\begin{tikzcd}
E & & !E \arrow[ll, "\mathsf{d}_E"'] \\
1 \arrow[u, "0"] \arrow[urr, "\overline{\mathsf{w}}_E"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
E & & !E \arrow[ll, "\mathsf{d}_E"'] \\
!E \otimes !E \arrow[u, "\mathsf{d}_E \otimes \mathsf{w}_E + \mathsf{w}_E \otimes \mathsf{d}_E"] \arrow[urr, "\overline{\mathsf{c}}_E"'] & &
\end{tikzcd}
\qquad \text{(Linearity)}$$

And more properties.

Scott semantics (Huth): the category **Scott**

Objects are preorders $S = (|S|, \leq_S)$ ($\leq_S$ preorder relation on $|S|$). Define $\mathcal{I}(S) = \{u \subseteq |S| \mid \forall a, a' \in |S|\ a \leq_S a' \in u \Rightarrow a \in u\}$.

Scott(S, T) is the set of linear maps $\mathcal{I}(S) \to \mathcal{I}(T)$, that is, maps preserving all unions.

- $S^\perp = S^{\mathrm{op}} = (|S|, \geq_S)$
- $S \otimes T = S \parr T = (|S| \times |T|, \leq_S \times \leq_T)$
- $\&_{i \in I} S_i = (\bigcup_{i \in I} \{i\} \times |S_i|, \leq)$ with $(i, a) \leq (j, b)$ if $i = j$ and $a \leq_{S_i} b$
- $\bigoplus_{i \in I} S_i = \&_{i \in I} S_i$.

$!S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq_{!S})$ with $u \leq_{!S} u'$ if $\forall a \in u\, \exists a' \in u'\ a \leq_S a'$.
$?S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq_{?S})$ with $u \leq_{?S} u'$ if $\forall a' \in u'\, \exists a \in u\ a \leq_S a'$.

A linear map $S \to T$ can be seen as an element of $\mathcal{I}(S^\perp \mathbin{⅋} T) = \mathcal{I}(S^{\mathsf{op}} \times T)$, and composition of maps corresponds to composition of relations.

$\mathsf{Id}_S = \{(a, a') \in |S| \times |S| \mid a' \leq_S a\}$.

As usual $S \multimap T = S^\perp \mathbin{⅋} T$.

For $a \in X$ and $m \in !X$, "$a \in m$" means that a occurs at least once in m.

Notice: what is important is $\mathcal{I}(S)$, not S itself, and if we set $!S = (\mathcal{M}_{\mathrm{fin}}(|S|), \leq_{!S})$ with $m \leq_{!S} m'$ if $\forall a \in m \, \exists a' \in m' \; a \leq_S a'$, then we define the same $\mathcal{I}(!S)$. This is what we do now.

If $t \in \mathcal{I}(S \multimap T)$, we define $!t \in \mathcal{I}(!S \multimap !T)$ by

$$!t = \{(m, p) \in |!S \multimap !T| \mid \forall b \in p \exists a \in m \; (a, b) \in t\}.$$

- $\mathrm{d}_S = \{(m, a) \mid \exists a' \in m \; a \leq_S a'\} \in \mathcal{I}(!S \multimap S)$ (dereliction)
- $\mathrm{p}_S = \{(m, [m_1, \dots, m_n]) \mid n \in \mathbb{N} \, \forall i \; m_i \leq_{!S} m\} \in \mathcal{I}(!S \multimap !!S)$ (digging)
- $\mathrm{w}_S = \{(m, *) \mid \in |!S|\} \in \mathcal{I}(!S \multimap 1)$ (weakening)
- $\mathrm{c}_S = \{(m, (l, r)) \mid l \leq_{!S} m \text{ and } r \leq_{!S} m\} \in \mathcal{I}(!S \multimap !S \otimes !S)$ (contraction).

Scott is not a model of differential linear logic

There is no *natural transformation* $\overline{\mathrm{d}}_S \in \mathcal{I}(S \multimap\, !S)$ such that $\mathrm{d}_S \circ \overline{\mathrm{d}}_S = \mathsf{Id}_S$.

Not surprising: differential LL is essentially sensitive to the quantitative features of proofs, and Scott semantics completely cancels this aspect.

Morphisms in the Kleisli category

In the Kleisli category of this comonad (over the category of preorders and linear maps), a morphism $S \to T$ is exactly a Scott continuous function $\mathcal{I}(S) \to \mathcal{I}(T)$.

This Kleisli category is a CCC.

Comparison of LL formulae interpretations

We interpret the atomic formulae of LL in the same way in both models (if α is interpreted by the set E in **Rel**, we represent it as $(E, =_E)$ in the Scott model).

If A is a formula of LL,

- E its interpretation in **Rel**
- and S its interpretation in the Scott model,

then $|S| = E$. This is due to our definition of $!S$.

Scott semantics as downwards closure

If π is a proof of A,

- $x \in \mathcal{P}(E)$ its relational interpretation
- and $u \in \mathcal{I}(S) \subseteq \mathcal{P}(E)$ its Scott interpretation,

what is the connection between these two sets?

If π is cut-free, then one sees easily that $u = \downarrow_S x$. So this equation extends to arbitrary LL proofs by cut-elimination and invariance of the semantics.

How to extend this to extensions of LL where not all programs are terminating (pure lambda-calculus, PCF etc)?

Preorders with projections

The idea is to work with mixed objects $X = (\langle X\rangle, \mathrm{D}(X))$ where

- $\langle X\rangle$ is a preorder
- $\mathrm{D}(X)$ is a set of subsets of $\mathcal{P}(|\langle X\rangle|)$.

It will make sense to compute $\downarrow x$ only for $x \subseteq |\langle X\rangle|$ such that $x \in \mathrm{D}(X)$.

Given a "proof" π of X,

- the relational semantics x of π will belong to $\mathrm{D}(X)$
- and its Scott semantics will be $\downarrow x \in \mathcal{I}(\langle X\rangle)$.

Main definition: orthogonality condition

Let S be a preorder and let $x, x' \subseteq |S|$.

We say that x and x' are S-dual if

$$x' \cap \downarrow_S x \neq \emptyset \Rightarrow x' \cap x \neq \emptyset \,.$$

This is equivalent to

$$\uparrow_S x' \cap x \neq \emptyset \Rightarrow x' \cap x \neq \emptyset \,.$$

so that x and x' are S-dual iff x' and x are $S^\perp$-dual.

Let $D \subseteq \mathcal{P}(|S|)$, we set

$$D^{\perp(S)} = \{x' \subseteq |S| \mid \forall x \in D\ x \text{ and } x' \text{ are } S\text{-dual}\}$$

so that one has $D \subseteq D' \Rightarrow D'^{\perp(S)} \subseteq D^{\perp(S)}$ and $D \subseteq D^{\perp(S)\perp(S^{\mathrm{op}})}$ and hence $D^{\perp(S)} = D^{\perp(S)\perp(S^{\mathrm{op}})\perp(S)}$.

A *preorder with projection* (PP) is a pair $X = (\langle X\rangle, \mathrm{D}(X))$ where $\langle X\rangle$ is a preorder and $\mathrm{D}(X) \subseteq \mathcal{P}(|\langle X\rangle|)$ is such that

$$\mathrm{D}(X) = \mathrm{D}(X)^{\perp(S)\perp(S^{\mathrm{op}})} \quad (\text{that is } \mathrm{D}(X) \supseteq \mathrm{D}(X)^{\perp(S)\perp(S^{\mathrm{op}})})$$

So given $x \subseteq |\langle X\rangle|$, to check that $x \in \mathrm{D}(X)$, it suffices to prove that

$$\forall x' \in \mathrm{D}(X)^{\perp(S)} \quad x' \cap \downarrow_{\langle X\rangle} x \neq \emptyset \Rightarrow x' \cap x \neq \emptyset\,.$$

Some immediate consequences of the definition

- $\mathcal{I}(\langle X\rangle) \subseteq \mathsf{D}(X) \subseteq \mathcal{P}(|\langle X\rangle|)$.
- If $A \subseteq \mathsf{D}(X)$ then $\cup A \in \mathsf{D}(X)$.
- $|\langle X\rangle| \in \mathsf{D}(X)$.

Morphisms of PP

Let X and Y be PPs. We define a PP $X \multimap Y$ by

- $\langle X \multimap Y\rangle = \langle X\rangle \multimap \langle Y\rangle = \langle X\rangle^{\mathrm{op}} \times \langle Y\rangle$ (product preorder)
- and, given $t \subseteq |\langle X\rangle| \times |\langle Y\rangle|$, one has $t \in \mathrm{D}(X \multimap Y)$ if, for all $x \in \mathrm{D}(X)$ and for all $y' \in \mathrm{D}(Y)^{\perp(\langle Y\rangle)}$,

$$t \cap (\downarrow_{\langle X\rangle} x \times \uparrow_{\langle Y\rangle} y') \neq \emptyset \Rightarrow t \cap (x \times y') \neq \emptyset\,.$$

In other words

$$\mathrm{D}(X \multimap Y) = \{x \times y' \mid x \in \mathrm{D}(X) \text{ and } y' \in \mathrm{D}(Y)^{\perp(\langle Y\rangle)}\}^{\perp(\langle X\rangle \times \langle Y\rangle^{\mathrm{op}})}$$

so that $X \multimap Y$ defined in that way is a PP.

Lemma

Let $t \subseteq |\langle X\rangle| \times |\langle Y\rangle|$. One has $t \in \mathrm{D}(X \multimap Y)$ iff, for any $x \in \mathrm{D}(X)$,

- $t\,x \in \mathrm{D}(Y)$ *(where $t\,x = \{b \mid \exists a \in x\ (a,b) \in t\}$)*
- $t\,(\downarrow_{\langle X\rangle} x) \subseteq \downarrow_{\langle Y\rangle}(t\,x)$.

From this, it follows easily that

- $\mathrm{Id}_{|\langle X\rangle|} \in \mathrm{D}(X \multimap X)$ (diagonal relation)
- and, given $s \in \mathrm{D}(X \multimap Y)$ and $t \in \mathrm{D}(Y \multimap Z)$, $t \circ s \in \mathrm{D}(X \multimap Z)$ (relational composition).

So we have defined a category **PP** with $\mathbf{PP}(X,Y) = \mathrm{D}(X \multimap Y)$.

PP is a model of LL

- $X^\perp = (\langle X\rangle^{\mathrm{op}}, \mathrm{D}(X)^{\perp(\langle X\rangle)})$
- $X \otimes Y = (X \multimap Y^\perp)^\perp$, in other words $\mathrm{D}(X \otimes Y) = \{x \times y \mid x \in \mathrm{D}(X) \text{ and } y \in \mathrm{D}(Y)\}^{\perp(\langle X\rangle\times\langle Y\rangle)\perp(\langle X\rangle^{\mathrm{op}}\times\langle Y\rangle^{\mathrm{op}})}$
- $X \parr Y = X^\perp \multimap Y$

Next we define $!X = \{x^! \mid x \in \mathrm{D}(X)\}^{\perp(!\langle X\rangle)\perp((!\langle X\rangle)^{\mathrm{op}})}$ where $x^! = \mathcal{M}_{\mathrm{fin}}(x)$.

It is essential to observe that $(\downarrow_{\langle X\rangle} x)^! = \downarrow_{\langle !X\rangle}(x^!)$.

Functional characterization of Kleisli morphisms

Lemma

Let $t \subseteq |\langle !X \multimap Y\rangle|$, one has $t \in \mathrm{D}(!X \multimap Y)$ iff, for any $x \in \mathrm{D}(X)$,

- $t\, x^! \in \mathrm{D}(Y)$
- *and* $t\,(\downarrow_{\langle X\rangle} x)^! \subseteq \downarrow_{\langle Y\rangle}(t\, x^!)$.

Using this characterization, one shows that

- If $t \in \mathrm{D}(X \multimap Y)$ then $!t = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N}\, \forall i\ (a_i, b_i) \in t\} \in \mathrm{D}(!X \multimap\, !Y)$

- $\mathsf{d}_{|\langle X\rangle|} = \{([a], a) \mid a \in |\langle X\rangle|\} \in \mathrm{D}(!X \multimap X)$
- $\mathsf{p}_{|\langle X\rangle|} = \{(m_1 + \cdots + m_n, [m_1, \dots, m_n]) \mid n \in \mathbb{N},\ m_1, \dots, m_n \in |\langle !X\rangle|\} \in \mathrm{D}(!X \multimap !!X)$
- $\mathsf{w}_{|\langle X\rangle|} = \{([], *)\} \in \mathrm{D}(!X \multimap 1)$
- $\mathsf{c}_{|\langle X\rangle|} = \{(l + r, (l, r)) \mid l, r \in |\langle !X\rangle|\} \in \mathrm{D}(!X \multimap !X \otimes !X)$

But one can check that, in general, $\overline{\mathsf{d}}_{|\langle X\rangle|} \notin \mathrm{D}(X \multimap !X)$: **PP** is not a model of differential LL.

Realizability

Let X be a **PP** and let $x \subseteq |\langle X \rangle|$ and $u \in \mathcal{I}(\langle X \rangle)$. Then we say that x realizes u, notation $x \Vdash_X u$, if $x \in \mathrm{D}(X)$ and $\downarrow_{\langle X \rangle} x = u$.

Lemma

Let $t \subseteq |\langle !X \multimap Y \rangle|$ and $w \in \mathcal{I}(\langle !X \multimap Y \rangle)$. One has $t \Vdash_{!X \multimap Y} w$ iff, for all $x \subseteq |\langle X \rangle|$ and all $u \in \mathcal{I}(\langle X \rangle)$

$$x \Vdash_X u \Rightarrow t\, x^! \Vdash_Y w\, u^!$$

w represents a Scott continuous function $f : \mathcal{I}(X) \to \mathcal{I}(Y)$, and $w\, u^! = f(u)$. Using this lemma one can show that "The Scott model of LL is the extensional collapse of the relational model".

Substructures

Given preorders S and S', we write $S \subseteq S'$ if $|S| \subseteq |S'|$, and

$$\forall a, b \in |S| \quad a \leq_S b \Leftrightarrow a \leq_{S'} b\,.$$

Given PPs X and X', we write $X \subseteq X'$ if

- $\langle X \rangle \subseteq \langle X' \rangle$
- $\mathrm{D}(X) \subseteq \mathrm{D}(X')$
- $\forall x' \in \mathrm{D}(X') \quad x' \cap |\langle X \rangle| \in \mathrm{D}(X)$ and $(\downarrow_{\langle X' \rangle} x') \cap |\langle X \rangle| \subseteq \downarrow_{\langle X \rangle}(x' \cap |\langle X \rangle|)$.

Then $\subseteq$ is a partial order relation on PPs and the class of PPs, ordered by this relation, is complete (all directed lubs exist).

All linear logic constructs are continuous wrt. this partial order, including linear negation.

As an example, consider the operation $\Phi(X) = (!X \multimap !X)$. Then Φ has a least fixpoint U which satisfies $U = !U \multimap !U$.

It is well known that such an object (in any categorical model of LL) is a model of the cbv lambda-calculus (Girard's "boring translation", Wadler).

A cbv lambda-calculus

Values (notation P, $Q\dots$)

- If x is a variable then x is a value
- if M is a term and x is a value then $\lambda x\, M$ is a value

and terms (notation M, $N\dots$)

- if P is a value then $\langle P\rangle$ is a term
- if M and N are terms then $M\,N$ is a term.

Substitution: if E is a term (resp. a value) and P is a value then $E\,[P/x]$ is a term (resp. a value).

Reduction: $\langle\lambda x\, M\rangle\,\langle P\rangle\ \beta_{\mathsf{V}}\ M\,[P/x]$.

Linear categorical semantics

Let $\mathcal{L}$ be a categorical model of LL (cartesian $*$-autonomous category $\mathcal{L}$ with a comonad $(!, \mathsf{d}_X : !X \to X, \mathsf{p}_X : !X \to !!X)$ and Seely isomorphisms $!(X \mathbin{\&} Y) \simeq !X \otimes !Y)$.

A linear model of cbv is a triple $(U, \mathsf{app}, \mathsf{lam})$ where U is an object of $\mathcal{L}$, $\mathsf{app} \in \mathcal{L}(U, !U \multimap !U)$, $\mathsf{lam} \in \mathcal{L}(!U \multimap !U, U)$ with $\mathsf{app} \circ \mathsf{lam} = \mathsf{Id}_{!U \multimap !U}$.

Remark: the Kleisli category $\mathcal{L}_!$ is a CCC and ! defines a strong monad on $\mathcal{L}_!$ so that this notion of model of cbv is compatible with models based on strong monads.

Interpretation of expressions

Given an expression E and a list of variables $\vec{x} = (x_1, \ldots, x_n)$ which is repetition free and contains all free variables of E, one defines $[E]^{\vec{x}} \in \mathcal{L}(!U^{\otimes n}, X)$ where $X = U$ if E is a value and $X = !U$ if E is a term.

Example: interpreting abstractions and value terms

- If M is a term, then by inductive hypothesis $[M]^{\vec{x},x} : !U^{\otimes n} \otimes !U \to !U$. Using monoidal closedness we have $\lambda([M]^{\vec{x},x}) : !U^{\otimes n} \to (!U \multimap !U)$ and one sets

$$[\lambda x\, M]^{\vec{x}} = \mathrm{lam} \circ \lambda([M]^{\vec{x},x}) : !U^{\otimes n} \to U$$

- If P is a value then by inductive hypothesis $[P]^{\vec{x}} : !U^{\otimes n} \to U$ and one sets

$$[\langle P \rangle]^{\vec{x}} = ![P]^{\vec{x}} \circ p : !U^{\otimes n} \to !U$$

  where $p : !U^{\otimes n} \to !(!U^{\otimes n})$ is a generalization of the digging morphism $\mathrm{p}_X : !X \to !!X$.

If $E\ \beta_{\mathsf{V}}\ E'$ then $[E]^{\vec{x}} = [E']^{\vec{x}}$.

A cbv finite resource calculus

We define simple expressions (values and terms).

- If x is a variable then x is a simple value.
- If t is a simple term and x is a variable then $\lambda x\, t$ is a simple value.
- If $p_1, \dots, p_n$ are simple values then $\langle p_1, \dots, p_n \rangle$ is a simple term (the order does not matter).
- If s and t are simple terms then $s\, t$ is a simple term.

A term is a finite sum of simple terms and a value is a finite sum of simple values.

All constructions are extended to terms and values by linearity, e.g. $\langle p_1 + p_2, q \rangle = \langle p_1, q \rangle + \langle p_2, q \rangle$.

Reduction in the cbv resource calculus

Simple expressions reduce to expressions (of the same kind).

- $\langle p_1, \dots, p_n \rangle \, t \; \beta_{\mathsf{V}} \; 0$ if $n \neq 1$
- $\langle \lambda x \, t \rangle \, \langle p_1, \dots, p_n \rangle \; \beta_{\mathsf{V}} \; 0$ is $n \neq$ the number of free occurrences of x in t.
- $\langle \lambda x \, t \rangle \, \langle p_1, \dots, p_n \rangle \; \beta_{\mathsf{V}} \; \sum_{f \in \mathfrak{S}_n} t \left[p_1 / x_{f(1)}, \dots, p_n / x_{f(n)} \right]$ if t has exactly n occurrences $x_1, \dots, x_n$ of x.

Lemma

This calculus is Church-Rosser and strongly normalizing.

Assume that $\mathcal{L}$ is also a "weak model" of differential LL, which means that:

- products and coporducts coincide in $\mathcal{L}$ so that each homset $\mathcal{L}(X, Y)$ has a canonical structure of commutative monoid ($\mathcal{L}$ is additive).
- There is a codereliction natural transformation $\overline{\mathsf{d}}_X \in \mathcal{L}(X, !X)$ such that $\mathsf{d}_X \circ \overline{\mathsf{d}}_X = \mathsf{Id}_X$.

Then coweakening $\overline{\mathsf{w}}_X : 1 \to !X$ and $\overline{\mathsf{c}}_X : !X \otimes !X \to !X$ can be defined using additivity of $\mathcal{L}$ and the Seely isomorphisms.

The fact that $(!X, \overline{\mathrm{w}}_X, \overline{\mathrm{c}}_X, \mathrm{w}_X, \mathrm{c}_X)$ is a commutative bialgebra and the required diagrams

$$
\begin{tikzcd}
X \arrow[rd, "\overline{\mathrm{d}}_X"] \arrow[dd, "0"'] & \\
& !X \arrow[ld, "\mathrm{w}_X"] \\
1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rd, "\overline{\mathrm{d}}_X"] \arrow[dd, "\overline{\mathrm{d}}_X \otimes \overline{\mathrm{w}}_X + \overline{\mathrm{w}}_X \otimes \overline{\mathrm{d}}_X"'] & \\
& !X \arrow[ld, "\mathrm{c}_X"] \\
!X \otimes !X &
\end{tikzcd}
\quad \text{(Leibniz)}
$$

$$
\begin{tikzcd}
X & \\
& !X \arrow[lu, "\mathrm{d}_X"'] \\
1 \arrow[uu, "0"] \arrow[ru, "\overline{\mathrm{w}}_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X & \\
& !X \arrow[lu, "\mathrm{d}_X"'] \\
!X \otimes !X \arrow[uu, "\mathrm{d}_X \otimes \mathrm{w}_X + \mathrm{w}_X \otimes \mathrm{d}_X"] \arrow[ru, "\overline{\mathrm{c}}_X"'] &
\end{tikzcd}
\quad \text{(Linearity)}
$$

are all for free: they result from the naturality of d_X and $\overline{\mathrm{d}}_X$.

Why "weak"? Because we don't need here to say anything about the interaction between $\overline{\mathrm{d}}_E$ and p_E (chain rule).

Given a cbv model $(U, \mathrm{app}, \mathrm{lam})$ in a weak model of differential LL, we can interpret the resource calculus above.

Main tool: using cocontraction, coweakening and codereliction, we can define for all n a morphism $\overline{\mathrm{d}}_X^{(n)} : X^{\otimes n} \to !X$.

Given a simple expression e and a list of variables $\vec{x} = (x_1, \dots, x_n)$ which is repetition free and contains all free variables of e, one defines $[e]^{\vec{x}} \in \mathcal{L}(!U^{\otimes n}, X)$ where $X = U$ if e is a value and $X = !U$ if e is a term.

For $e = \sum_{i=1}^{k} e_i$ sum of simple terms, we set of course $[e]^{\vec{x}} = \sum_{i=1}^{k} [e_i]^{\vec{x}}$.

- If t is a simple term, then by inductive hypothesis $[t]^{\vec{x},x} : !U^{\otimes n} \otimes !U \to !U$. Using monoidal closedness we have $\lambda([t]^{\vec{x},x}) : !U^{\otimes n} \to (!U \multimap !U)$ and one sets

$$[\lambda x\, t]^{\vec{x}} = \mathsf{lam} \circ \lambda([t]^{\vec{x},x}) : !U^{\otimes n} \to U$$

- If $p_1, \dots, p_k$ are simple values then by inductive hypothesis $[p_i]^{\vec{x}} : !U^{\otimes n} \to U$ and one sets

$$[\langle p_1, \dots, p_k \rangle]^{\vec{x}} = \overline{\mathrm{d}}_U^{(k)} \circ ([p_1]^{\vec{x}} \otimes \cdots \otimes [p_k]^{\vec{x}}) \circ c$$

  where $c : !U^{\otimes n} \to (!U^{\otimes n})^{\otimes k}$ is a generalization of the contraction morphism $\mathrm{c}_X : !X \to !X \otimes !X$.

One proves easily that $e \;\beta_{\mathsf{V}}\; e' \Rightarrow [e]^{\vec{x}} = [e']^{\vec{x}}$.

In **PP**, we have built an object U such that $U = !U \multimap !U$.

So U is a model of cbv in **PP** with $\mathsf{app} = \mathsf{Id}$ and $\mathsf{lam} = \mathsf{Id}$ and

- $U_r = |\langle U \rangle|$ is a model of cbv in **Rel** (which is a weak model of differential linear logic)
- $U_s = \langle U \rangle$ is a model of cbv in **Scott** (which is *not* a weak model of differential linear logic)

In U_r, we can interpret the cbv lambda-calculus, and also the cbv finite resource calculus, and the Taylor expansion formula holds (in **Rel**, sum of morphisms is just union). Here, it reads

$$[E]_r^{\vec{x}} = \bigcup \{[e]^{\vec{x}} \mid e \in \mathcal{T}(E)\}$$

where $\mathcal{T}(E)$ is defined as

- $\mathcal{T}(x) = \{x\}$
- $\mathcal{T}(\lambda x\, M) = \{\lambda x\, t \mid t \in \mathcal{T}(M)\}$
- $\mathcal{T}(\langle P \rangle) = \{\langle p_1, \dots, p_n \rangle \mid n \in \mathbb{N} \text{ and } p_1, \dots, p_n \in \mathcal{T}(P)\}$
- $\mathcal{T}(M\, N) = \{s\, t \mid s \in \mathcal{T}(M) \text{ and } t \in \mathcal{T}(N)\}$

Corollary: a cbv sensitivity theorem

Let M be a closed term in the cbv lambda-calculus, and assume that $[M]_s \neq \emptyset$.

Since U is a model of cbv in **PP**, we have $[M]_r = [M] \in \mathrm{D}(!U)$ and, *as a consequence*,

$$[M]_s = \downarrow_{\langle !U \rangle}([M]_r)\,.$$

Therefore $[M]_r \neq \emptyset$.

But $[M]_r = \bigcup\{[t] \mid t \in \mathcal{T}(M)\}$, and hence there exists $t \in \mathcal{T}(M)$ such that $[t] \neq \emptyset$.

Hence the normal form t_0 of t, which is a finite set of normal values of the cbv resource calculus, is non-empty. Indeed $[t_0] = [t] \neq \emptyset$.

It follows easily (by mimicking on M the reduction of t) that M reduces to a term of shape $\langle P \rangle$ where P is a closed value (abstraction).

This result can be proved by a reducibility method, that is, using a logical relation which involves both syntax (the cbv lambda-calculus) and semantics (the Scott model U_s).

Here, reducibility is completely semantical (the main definition is that of $\mathrm{D}(X)$). It induces a well-behaved functor $\mathbf{PP} \to \mathbf{Scott}$. Using this functor we get the equation $[M]_s = \downarrow_{\langle !U \rangle}([M]_r)$. The rest of the proof is completely elementary and relies on Taylor expansion and strong normalization of the resource calculus.

The same model **PP** can be used to prove similar results for the Scott semantics of various lambda-calculi (PCF, cbn etc).