INTEGRATION IN CONES

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January 16, 2023

Abstract. Measurable cones, with linear and measurable functions as morphisms, are a model of intuitionistic linear logic and of probabilistic PCF which accommodates "continuous data types" such as the real line. So far they lacked however a major feature to make them a model of other probabilistic programming languages: a general and versatile theory of integration which is the key ingredient for interpreting the sampling programming primitives. The goal of this paper is to develop such a theory based on a notion of integrable cones and of integral preserving linear maps: our definition of integrals is an adaptation to cones of Pettis integrals in topological vector spaces. We prove that we obtain again in that way a model of Linear Logic for which we develop two exponential comonads: the first based on a notion of stable functions introduced in earlier work and the second based on a new notion of integrable analytic function on cones.

An updated version of this document can be found here:
`https://www.irif.fr/~ehrhard/pub/integrable-cones.pdf`

Introduction

There are several approaches in the denotational semantics of functional probabilistic programming languages that we can summarize as follows:

- quasi-Borel spaces (QBS) [VKS19] which are, roughly speaking, separated presheaves on the cartesian category of measurable spaces and measurable functions (or on a full cartesian sub-category thereof);
- probabilistic games [DH00] which are similar to deterministic games apart that now strategies are probability distributions on plays;
- models based on categories of domains, possibly equipped with a probabilistic monad, and where morphisms are Scott continuous functions;
- probabilistic coherence spaces [DE11] (PCS) which can be understood as a refinement of the relational model of Linear Logic LL where formulas (types) are interpreted as sets and programs as families of non-negative real numbers indexed by these sets (not necessarily probability distributions). In this model morphisms are either linear functions or analytic functions (in the CCC used for interpreting the programming languages). This approach can be understood as extending to higher types the basic idea of [Koz81] which is to interpret programs as probability distribution transformers.

Modern probabilistic programming languages deal with probability distributions on continuous data-types such as the real line, and PCSs are not able to represent such types: PCSs are fundamentally of a discrete nature. In recent works [EPT18, Ehr20] we have developed a "continuous" extension of the PCS semantics based on a notion of *positive cone* introduced by Selinger in [Sel04] (we will often drop the adjective "positive"). Cones are similar to Banach spaces, with the difference that, in a cone, "everything is positive"; for instance the coefficients are taken in $\mathbb{R}_{\geq 0}$ and not in $\mathbb{R}$ and $x + y = 0$ is possible only if $x = y = 0$. For that reason cones are naturally ordered and are required to satisfy a completeness property expressed *à la* Scott, in terms of the norm and of this order relation, and not as Cauchy-completeness wrt. a distance induced by the norm as in ordinary Banach spaces. This has the major benefit of making easier the interpretation of recursive programs (no need of contractiveness assumptions). Cones are naturally equipped with a notion of linear morphisms, which are also assumed to be Scott continuous, and with a notion of non-linear morphism introduced in [EPT18], called stable functions and characterized by a *total monotonicity* condition (plus Scott continuity) which allow to define a cartesian closed category where fixpoint operators are available at all types. With these morphisms, cones are a conservative extension of the category of PCSs and analytic functions as shown in [Cru18].

The most essential feature of any probabilistic programming language is the possibility of *sampling* a value according to a given probability distribution. In the presence of continuous data-types this requires some form of integration and therefore the morphisms (here, the linear or the stable functions between cones) must satisfy a suitable measurability condition. In [EPT18, Ehr20] the cones were accordingly equipped with a *measurability structure* defined in reference to a collection of basic measurability spaces (such a collection can be simply $\{\mathbb{R}\}$, what we assume in this introduction for simplicity). Given a cone P equipped with such a measurability structure $\mathcal{M}$ it is then possible to define a class of bounded[1] functions $\mathbb{R} \to P$ that we call the *measurable paths* of P. And then a (linear or stable) function $P \to Q$ is *measurable* from $(P, \mathcal{M})$ to $(Q, \mathcal{N})$ if its pre-composition with any $\mathcal{M}$-measurable path of P gives a $\mathcal{N}$-measurable path of Q. Equipped with their measurability paths, these measurable cones (more precisely, their unit balls) can be considered as QBSs, and the condition above of measurable path preservation is exactly the same as the definition of a morphism of QBSs (however notions such as linearity, stability or analyticity, which are crucial for us, do not arise naturally in the framework of QBSs).

These measurable cones were sufficient in [EPT18] to allow sampling over the type of, say, real numbers in a probabilistic extension of PCF because all types in such a language can be written $A_1 \Rightarrow \cdots \Rightarrow A_n \Rightarrow \mathbb{R}$ and hence integrability for paths valued in such a type boils down to the integrability of $\mathbb{R}$-valued paths (with additional parameters in $A_1, \dots, A_n$) which is possible by our measurability assumptions. But if we want to interpret a call-by-value (or even call-by-push-value) language then we face the problem of integrating functions valued in more general cones such as for instance $!\mathbb{R}$ (in the sense of Linear Logic). So we must deal with cones where measurable paths can be integrated. Fortunately it turns out that, thanks to the properties of the measurability structure $\mathcal{M}$ of a cone P, it is easy to define the integral of a P-valued path $\gamma : \mathbb{R} \to P$ wrt. a finite measure μ on $\mathbb{R}$: it is an $x \in P$ such that, for any measurability test m on P, the real number $m(x)$ is equal to the standard Lebesgue integral $\int m(\gamma(r))\mu(dr)$ which is well defined and belongs to $\mathbb{R}_{\geq 0}$ since $m \circ \gamma$ is

[1]With respect to the norm of P.

measurable and bounded, and μ is finite. And when such an x exists it is unique by our assumptions that the measurability tests associated with a cone separate it. So we can define a cone to be integrable if such integrals always exist, whatever be the choices of γ and μ.

In that way we are able to define a category of *integrable cones* and *linear and integrable maps*, that is, linear and measurable maps of cones which moreover commute with all integrals. Such linear maps will sometimes be called integrable. It is rather easy to prove that this locally small category is complete, has a cogenerator and is well-powered so that we can apply the special adjoint functor theorem to any continuous functor from this category to any other locally small category. This allows first to equip our category with a tensor product: given two integrable cones B, C (we keep the measurability structures implicit), we can form the integrable cone $B \multimap C$ whose elements are the linear integrable maps from B to C, addition is defined pointwise and the norm is defined by $\|f\| = \sup_{\|x\|\leq 1} \|f(x)\|$. Then the functor $B \multimap \_$ is easily seen to preserve all limits and hence has a left adjoint $\_ \otimes B$. And we can prove that one defines in that way a tensor product $\_ \otimes \_$ which makes our category symmetric monoidal closed[2].

There is a very important faithful functor from the category of measurable spaces and sub-probability kernels to the category of measurable cones which maps a measurable space $\mathcal{X}$ to the cone $\mathsf{Meas}(\mathcal{X})$ of finite non-negative measures on $\mathcal{X}$. As already explained in [Geo21] (in a slightly different context) the integral preservation property that we enforce on linear morphisms on cones has the major benefit of making this functor not only faithful but also full.

In a second part of the paper we define two cartesian closed categories of integrable cones and non-linear morphisms which are Scott continuous and measurable. We also develop the associated notions of resource comonad (in the sense of the semantics of LL, see for instance [Mel09]) applying the special adjoint functor theorem to the continuous inclusion functor from the category of integrable cones and integrable linear functions to the non-linear category.

- In the first case the non-linear morphisms between integrable cones are the *measurable and stable functions* that were introduced in [EPT18]. These morphisms are Scott continuous functions satisfying a “total monotonicity” condition, which is an iterated form of monotonicity (plus preservation of measurable paths by post-composition of course). A peculiarity of this construction is that apparently no integral preservation condition is imposed on these morphisms[3].
- We can consider this fact as an issue for which we propose a solution by defining a notion of *analytic morphism* as the bounded limits of polynomial functions which are themselves described as finite sums of functions of shape $x \mapsto f(x, \dots, x)$ where f is an n-linear symmetric integrable and measurable function. These analytic functions are of course stable but not all stable functions are analytic because this latter notion is based on integrable linearity[4].

For any measurable space $\mathcal{X}$, we show that for both exponential comonads described above, the integrable cone $\mathsf{Meas}(\mathcal{X})$ has a canonical structure of coalgebra, which means that this

[2]In [Ehr20] we used the fact that PCSs are dense in cones to prove this result but this is actually not necessary, thanks to a slightly stronger assumption on the measurability structure of cones.

[3]Notice that it is not possible to expect that non-linear morphisms will preserve integrals but one could expect that they satisfy a weakened version of this condition.

[4]An n-ary integrable multilinear function is a function with n-arguments which is linear and integrable in each parameter.

cone can be considered as a *data-type* in the sense of [Kri90] or in the sense of the *positive formulas* of Polarized Linear Logic [Gir91, LR03, Ehr16]. The associated `let` operator can also be understood as a sampling construct, it is interpreted using this coalgebra structure which is defined using integration in the integrable cone $!\mathsf{Meas}(\mathcal{X})$. Combined with the fact that the Kleisli categories of these comonads are cartesian closed and ω-cpo enriched, this means that integrable cones provide a semantics for a large number of functional programming languages with continuous data types and basic probability features.

Besides measurable cones, one major source of inspiration of this work is [Geo21], which introduces the notion of *convex QBSs*, which are a particular class of algebras on the Giry monad of sub-probability measures in the category of QBSs. In other words, a convex QBS is a QBS equipped with an abstract, algebraic operation of "integration" from which all elementary operations of a cone can be derived. The main differences with respect to the present setting are, first, that linear negation in convex QBSs is involutive (because they are defined as dual pairs), and second, that measurability in convex QBSs is axiomatized in the QBS manner, by equipping each object with a collection of "measurable paths" from $\mathbb{R}$ to this object, satisfying sheaf-like conditions[5]. In integrable cones, following [EPT18], measurability is axiomatized by means of a "measurability structure", *ie.* a collection of "test functions" that map a real number and an element of the cone to an a non-negative real number, measurably with respect to the first variable, and linearly and continuously with respect to the second. In turn, this measurability structure induces a class of measurable maps from $\mathbb{R}$ to the cone, turning the latter into a QBS: a map from $\mathbb{R}$ to the cone is measurable if and only if its composition with any test function is a measurable map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}_{\geq 0}$ (by composition, we mean that the second argument of the test function is replaced by the map, and the first argument is left alone). A map between integrable cones is measurable when it is a morphism of QBSs. This means that, from the point of view of measurability alone (*ie.* if we forget the algebraic structure), integrable cones can be seen as a particular class of QBSs whose QBS structure can be defined as the "dual" of a set of test functions. This restriction has the pleasant consequence of making the theory of measurability and integration in cones quite easy, reducing it to standard Lebesgue integration by means of post-composition with tests.

Similarly defined integrals of functions ranging in topological vector spaces separated by their topological duals have been introduced by Pettis a long time ago [Pet38], and are also known as *weak integrals* or *Gelfand-Pettis integrals.* The transposition of this definition in our positive cone setting turns out to be quite suitable, thanks to its compatibility with categorical limits.

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[5]In fact, these two differences are closely linked: negation in convex QBSs can be involutive precisely because their measurability is axiomatized in the QBS manner, without restrictions on the QBS-structure.

1. Preliminaries

1.1. **Notations.** In the whole paper, we say that a set is countable if it is finite or has the same cardinality as $\mathbb{N}$.

If $\mathcal{X}$ is a measurable space, μ a non-negative measure on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}_{\geq 0}$ a non-negative measurable function, we use

$$\int f(r)\mu(dr)$$

for the integral $\in \overline{\mathbb{R}_{\geq 0}}$ rather than the more usual $\int f(r)d\mu(r)$. The reason of this choice is that it is much more convenient when the measure arises as the image of a kernel $\kappa : \mathcal{Y} \rightsquigarrow \mathcal{X}$ in which case we can use the non ambiguous notation $\int f(r)\kappa(s, dr)$. This notation is also intuitively compelling if we see dr as representing metaphorically an "infinitesimal" measurable subset of $\mathcal{X}$.

If a is an element and $n \in \mathbb{N}$ we use $\overline{a}^n$ for the n-tuple $(a, \dots, a)$.

We use $\mathbb{N}^+$ for $\mathbb{N} \setminus \{0\}$.

If $n \in \mathbb{N}$ we set $\overline{n} = \{1, \dots, n\}$.

We use notations borrowed to the lambda-calculus to denote mathematical functions: if e is a mathematical expression for an element of B depending on a parameter $x \in A$, we use $\boldsymbol{\lambda} x \in A \cdot e$ for the corresponding function $A \to B$.

1.2. **Categories.** The following is a standard and very useful Yoneda-like lemma which gives a simple tool for proving that two functors are naturally isomorphic by checking that two associated indexed classes of homsets are in natural bijective correspondence.

Lemma 1.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $F, G : \mathbf{C} \to \mathbf{D}$ *be functors and let* $\psi_{C,D} : \mathbf{D}(F(C), D) \to \mathbf{D}(G(C), D)$ *be a natural bijection. Then the family of morphisms* $\eta_C = \psi_{C,F(C)}(\mathsf{Id}_{F(C)}) \in \mathbf{D}(G(C), F(C))$ *is a natural isomorphism whose inverse is the family of morphisms* $\theta_C = \psi_{C,G(C)}{}^{-1}(\mathsf{Id}_{G(C)}) \in \mathbf{D}(F(C), G(C))$.

Proof. We prove first naturality of η, so let $f \in \mathbf{C}(C, C')$, we have

$$\begin{aligned} F(f)\,\eta_C &= (\mathbf{D}(G(C), F(f)) \circ \psi_{C,F(C)})(\mathsf{Id}_{F(C)}) \\ &= (\psi_{C,F(C')} \circ \mathbf{D}(F(C), F(f)))(\mathsf{Id}_{F(C)}) \\ &= \psi_{C,F(C')}(F(f)) \\ &= (\psi_{C,F(C')} \circ \mathbf{D}(F(f), F(C')))(\mathsf{Id}_{F(C')}) \\ &= (\mathbf{D}(G(f), F(C')) \circ \psi_{C',F(C')})(\mathsf{Id}_{F(C')}) \\ &= \eta_{C'}\, G(f) \end{aligned}$$

by commutation of the diagrams

$$\begin{tikzcd} \mathbf{D}(F(C), F(C)) \arrow[r, "\psi_{C,F(C)}"] \arrow[d, "\mathbf{D}(F(C), F(f))"'] & \mathbf{D}(G(C), F(C)) \arrow[d, "\mathbf{D}(G(C), F(f))"] \\ \mathbf{D}(F(C), F(C')) \arrow[r, "\psi_{C,F(C')}"] & \mathbf{D}(G(C), F(C')) \\ \mathbf{D}(F(C'), F(C')) \arrow[r, "\psi_{C',F(C')}"] \arrow[u, "\mathbf{D}(F(f), F(C'))"] & \mathbf{D}(G(C'), F(C')) \arrow[u, "\mathbf{D}(G(f), F(C'))"'] \end{tikzcd}$$

and naturality of θ is similar. Next, by naturality of ψ and definition of θ_C we have

$$\begin{aligned}\theta_C\,\eta_C &= \theta_C\,\psi_{C,F(c)}(\mathsf{Id}_{F(C)})\\ &= \mathbf{D}(G(C),\theta_C)\circ\psi_{C,F(C)})(\mathsf{Id}_{F(C)})\\ &= (\psi_{C,G(C)}\circ\mathbf{D}(F(C),\theta_C))(\mathsf{Id}_{F(C)})\\ &= \psi_{C,G(C)}(\theta_C) = \mathsf{Id}_{G(C)}\end{aligned}$$

The equation $\eta_C\,\theta_C = \mathsf{Id}_{F(C)}$ is proven similarly. □

2. Cones

A *precone* is a $\mathbb{R}_{\geq 0}$-semimodule P which satisfies

(**Simpl**) $\forall x_1, x_2, x \in P\ x_1 + x = x_2 + x \Rightarrow x_1 = x_2$

(**Pos**) $\forall x_1, x_2 \in P\ x_1 + x_2 = 0 \Rightarrow x_1 = 0$

Given $x_1, x_2 \in P$, we stipulate that $x_1 \leq x_2$ if $\exists x \in P\ x_2 = x_1 + x$. By (**Simpl**) and (**Pos**) this defines a partial order relation on P: *the canonical order* of P. Moreover when $x_1 \leq x_2$ there is exactly one $x \in P$ such that $x_2 = x_1 + x$, that we denote as $x_2 - x_1$. Notice that this subtraction between elements of P is only partially defined, and that it satisfies all the usual laws of subtraction.

A *cone* is a precone P equipped with a function $\|\_\|_P : P \to \mathbb{R}_{\geq 0}$ (or simply $\|\_\|$), called the *norm of* P, which satisfies the following properties.

(**Normh**) $\forall\lambda \in \mathbb{R}_{\geq 0}\forall x \in P\ \ \|\lambda x\| = \lambda\,\|x\|$

(**Normz**) $\forall x \in P\ \ \|x\| = 0 \Rightarrow x = 0$

(**Normt**) $\forall x_1, x_2 \in P\ \ \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

(**Normp**) $\forall x_1, x_2 \in P\ \ \|x_1\| \leq \|x_1 + x_2\|$ or, equivalently $\forall x_1, x_2 \in P\ x_1 \leq x_2 \Rightarrow \|x_1\| \leq \|x_2\|$.

This condition expresses the positiveness of P and implies (**Pos**), but it is seems more sensible to require (**Pos**) at the beginning because of its purely algebraic nature, and because this allows to define the useful notion of precone.

(**Normc**) Any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of P which is monotone (for the canonical order relation of P) and satisfies $\forall n \in \mathbb{N}\ \ \|x_n\| \leq 1$ has a lub $x = \sup_{n\in\mathbb{N}} x_n$ in P which satisfies $\|x\| \leq 1$.

A subset A of P is

- *bounded* if it is bounded in the sense of the norm, that is $\exists\lambda \in \mathbb{R}_{\geq 0}\forall x \in A\ \ \|x\| \leq \lambda$. We set $\mathcal{B}P = \{x \in P \mid \|x\| \leq 1\}$. Then A bounded means $\exists\lambda \in \mathbb{R}_{\geq 0}\ A \subseteq \lambda\mathcal{B}P$.
- *Scott closed* if $\forall x_1, x_2 \in P\ (x_1 \leq x_2$ and $x_2 \in A) \Rightarrow x_1 \in A$ and for any bounded monotone sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A one has $\sup_{n\in\mathbb{N}} x_n \in A$.

Notice that P and $\mathcal{B}P$ are Scott closed subsets of P.

We introduce now a terminology which will be used everywhere. A function $f : S \to P$ from a set S to a cone P is *bounded* if $f(S)$ is bounded in P.

Definition 2.1. Let P and Q be cones, let $A \subseteq P$ be Scott closed and let $f : A \to Q$ be a function.

- f is *monotone* if $\forall x_1, x_2 \in A\ x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. Notice that if f is monotone and $(x_n)_{n\in\mathbb{N}}$ is a bounded and monotone sequence in A then the sequence $(f(x_n))_{n\in\mathbb{N}}$ is bounded by $\|f(\sup_{n\in\mathbb{N}} x_n)\|$ in Q, by (**Normp**) and monotonicity of f.
- f is *Scott continuous*, or simply continuous (no other notion of continuity will be considered in this paper), if for any bounded monotone sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A, one has $f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$, that is $f(\sup_{n\in\mathbb{N}} x_n) \leq \sup_{n\in\mathbb{N}} f(x_n)$ since the converse holds by monotonicity of f.
- f is *linear* if $A = P$, $f(\lambda x) = \lambda f(x)$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$, for all $\lambda \in \mathbb{R}_{\geq 0}$ and $x, x_1, x_2 \in P$. Notice that if f is linear then f is monotone because, given $x_1, x_2 \in P$, if $x_1 \leq x_2$ then $f(x_2 - x_1) + f(x_1) = f(x_2)$. One says that f is linear and continuous if it is linear and Scott continuous.
- If $f : P \to Q$ is linear, on says that f is *bounded* if its restriction to $\mathcal{B}P$ is a bounded function.

Lemma 2.2. *Let P and Q be cones and let $f : P \to Q$ be linear and continuous. If f is bijective then f^{-1} is linear and continuous.*

Proof. Linearity follows from the injectivity of f: let $y_1, y_2 \in Q$, $x_1 = f^{-1}(y_1 + y_2)$ and $x_2 = f^{-1}(y_1) + f^{-1}(y_2)$, we have $f(x_1) = y_1 + y_2$ and $f(x_2) = y_1 + y_2$ by linearity of f, hence $x_1 = x_2$. Scalar multiplication is dealt with similarly. Since f^{-1} is linear, it is monotone.

Let $(y_n \in \mathcal{B}Q)_{n=1}^{\infty}$ be a monotone sequence and let $y \in \mathcal{B}Q$ be its lub. The sequence $(f^{-1}(y_n) \in P)_{n=1}^{\infty}$ is monotone and upper bounded by $f^{-1}(y)$ and hence bounded in norm by $\left\|f^{-1}(y)\right\|_P$, so it has a lub $x \in P$ such that $x \leq f^{-1}(y)$. By continuity of f we have $f(x) = f(\sup_{n=1}^{\infty} f^{-1}(y_n)) = \sup_{n=1}^{\infty} y_n = y$ and hence $x = f^{-1}(y)$ which shows that f^{-1} is continuous. □

Lemma 2.3. *Let P and Q be cones, let $A \subseteq P$ be Scott closed and let $f, g : A \to Q$ be functions such that f is monotone, g is Scott-continuous, $\forall x \in P\ f(x) \leq g(x)$ and the function $g - f = \boldsymbol{\lambda} x \in P \cdot (g(x) - f(x))$ is monotone. Then $g - f$ is Scott continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded monotone sequence in A and let $x = \sup_{n\in\mathbb{N}} x_n$. For all $n \in \mathbb{N}$ we have $f(x_n) \leq f(x)$ and hence $g(x_n) \leq f(x) + g(x_n) - f(x_n)$. The sequence $(f(x) + g(x_n) - f(x_n))_{n\in\mathbb{N}}$ is monotone by our assumption that $g - f$ is monotone, and it is bounded by $f(x) + g(x)$. We have

$$\begin{aligned}
g(x) &= g(\sup_{n\in\mathbb{N}} x_n) \\
&= \sup_{n\in\mathbb{N}} g(x_n) \quad \text{since } g \text{ is Scott continuous} \\
&\leq \sup_{n\in\mathbb{N}} (f(x) + g(x_n) - f(x_n)) \\
&= f(x) + \sup_{n\in\mathbb{N}} (g(x_n) - f(x_n)) \quad \text{by continuity of addition}
\end{aligned}$$

and hence $g(x) - f(x) \leq \sup_{n\in\mathbb{N}} (g(x_n) - f(x_n))$. □

There is a cone $1 = \perp$ whose set of elements is $\mathbb{R}_{\geq 0}$ and $\|x\| = x$. There is also a cone $0 = \top$ whose only element is 0.

Lemma 2.4. *If $f : P \to Q$ is linear then f is bounded, that is the set $f(\mathcal{B}P)$ is bounded. We set $\|f\| = \sup_{x\in\mathcal{B}P} \|f(x)\| \in \mathbb{R}_{\geq 0}$.*

Proof. See [Sel04], we give the proof because it is short and interesting. If the lemma does not hold there is a monotone sequence $(x_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ \|x_n\| \leq 1$ and $\forall n \in \mathbb{N}\ \|f(x_n)\| \geq 4^n$. Then let $y_n = \sum_{k=1}^{n} \frac{1}{2^k} x_k \in P$, we have $\|y_n\| \leq \sum_{k=1}^{n} \frac{1}{2^n} \|x_n\| \leq 1$ and $(y_n)_{n\in\mathbb{N}}$ is a monotone sequence which therefore has a lub $y \in \mathcal{B}P$, and we have $\|f(y)\| \geq \|f(y_n)\| \geq \frac{1}{2^n} \|f(x_n)\| \geq 2^n$ by (**Normc**) and linearity of f. Since this holds for all $n \in \mathbb{N}$ we have a contradiction. □

Remark 2.5. We have obtained this property without even requiring f to be Scott continuous. This is quite surprising because, in Banach spaces, continuity of linear maps can be equivalently expressed as a completely similar boundedness property. The point seems to be that Scott continuity for linear maps is a much stronger property than continuity wrt. the distance induced by the norm[6]. The fact that Lemma 2.3 does not require f to be continuous is similarly surprising.

The set $P \multimap Q$ of all linear and continuous maps from P to Q, equipped with obvious pointwise defined algebraic operations, is a precone. Notice that $f_1 \leq f_2$ simply means that $\forall x \in P\ f_1(x) \leq f_2(x)$ and then the difference is given by $(f_2 - f_1)(x) = f_2(x) - f_1(x)$: it suffices to check that this latter map is linear which is obvious, and that it is continuous which results from Lemma 2.3. In these proofs we also use the fact that addition $P \times P \to P$ and scalar multiplication $\mathbb{R}_{\geq 0} \times P \to P$ are Scott continuous, which easily follows from the axioms on the norm of P, see [Sel04].

Lemma 2.6. *Equipped with the norm defined in Lemma 2.4, the precone $P \multimap Q$ is a cone.*

The proof is easy. By definition of $\|f\|$ and by (**Normh**) we have

$$\forall x \in P \quad \|f(x)\| \leq \|f\| \, \|x\| \ .$$

We set $P' = (P \multimap \perp)$. If $x \in P$ and $x' \in P'$ we write $\langle x, x' \rangle = x'(x) \in \mathbb{R}_{\geq 0}$, and notice that $\|x'\| = \sup_{x\in\mathcal{B}P} \langle x, x' \rangle$.

The category **Cones** has the cones as objects, and **Cones**(P, Q) is the set of all linear and continuous $f : P \to Q$ such that $\|f\| \leq 1$.

Theorem 2.7. *The category* **Cones** *has all small products, the terminal object is $\top$ and, given a family $(P_i)_{i\in I}$ of cones (with no cardinality restrictions on I), their cartesian product $(\&_{i\in I} P_i, (\mathsf{pr}_i)_{i\in I})$ is defined as follows:*

- *$\&_{i\in I} P_i$ is the set of all $\vec{x} = (x_i)_{i\in I} \in \prod_{i\in I} P_i$ such that the family $(\|x_i\|)_{i\in I}$ is bounded, equipped with the obvious algebraic operations defined componentwise*
- *and $\|\vec{x}\| = \sup_{i\in I} \|x_i\|$.*

The projections are the standard projections of the cartesian product in **Set**. *Given a family of morphisms $(f_i \in \mathbf{Cones}(Q, P_i))_{i\in I}$, the associated morphism $f = \langle f_i \rangle_{i\in I} \in \mathbf{Cones}(Q, \&_{i\in I} P_i)$ is characterized by $f(y) = (f_i(y))_{i\in I}$.*

The canonical order of $\&_{i\in I} P_i$ is the product of the canonical orders of the P_i's.

See [Sel04]. There is a clear similarity with the ℓ^∞ construct of Banach spaces.

[6]Even if the standard formula $d(x, y) = \|x - y\|$ is meaningless here because $x - y$ is not always defined when $x, y \in P$, there is a simple way of using the norm to define a distance in P which always turns it into a complete metric space.

3. Measurable cones

Let $\mathbf{ar}$ be a pointed magma, that is a *set*[7] equipped with a distinguished element 0 and a binary operation + on which we make no assumptions, whose elements can be called arities, or dimensions, and assume that with any $d \in \mathbf{ar}$ is associated a measurable space $\overline{d}$ in such a way that $\overline{0}$ is the terminal object of $\mathbf{Meas}$ (the one element measurable space) and $\overline{d+e} = \overline{d} \times \overline{e}$. Then $\mathbf{ar}$ becomes a cartesian category by setting $\mathbf{ar}(d,e) = \mathbf{Meas}(\overline{d},\overline{e})$. Most often we will consider $\mathbf{ar}$ as a commutative monoid, considering the monoidal isos induced by this cartesian structure as identities; this is a convenient abuse of notations. If $\varphi_i \in \mathbf{ar}(d,e_i)$ for $i = 1,2$, we use $\langle \varphi_1, \varphi_2 \rangle$ for the element of $\mathbf{ar}(d, e_1 + e_2)$ which maps r to $(\varphi_1(r), \varphi_2(r))$.

Remark 3.1. In most situations we could assume $\mathbf{ar}$ to be the free magma generated by one element 1, that is the set of binary trees whose leaves are labeled by 1, identifying the empty tree with the element $0 \in \mathbf{ar}$. Then we can take $\overline{d} = \mathbb{R}$ for all $d \in \mathbf{ar} \setminus \{0\}$ and $\overline{0}$ to be the one element measurable space. This makes sense because, in the category of measurable spaces, $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ are isomorphic[8] so its subcategory whose only objects are $\overline{0}$ and $\mathbb{R}$ is cartesian.

A measurability structure on a cone P is a family $\mathcal{M} = (\mathcal{M}_d)_{d \in \mathbf{ar}}$ with $\mathcal{M}_d \subseteq (P')^{\overline{d}}$ (where we recall that $P' = (P \multimap \perp)$) satisfying the four next conditions (**Msmeas**), (**Mscomp**), (**Msnorm**), (**Mssep**) and (**Msnorm**). When $d = 0$ we consider $m \in \mathcal{M}_d$ as an element of P'. We use σ_d for the σ-algebra of $\overline{d}$.

(**Msmeas**) For any $m \in \mathcal{M}_d$ and $x \in \mathcal{B}P$, one has $\boldsymbol{\lambda} r \in \overline{d} \cdot m(r,x) \in \mathbf{Meas}(\overline{d}, [0,1])$ where $[0,1] \subseteq \mathbb{R}$ is equipped with its standard Borelian σ-algebra.

(**Mscomp**) For any $m \in \mathcal{M}_d$ and $\varphi \in \mathbf{ar}(e,d)$ one has $\boldsymbol{\lambda}(s,x) \in (\overline{e} \times P) \cdot m(\varphi(s),x) = m \circ (\varphi \times P) \in \mathcal{M}_e$.

In particular, since 0 is the terminal object of $\mathbf{ar}$, any element $m \in \mathcal{M}_0$ induces an element $\boldsymbol{\lambda}(r,x) \in (\overline{d} \times P) \cdot m(x) \in \mathcal{M}_d$ and in this way we consider $\mathcal{M}_0$ as a subset of $\mathcal{M}_d$ for any $d \in \mathbf{ar}$.

(**Mssep**) If $x_1, x_2 \in P$ satisfy $\forall m \in \mathcal{M}_0\ m(x_1) = m(x_2)$ then $x_1 = x_2$.

Remark 3.2. We could also consider the following stronger separation condition: if for all $m \in \mathcal{M}_0$ one has $m(x) \leq m(y)$ then $x \leq y$. However this would complicate the definition of the measurability structures of the spaces of stable functions in Section 8.2 and of analytic functions in Section 9.3. This stronger separability does not seem to be necessary (at least for the purpose of what we do in this paper) but one should keep in mind that all our constructions could be performed within this restricted class.

(**Msnorm**) For all $x \in P$, one has $\|x\| = \sup\{m(x)/\|m\| \mid m \in \mathcal{M}_0 \text{ and } m \neq 0\}$ or, equivalently, $\|x\| \leq \sup\{m(x)/\|m\| \mid m \in \mathcal{M}_0 \text{ and } m \neq 0\}$.

Indeed, for any $x' \in P' \setminus \{0\}$ and $x \in P$ one has $\|x\| \geq \frac{\langle x, x' \rangle}{\|x'\|}$.

Remark 3.3. The condition (**Msnorm**) was missing in [Ehr20] and can also be formulated as follows: for any $x \in P \setminus \{0\}$ and for any $\varepsilon > 0$ there exists $m \in \mathcal{M}_0 \setminus \{0\}$ such that

[7] This assumption is crucial for making the use of the special adjoint functor theorem possible.

[8] Such isomorphisms involve however non canonical encodings so we prefer to avoid using this property explicitly.

$\|x\| \leq \frac{m(x)}{\|m\|} + \varepsilon$. The condition that $x \neq 0$ is required because we could possibly have $\mathcal{M}_0 = \{0\}$, but in that situation, by (**Mssep**) we must have $P = \{0\}$.

Definition 3.4 (Measurable cone)**.** A *measurable cone* is a pair $C = (\underline{C}, \mathcal{M}^C)$ where $\underline{C}$ is a cone and $\mathcal{M}^C$ is a measurability structure on $\underline{C}$.

Definition 3.5 (Measurable path)**.** Let $d \in \mathbf{ar}$ and let C be a measurable cone. A *(measurable) path* of arity d is a function $\gamma : \overline{d} \to \underline{C}$ which is bounded (that is $\gamma(\overline{d})$ is bounded in $\underline{C}$) such that, for any $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^C$, the function $\boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot m(s, \gamma(r)) : \overline{e+d} \to \mathbb{R}_{\geq 0}$ is measurable.

Lemma 3.6. *Let $x \in \underline{C}$ and $\gamma = \boldsymbol{\lambda} r \in \overline{d} \cdot x : \overline{d} \to \underline{C}$ be the constant function. Then γ is a measurable path.*

This immediately results from the definitions.

Lemma 3.7. *Let $\gamma : \overline{d} \to \underline{C}$ be a measurable path and let $\varphi \in \mathbf{ar}(e, d)$ for some $e \in \mathbf{ar}$. Then $\gamma \circ \varphi : \overline{e} \to \underline{C}$ is also a measurable path.*

Proof. Let $e' \in \mathbf{ar}$ and $m \in \mathcal{M}_{e'}^C$, we have $\boldsymbol{\lambda}(s', s) \in \overline{e'+e} \cdot m(s', \gamma(\varphi(s))) = (\boldsymbol{\lambda}(s', r) \in \overline{e'+d} \cdot m(s', \gamma(r))) \circ (\overline{e'} \times \varphi)$ which is measurable as the composition of two measurable maps. $\square$

We turn the cone $1 = \perp$ into a measurable cone by defining $\mathcal{M}_0^1$ as the singleton $\{\mathsf{Id}\}$, and $\mathcal{M}_d^1 = \mathcal{M}_0^1$ for all $d \in \mathbf{ar}$.

Theorem 3.8. *Let $x \in \underline{B}$. Then*

$$\|x\| = \sup_{x' \in \mathcal{B}B'} \langle x, x' \rangle .$$

Proof. By definition of the norm in B' we have $\|x\| \geq \langle x, x' \rangle$ for all $x' \in \mathcal{B}B'$. We can assume that $x \neq 0$ since otherwise the announced equation trivially holds. Let $\varepsilon > 0$ and $m \in \mathcal{M}_0^B \setminus \{0\}$ be such that $\|x\| \leq \frac{m(x)}{\|m\|} + \varepsilon$. Let $x' = m / \|m\|$, we have $x' \in \underline{B'}$. Let indeed $d \in \mathbf{ar}$ and $\beta \in \mathsf{Path}(d, C)$ we know that $\boldsymbol{\lambda} r \in \overline{d} \cdot m(\beta(r))$ is measurable by definition of a path and hence $\boldsymbol{\lambda} r \in \overline{d} \cdot \langle \beta(r), x' \rangle$ is measurable. Since $\|x\| \leq \langle x, x' \rangle + \varepsilon$ our contention is proven. $\square$

Remark 3.9. One main purpose of the condition (**Msnorm**) is to get the above highly desirable property. We could have expected to get it for free by means of a Hahn Banach theorem for cones like in [Sel04]. However the very nice Hahn Banach theorem proven in that paper relies on the assumption that cones are continuous domains, an assumption that we cannot afford here because we need our cones to define a complete category in order to apply the special adjoint functor theorem. So we take this Hahn Banach separation property as one of our axioms and fortunately, due to our use of measurability tests, it is preserved by all limits without inducing noticeable technical difficulties.

We can now define our first main category of interest.

Definition 3.10. The category **MCones** has measurable cones as objects and an element of $\mathbf{MCones}(B, C)$ is an $f \in \mathbf{Cones}(\underline{B}, \underline{C})$ such that for all $d \in \mathbf{ar}$ and all measurable path $\beta : \overline{d} \to \underline{B}$ the function $f \circ \beta$ is a measurable path. Equivalently

$$\forall e \in \mathbf{ar}\, \forall m \in \mathcal{M}_e^C \quad \boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot m(s, f(\beta(r)) \text{ is measurable.}$$

This equivalence results from Lemma 2.4.

Remark 3.11. An isomorphism $f \in \mathbf{MCones}(B,C)$ is a bijection $f : \underline{B} \to \underline{C}$ which is linear, Scott continuous, satisfies $\forall x \in \underline{B}\ \ \|f(x)\|_C = \|x\|_B$ and, for all $d \in \mathbf{ar}$ and all function $\beta : \overline{d} \to \underline{B}$, one has $\beta \in \underline{\mathsf{Path}(d,B)} \Leftrightarrow f \circ \beta \in \underline{\mathsf{Path}(d,C)}$. This means that B and C can be isomorphic even if the measurability structures $\mathcal{M}^C$ and $\mathcal{M}^D$ are quite different: it suffices that they induce the same measurable paths.

4. The cone of paths and the cone of measures

4.1. **The measurable cone of paths.** Let C be an object of **MCones** and $d \in \mathbf{ar}$. Let P be the set of all measurable paths $\gamma : \overline{d} \to \underline{C}$. We turn P into a precone by defining the algebraic laws in the obvious pointwise manner. For instance let $\gamma_1, \gamma_2 \in P$, we define $\gamma = \gamma_1 + \gamma_2$ by $\gamma(r) = \gamma_1(r) + \gamma_2(r)$ which is bounded by (**Normt**). To check measurability, take $m \in \mathcal{M}^C_d$, we have $\boldsymbol{\lambda} r \in \overline{d} \cdot m(r, \gamma(r)) = \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, \gamma_1(r)) + \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, \gamma_2(r))$ (pointwise addition) by linearity of m in its second parameter, which is measurable in r by continuity of addition on $\mathbb{R}_{\geq 0}$.

Then we have $\gamma_1 \leq \gamma_2$ iff $\forall r \in \overline{d}\ \gamma_1(r) \leq \gamma_2(r)$: it suffices to check that, when this latter condition holds, the map $\boldsymbol{\lambda} r \in \overline{d} \cdot (\gamma_2(r) - \gamma_1(r))$ is a path which results from the continuity (and hence measurability) of subtraction of real numbers.

Given $\gamma \in P$ we set

$$\|\gamma\| = \sup_{r \in \overline{d}} \|\gamma(r)\| \in \mathbb{R}_{\geq 0}$$

which is well defined by our assumption that γ is bounded. This satisfies all the required conditions for turning P into a cone, the only non obvious one being (**Normc**). So let $(\gamma_n)_{n \in \mathbb{N}}$ be a monotone sequence of elements of P such that $\forall n \in \mathbb{N}\, \forall r \in \overline{d}\ \ \|\gamma_n(r)\| \leq 1$. We define $\gamma : \overline{d} \to P$ by $\gamma(r) = \sup_{n \in \mathbb{N}} \gamma_n(r) \in \mathcal{B}\underline{C}$ which is well defined since for each $r \in \overline{d}$ the sequence $(\gamma_n(r))_{n \in \mathbb{N}}$ is monotone in $\mathcal{B}\underline{C}$. It suffices to check that γ satisfies the measurability condition, so let $e \in \mathbf{ar}$ and $m \in \mathcal{M}^C_e$, we have by Scott continuity of m in its second argument

$$\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, \gamma(r)) = \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \sup_{n \in \mathbb{N}} m(s, \gamma_n(r))$$

which is measurable by the monotone convergence theorem of measure theory (observing that $(\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, \gamma_n(r)))_{n \in \mathbb{N}}$ is a monotone sequence of measurable function $\overline{e+d} \to [0,1]$).

Remark 4.1. It is for being able to prove this kind of properties that we assume the unit balls of cones to be complete only for increasing chains and not for arbitrary directed sets.

Let $e \in \mathbf{ar}$, $\varphi \in \mathbf{ar}(e,d)$ and $m \in \mathcal{M}^C_e$, we define

$$\begin{aligned} \varphi \triangleright m : \overline{e} \times P &\to \mathbb{R}_{\geq 0} \\ (s, \gamma) &\mapsto m(s, \gamma(\varphi(s)))\,. \end{aligned}$$

Observe first that for any $s \in \overline{e}$, the map $\boldsymbol{\lambda} \gamma \in P \cdot (\varphi \triangleright m)(s, \gamma)$ is linear and continuous by linearity and continuity of m in its second argument. We check that the family $(\mathcal{M}_e \subseteq$

$P'^{\overline{e}})_{e\in\mathbf{ar}}$ defined by $\mathcal{M}_e = \{\varphi \triangleright m \mid \varphi \in \mathbf{ar}(e,d) \text{ and } m \in \mathcal{M}^C_e\}$ is a measurability structure on P.

▶ (**Msmeas**). Let $p \in \mathcal{M}_e$ and $\gamma \in \mathcal{B}P$, so that $p = \varphi \triangleright m$ for some $\varphi \in \mathbf{ar}(e,d)$ and $m \in \mathcal{M}^C_e$, then let $\theta = \boldsymbol{\lambda} s \in \overline{e} \cdot p(s,\gamma) = \boldsymbol{\lambda} s \in \overline{e} \cdot m(s,\gamma(\varphi(s)))$. We know that $\psi = \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s,\gamma(r))$ is measurable $\overline{e+d} \to [0,1]$ and hence $\theta = \psi \circ \langle \overline{e}, \varphi\rangle$ is measurable $\overline{e} \to [0,1]$ since φ is measurable.

▶ (**Mscomp**). Let $p \in \mathcal{M}_e$ and $\psi \in \mathbf{ar}(e',e)$. We have $p = \varphi \triangleright m$ for some $\varphi \in \mathbf{ar}(e,d)$ and $m \in \mathcal{M}^C_e$. Then we have $p \circ (\psi \times P) = (\varphi \circ \psi) \triangleright (m \circ (\psi \times \underline{C})) \in \mathcal{M}_{e'}$.

▶ (**Mssep**). Let $\gamma_1, \gamma_2 \in P$ and assume that $\forall p \in \mathcal{M}_0\ p(\gamma_1) = p(\gamma_2)$. Let $r \in \overline{d}$ that we consider as an element of $\mathbf{ar}(0,d)$. Let $m \in \mathcal{M}^C_0$, by our assumption we have $(r \triangleright m)(\gamma_1) = (r \triangleright m)(\gamma_2)$, that is $m(\gamma_1(r)) = m(\gamma_2(r))$ and since this holds for all $m \in \mathcal{M}^C_0$ we have $\gamma_1(r) = \gamma_2(r)$ by (**Mssep**) in C.

▶ (**Msnorm**). Let $\gamma \in P \setminus \{0\}$ and $\varepsilon > 0$. We can find $r \in \overline{d}$ such that $\gamma(r) \neq 0$ and $\|\gamma\| \leq \|\gamma(r)\| + \frac{\varepsilon}{2}$. By (**Msnorm**) holding in C we can find $m \in \mathcal{M}^C_0 \setminus \{0\}$ such that $\|\gamma(r)\| \leq \frac{m(\gamma(r))}{\|m\|} + \frac{\varepsilon}{2}$. Remember that $r \triangleright m \in \mathcal{M}_0$ and notice that $\|r \triangleright m\| = \sup\{m(\delta(r)) \mid \delta \in \mathcal{B}P\} = \|m\|$ by Lemma 3.6. So we have

$$\|\gamma\| \leq \|\gamma(r)\| + \frac{\varepsilon}{2} \leq \frac{(r \triangleright m)(\gamma)}{\|r \triangleright m\|} + \varepsilon$$

and hence $\|\gamma\| = \sup\{\frac{p(\gamma)}{\|p\|} \mid p \in \mathcal{M}_0 \text{ and } p \neq 0\}$ as required since $\mathcal{M}_0 = \{r \triangleright m \mid r \in \overline{d} \text{ and } m \in \mathcal{M}^C_0\}$.

We use $\mathsf{Path}(d,C)$ for the measurable cone $(P,\mathcal{M})$ defined above.

Lemma 4.2. *Let B be a cone and $d,e \in \mathbf{ar}$. There is an iso*

$$\mathsf{fl}_{d,e} \in \mathbf{MCones}(\mathsf{Path}(d,\mathsf{Path}(e,B)),\mathsf{Path}(d+e,B))$$

which "flattens" $\eta \in \underline{\mathsf{Path}(d,\mathsf{Path}(e,B))}$ into $\mathsf{fl}_{d,e}(\eta) = \boldsymbol{\lambda}(r,s) \in \overline{d+e} \cdot \eta(r)(s)$. As a consequence

$$\mathsf{fl}^{-1}_{e,d}\,\mathsf{fl}_{d,e} \in \mathbf{MCones}(\mathsf{Path}(d,\mathsf{Path}(e,B)),\mathsf{Path}(e,\mathsf{Path}(d,B)))\,,$$

the function which swaps the parameters of a path of paths is an iso.

Proof. Let $\eta \in \underline{\mathsf{Path}(d,\mathsf{Path}(e,B))}$, we need first to prove that $\eta' = \mathsf{fl}(\eta) \in \underline{\mathsf{Path}(d+e,B)}$ so let $e' \in \mathbf{ar}$ and let $m \in \mathcal{M}^B_{e'}$, we must prove that

$$\varphi = \boldsymbol{\lambda}(s',r,s) \in \overline{e'+d+e} \cdot m(s',\eta'(r,s)) = \boldsymbol{\lambda}(s',r,s) \in \overline{e'+d+e} \cdot m(s',\eta(r)(s))$$

is measurable. Let $m' = m \circ (\mathsf{pr}_1 \times \underline{B}) \in \mathcal{M}^B_{e'+e}$ (that is $m'(s',s,x) = m(s',x)$) so that $\mathsf{pr}_2 \triangleright m' \in \mathcal{M}^{\mathsf{Path}(e,B)}_{e'+e}$, we know that $\boldsymbol{\lambda}(s',s,r) \in \overline{e'+e+d} \cdot (\mathsf{pr}_2 \triangleright m')(s',s,\eta(r)) = \boldsymbol{\lambda}(s',s,r) \in \overline{e'+e+d} \cdot m(s',\eta(r)(s))$ is measurable from which it follows that φ is measurable. Moreover it is clear that $\eta'(\overline{d+e}) \subseteq \|\eta\|\,\mathcal{B}\underline{B}$ is bounded in $\underline{B}$ and hence $\eta' \in \underline{\mathsf{Path}(d+e,B)}$ as announced.

The linearity and Scott continuity of fl are clear so we check its measurability. Let $e' \in \mathbf{ar}$ and let $\eta \in \underline{\mathsf{Path}(e',\mathsf{Path}(d,\mathsf{Path}(e,B)))}$, we must prove that

$$\mathsf{fl} \circ \eta \in \underline{\mathsf{Path}(e',\mathsf{Path}(d+e,B))}\,.$$

So let $e'' \in \mathbf{ar}$ and let $p \in \mathcal{M}_{e''}^{\mathsf{Path}(d+e,B)}$. Let $\varphi' = \langle\varphi,\psi\rangle \in \mathbf{ar}(e'', d+e)$ and $m \in \mathcal{M}_{e''}^{B}$ be such that $p = \varphi' \triangleright m$, we have that

$$\begin{aligned}\varphi'' &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot p(s'', \mathsf{fl}(\eta(s')))\\ &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot m(s'', \mathsf{fl}(\eta(s'))(\varphi(s''),\psi(s'')))\\ &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot m(s'', \eta(s')(\varphi(s''))(\psi(s'')))\end{aligned}$$

is measurable because

$$\psi = \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot (\varphi \triangleright (\psi \triangleright m))(s'', \eta(s'))$$

and by our assumption about η. Last notice that $\|\mathsf{fl}(\eta)\| = \|\eta\|$ which shows that $\mathsf{fl} \in \mathbf{MCones}(\mathsf{Path}(d, \mathsf{Path}(e,B)), \mathsf{Path}(d+e,B))$.

As to the converse direction, given $\eta \in \underline{\mathsf{Path}(d+e,B)}$ let $\mathsf{fl}'(\eta) = \boldsymbol{\lambda} r \in \overline{d} \cdot \boldsymbol{\lambda} s \in \overline{e} \cdot \eta(r,s)$, we must first prove that $\mathsf{fl}'(\eta) \in \underline{\mathsf{Path}(d, \overline{\mathsf{Path}(e,B)})}$, we just check measurability, boundedness being obvious. Let $p \in \mathcal{M}_{e'}^{\mathsf{Path}(e,B)}$ for some $e' \in \mathbf{ar}$. Let $\varphi \in \mathbf{ar}(e', e)$ and $m \in \mathcal{M}_{e'}^{B}$ be such that $p = \varphi \triangleright m$, we must prove that $\psi = \boldsymbol{\lambda}(s', r) \in \overline{e'+d} \cdot p(s', \mathsf{fl}'(\eta)(r)) = \boldsymbol{\lambda}(s', r) \in \overline{e'+d} \cdot m(s', \eta(r, \varphi(s')))$ is measurable. This follows from the fact that φ and $\boldsymbol{\lambda}(s', r, s) \in \overline{e'+d+e} \cdot m(s', \eta(r,s))$ are measurable, the latter by our assumption about η.

Checking that fl' is a morphism in **MCones** follows exactly the same pattern as for fl, using the obvious bijection between $\mathcal{M}_{e'}^{\mathsf{Path}(d,\mathsf{Path}(e,B))}$ and $\mathcal{M}_{e'}^{\mathsf{Path}(d+e,B)}$ induced by the fact that **ar** is cartesian. Finally the observation that $\mathsf{fl}' = \mathsf{fl}^{-1}$ shows that fl is an iso in **MCones**. $\square$

4.2. **The measurable cone of finite measures.** Let $d \in \mathbf{ar}$, we define a cone P as follows:

- an element of P is a $\mathbb{R}_{\geq 0}$-valued measure μ on the measurable space $\overline{d}$ which is finite in the sense that $\mu(\overline{d}) < \infty$;
- the algebraic operations of P are defined in the obvious "pointwise" way, that is $(\mu_1 + \mu_2)(U) = \mu_1(U) + \mu_2(U)$ for all $U \in \sigma_d$ (the σ-algebra of measurable sets of $\overline{d}$) *etc.*;
- the norm is given by $\|\mu\| = \mu(\overline{d})$ (this is the total variation norm of μ since μ is non-negative);
- observing that $\mu_1 \leq \mu_2$ means $\forall U \in \sigma_d\ \mu_1(U) \leq \mu_2(U)$ it is clear that any monotone sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{B}P$ has a lub $\mu \in \mathcal{B}P$ which is computed pointwise: $\mu(U) = \sup_{n\in\mathbb{N}} \mu_n(U)$.

Given $U \in \sigma_d$ we define $\widetilde{U} : P \to \mathbb{R}_{\geq 0}$ by $\widetilde{U}(\mu) = \mu(U)$. Then we define $\mathcal{M}_0 = \{\widetilde{U} \mid U \in \sigma_d\}$ and more generally $\mathcal{M}_e = \mathcal{M}_0$ for each $e \in \mathbf{ar}$. Then $\mathsf{Meas}(d) = (P, (\mathcal{M}_d)_{d\in\mathbf{ar}})$ is clearly a measurable cone.

Notice that if $e \in \mathbf{ar}$ then an element κ of $\underline{\mathsf{Path}(e, \mathsf{Meas}(d))}$ is a kernel from $\overline{e}$ to $\overline{d}$ which is finite in the sense that $\{\kappa(s)(\overline{d}) \mid s \in \overline{e}\} \subseteq \mathbb{R}_{\geq 0}$ is bounded.

Let $\varphi \in \mathbf{ar}(d,e)$ (remember that this means that φ is a measurable function $\overline{d} \to \overline{e}$), then given $\mu \in \underline{\mathsf{Meas}(d)}$ we can define $\nu = \varphi_*(\mu) \in \mathsf{Meas}(e)$ by $\nu(V) = \mu(\varphi^{-1}(V))$ for each $V \in \sigma_e$ (the push-forward of μ along φ).

Lemma 4.3. *We have* $\varphi_* \in \mathbf{MCones}(\mathsf{Meas}(d), \mathsf{Meas}(e))$. *The operation* Meas *on measurable cones extends to a functor* $\mathsf{Meas} : \mathbf{ar} \to \mathbf{MCones}$*, acting on morphisms by measure push-forward:* $\mathsf{Meas}(\varphi) = \varphi_*$.

Proof. Linearity and continuity being obvious, as well as the fact that $\|\mathsf{Meas}(f)\| \leq 1$, we only have to check measurability. Let $\kappa \in \underline{\mathsf{Path}(e', \mathsf{Meas}(d))}$, we must prove that $\kappa' = \mathsf{Meas}(\varphi) \circ \kappa \in \underline{\mathsf{Path}(e', \mathsf{Meas}(e))}$. Let $p \in \mathcal{M}_{e''}^{\mathsf{Meas}(e)}$ for some $e'' \in \mathbf{ar}$, that is $p = \widetilde{V}$ for some $V \in \sigma_e$. We have $\boldsymbol{\lambda}(s'', s') \in \overline{e'' + e'} \cdot p(s'', \kappa'(s')) = \boldsymbol{\lambda}(s'', s') \in \overline{e'' + e'} \cdot \kappa'(s')(V) = \boldsymbol{\lambda}(s'', s') \in \overline{e'' + e'} \cdot \kappa(s')(\varphi^{-1}(V))$ which is measurable because κ is a kernel. Functoriality of Meas is obvious. □

Lemma 4.4. *Let $d, e \in \mathbf{ar}$. There is a unique bilinear, continuous and measurable function $\underline{\mathsf{Meas}(d)} \times \underline{\mathsf{Meas}(e)} \to \underline{\mathsf{Meas}(d+e)}$ which maps (μ, ν) to the unique measure $\mu \times \nu$ on $\overline{d+e} = \overline{d} \times \overline{e}$ which satisfies $(\mu \times \nu)(U \times V) = \mu(U)\nu(V)$ for all $U \in \sigma_d$ and $V \in \sigma_e$. Moreover this operation is a natural transformation $\mathsf{Meas} \times \mathsf{Meas} \Rightarrow \mathsf{Meas}$.*

Proof. The existence and uniqueness of this operation is a standard result in measure theory, based on the Caratheodory Theorem. We only prove measurability, the other properties resulting from the bilinearity and continuity of multiplication in $\mathbb{R}_{\geq 0}$. So we take finite kernels $\kappa \in \underline{\mathsf{Path}(e', \mathsf{Meas}(d))}$ and $\lambda \in \underline{\mathsf{Path}(e', \mathsf{Meas}(e))}$ and we contend that $\boldsymbol{\lambda} s' \in \overline{e'} \cdot (\kappa(s') \times \lambda(s'))(W)$ is measurable and bounded with norm $\leq \|\kappa\| \, \|\lambda\|$ which is easy to prove by induction on the first ordinal (height of W) for which W appears in the transfinite definition of the σ-algebra $\sigma_{d+e} = \sigma_d \otimes \sigma_e$ using the monotone convergence theorem in the case where $W = \sum_{i \in I} W_i$ is the countable disjoint union of the W_i's whose height is $<$ than that of W (the other case, when $W = \overline{d} \times \overline{e} \setminus W'$, is straightforward).

Let us check naturality so let $\varphi \in \mathbf{ar}(d', d)$ and $\psi \in \mathbf{ar}(e', e)$. Let $\mu' \in \underline{\mathsf{Meas}(d')}$ and $\nu' \in \underline{\mathsf{Meas}(e')}$. Given $U \in \sigma_d$ and $V \in \sigma_e$ we have

$$\begin{aligned}
(\varphi \times \psi)_*(\mu' \times \nu')(U \times V) &= (\mu' \times \nu')((\varphi \times \psi)^{-1}(U \times V)) \\
&= (\mu' \times \nu')(\varphi^{-1}(U) \times \psi^{-1}(V)) \\
&= \mu'(\varphi^{-1}(U))\nu'(\psi^{-1}(V)) \\
&= (\varphi_* \mu' \times \psi_* \nu')(U \times V)
\end{aligned}$$

and hence $(\varphi \times \psi)_*(\mu' \times \nu') = (\varphi_* \mu' \times \psi_* \nu')$ by uniqueness of the product of measures. □

Definition 4.5. Let B be a measurable cone and $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then αB is the measurable cone which is defined exactly as B apart for the norm which is given by $\|x\|_{\alpha B} = \alpha^{-1} \|x\|_B$.

Notice that $\mathcal{B}(\alpha B) = \alpha \mathcal{B} B = \{x \in \underline{B} \mid \|x\|_B \leq \alpha\}$.

5. Integrable cones

The following definition is quite similar to Definition 2.1 in [Pet38] of the integral of a function valued in a topological vector space. Our integrals are valued in cones instead of vector spaces.

Definition 5.1. Let B be a measurable cone, $d \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(d, B)}$ and $\mu \in \underline{\mathsf{Meas}(d)}$. An *integral of β over μ* is an element x of $\underline{B}$ such that, for all $m \in \mathcal{M}_0^B$, one has

$$m(x) = \int m(\beta(r))\mu(dr)\,.$$

Notice indeed that $m \circ \beta : \overline{d} \to \mathbb{R}_{\geq 0}$ is a bounded measurable function so that the integral above is well defined and finite (remember that the measure μ is finite). Notice also that by (**Mssep**) if such an integral x exists, it is unique, so we can introduce a notation for it, we write

$$x = \int \beta(r)\mu(dr)\,.$$

When we want to stress the cone B where this integral is computed we denote it as $\int^B \beta(r)\mu(dr)$ and when we want to insist on the measurable space on which the integral is computed we write $\int_{\overline{d}} \beta(r)\mu(dr)$ or $\int_d \beta(r)\mu(dr)$.

Lemma 5.2. *If $\beta \in \underline{\mathsf{Path}(d, B)}$ is integrable over $\mu \in \mathsf{Meas}(d)$ then*

$$\left\| \int \beta(r)\mu(dr) \right\|_B \leq \|\beta\|_{\mathsf{Path}(d,B)} \|\mu\|_{\mathsf{Meas}(d)}\,.$$

Proof. Let $x = \int \beta(r)\mu(dr)$. If $x = 0$ there is nothing to prove so assume that $x \neq 0$. Let $\varepsilon > 0$ and let $m \in \mathcal{M}_0^B \setminus \{0\}$ be such that $\|x\| \leq \varepsilon + \frac{m(x)}{\|m\|}$, that is

$$\|x\| \leq \varepsilon + \frac{1}{\|m\|} \int m(\beta(r))\mu(dr)\,.$$

For each $r \in \overline{d}$ we have $m(\beta(r)) \leq \|m\| \, \|\beta(r)\| \leq \|m\| \, \|\beta\|$. Our contention follows from $\|\mu\| = \mu(\overline{d}) = \int \mu(dr)$. $\square$

Definition 5.3. A measurable cone is *integrable* if, for all $d \in \mathbf{ar}$, any $\beta \in \underline{\mathsf{Path}(d, B)}$ has an integral in $\underline{B}$ over any measure $\mu \in \underline{\mathsf{Meas}(d)}$. When this is the case we use $\mathcal{I}_d^B$ for the uniquely defined function $\underline{\mathsf{Path}(d, B)} \times \underline{\mathsf{Meas}(d)} \to \underline{B}$ such that $\mathcal{I}_d^B(\beta, \mu) = \int \beta(r)\mu(dr)$.

Theorem 5.4. *The measurable cone $\mathsf{Meas}(d)$ is integrable.*

This is just a reformulation of the standard integration of a kernel.

Proof. Let $d \in \mathbf{ar}$, $\kappa \in \underline{\mathsf{Path}(e, \mathsf{Meas}(d))}$, which means that κ is a finite kernel $\overline{e} \rightsquigarrow \overline{d}$, and let $\nu \in \underline{\mathsf{Meas}(e)}$, which means that ν is a finite measure. We define $\mu : \sigma_d \to \mathbb{R}_{\geq 0}$ by

$$\forall U \in \sigma_d \quad \mu(U) = \int \kappa(s, U)\nu(ds) \in \mathbb{R}_{\geq 0}\,.$$

The fact that μ defined in that way is a finite measure is completely standard in measure theory and μ is the integral of κ by the very definition of $\mathcal{M}_0^{\mathsf{Meas}(d)}$. $\square$

In the sequel we assume that B is an integrable cone. We state and prove some basic expected properties of integration.

Lemma 5.5. *Let $\varphi : \overline{e + d} \to \mathbb{R}_{\geq 0}$ be measurable and bounded and let $\kappa : \overline{e} \to \underline{\mathsf{Meas}(d)}$ be a finite kernel. Then the function $\boldsymbol{\lambda} s \in \overline{e} \cdot \int \varphi(s, r)\kappa(s, dr)$ is measurable.*

Proof. The property is obvious when φ is simple[9], and the result follows from the monotone convergence theorem by the fact that any $\mathbb{R}_{\geq 0}$-measurable function is the lub of a monotone sequence of simple functions. $\square$

Lemma 5.6. *For each $d \in \mathbf{ar}$, the map $\mathcal{I}_d^B$ is bilinear, continuous and measurable.*

[9] A $\mathbb{R}$-valued measurable function is simple iff it ranges in a finite subset of $\mathbb{R}$.

Proof. Linearity results from the linearity of integration and from (**Mssep**) satisfied by B, let us prove separate continuity. Let $(\beta_n)_{n\in\mathbb{N}}$ be a monotone sequence in $\mathcal{B}\underline{\mathsf{Path}(d,B)}$ and let $\mu \in \underline{\mathsf{Meas}(d)}$. The sequence $(\mathcal{I}_d^B(\beta_n,\mu) \in \underline{B})_{n\in\mathbb{N}}$ is monotone by linearity of $\mathcal{I}_d^B$ and for all $n \in \mathbb{N}$ we have $\left\|\mathcal{I}_d^B(\beta_n,\mu)\right\| \le \|\beta_n\|\,\|\mu\| \le \|\mu\|$ so that $\sup_{n\in\mathbb{N}} \mathcal{I}_d^B(\beta_n,\mu) \in \underline{B}$ exists. Let $\beta = \sup_{n\in\mathbb{N}} \beta_n \in \mathcal{B}\underline{\mathsf{Path}(d,B)}$, that is $\forall r \in \overline{d}\ \beta(r) = \sup_{n\in\mathbb{N}} \beta_n(r)$. Let $m \in \mathcal{M}_0^B$, since $(m \circ \beta_n)_{n\in\mathbb{N}}$ is a monotone sequence of measurable functions by linearity of m and since $m \circ \beta = \sup_{n\in\mathbb{N}} m \circ \beta_n$ (pointwise) by continuity of m, we have

$$\int m(\beta(r))\mu(dr) = \sup_{n\in\mathbb{N}} \int m(\beta_n(r))\mu(dr)$$

by the monotone convergence theorem. That is $m(\mathcal{I}_d^B(\beta,\mu)) = \sup_{n\in\mathbb{N}} m(\mathcal{I}_d^B(\beta_n,\mu)) = m(\sup_{n\in\mathbb{N}} \mathcal{I}_d^B(\beta_n,\mu))$ by continuity of m. By (**Mssep**) we get $\mathcal{I}_d^B(\beta,\mu) = \sup_{n\in\mathbb{N}} \mathcal{I}_d^B(\beta_n,\mu)$ as required.

Let $\beta \in \underline{\mathsf{Path}(d,B)}$ and let $(\mu_n \in \mathcal{B}\mathsf{Meas}(d))_{n\in\mathbb{N}}$ be a monotone sequence with lub μ. It is a standard fact that for any measurable and bounded $\varphi : \overline{d} \to \mathbb{R}_{\ge 0}$ the sequence $(\int \varphi(r)\mu_n(dr))_{n\in\mathbb{N}}$ is monotone and has $\int \varphi(r)\mu(dr)$ as lub: this is due to the fact that $\int \varphi(r)\mu(dr)$ is the lub of all $\int \varphi_0(r)\mu(dr)$ where $\varphi_0 \le \varphi$ is simple, and to the fact that $\int \varphi_0(r)\mu(dr) = \sup_{n\in\mathbb{N}} \int \varphi_0(r)\mu_n(dr)$ holds trivially when φ_0 is simple. As above the sequence $(\mathcal{I}_d^B(\beta,\mu_n))_{n\in\mathbb{N}}$ is monotone with $\forall n \in \mathbb{N}\ \left\|\mathcal{I}_d^B(\beta,\mu_n)\right\| \le \|\beta\|\,\|\mu\|$ and therefore has a lub $\sup_{n\in\mathbb{N}} \mathcal{I}_d^B(\beta,\mu_n) \in \underline{B}$. Let $m \in \mathcal{M}_0^B$, we have

$$\begin{aligned}
m(\sup_{n\in\mathbb{N}} \mathcal{I}_d^B(\beta,\mu_n)) &= \sup_{n\in\mathbb{N}} m(\mathcal{I}_d^B(\beta,\mu_n)) \\
&= \sup_{n\in\mathbb{N}} \int m(\beta(r))\mu_n(dr) \\
&= \int m(\beta(r))\mu(dr) \\
&= m(\mathcal{I}_d^B(\beta,\mu))
\end{aligned}$$

and the announced continuity follows by (**Mssep**) in B.

Now we prove measurability, so let $e \in \mathbf{ar}$, $\eta \in \underline{\mathsf{Path}(e,\mathsf{Path}(d,B))}$ and let $\kappa \in \underline{\mathsf{Path}(e,\mathsf{Meas}(d))}$, we prove that the function $\beta = \mathcal{I}_d^B \circ \langle \eta,\kappa\rangle : \overline{e} \to \underline{B}$ belongs to $\underline{\mathsf{Path}(e,B)}$. The fact that $\beta(\overline{e})$ is bounded results from Lemma 5.2. Let $e' \in \mathbf{ar}$ and $m \in \mathcal{M}_{e'}^B$, we have

$$\begin{aligned}
\boldsymbol{\lambda}(s',s) \in \overline{e'+e} \cdot m(s',\beta(s)) &= \boldsymbol{\lambda}(s',s) \in \overline{e'+e} \cdot m(s',\mathcal{I}_d^B(\eta(s),\kappa(s))) \\
&= \boldsymbol{\lambda}(s',s) \in \overline{e'+e} \cdot \int m(s',\eta(s,r))\kappa(s,dr)
\end{aligned}$$

and this function is measurable by Lemma 5.5 and by our assumption about η. □

Lemma 5.7 (Change of variable). *Let $d,e \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(d,B)}$ and $\varphi \in \mathbf{ar}(e,d)$. We have*

$$\int \beta(\varphi(s))\mu(ds) = \int \beta(r)\varphi_*(\mu)(dr)\,.$$

In other words $\mathcal{I}_d^B$ is dinatural in d.

Proof. By the usual change of variable formula, through the use of measurability tests $m \in \mathcal{M}_0^B$ and (**Mssep**) for B. □

Lemma 5.8. *If B is an integrable cone and $\alpha \in \mathbb{R}$ is such that $\alpha > 0$ then the measurable cone αB is integrable, and has the same integrals as B.*

We can define now the category which is at the core of the present study.

Definition 5.9. The category **ICones** has integrable cones as objects and an element of $\mathbf{ICones}(B,C)$ is an $f \in \mathbf{MCones}(B,C)$ such that, for all $d \in \mathbf{ar}$ and all $\beta \in \underline{\mathsf{Path}(d,B)}$ and $\mu \in \underline{\mathsf{Meas}(d)}$ one has

$$f(\int \beta(r)\mu(dr)) = \int f(\beta(r))\mu(dr)\,.$$

This property of f will be called *integral preservation* and when it holds we often simply say that f is *integrable*.

Notice that the right hand term of the above equation is well defined because $f \circ \beta \in \underline{\mathsf{Path}(d,C)}$ by our assumption about f. It is obvious that we define a category in that way.

Lemma 5.10. *The functor* $\mathsf{Meas} : \mathbf{ar} \to \mathbf{MCones}$ *introduced in Lemma 4.3 is a functor* $\mathbf{ar} \to \mathbf{ICones}$.

Proof. Let $\varphi \in \mathbf{ar}(d,e)$ and $\kappa \in \underline{\mathsf{Path}(e',\mathsf{Meas}(d))}$ be a finite kernel. Given $\mu' \in \underline{\mathsf{Meas}(e')}$ and $V \in \sigma_e$ we have

$$\begin{aligned}
\varphi_*(\int \kappa(s')\mu'(ds'))(V) &= (\int \kappa(s')\mu'(ds'))(\varphi^{-1}(V)) \\
&= \int \kappa(s',\varphi^{-1}(V))\mu'(ds') \quad \text{by def. of integration in } \mathsf{Meas}(d) \\
&= \int \varphi_*(\kappa(s'))(V)\mu'(ds') \\
&= (\int \varphi_*(\kappa(s'))\mu'(ds'))(V) \quad \text{by def. of integration in } \mathsf{Meas}(e)
\end{aligned}$$

so that φ_* preserves integrals. □

5.1. **The integrable cone of paths and a Fubini theorem for cones.**

Theorem 5.11. *For any $d \in \mathbf{ar}$, the measurable cone $\mathsf{Path}(d,B)$ is integrable.*

Proof. Let $e \in \mathbf{ar}$, $\eta \in \underline{\mathsf{Path}(e,\mathsf{Path}(d,B))}$ and $\nu \in \underline{\mathsf{Meas}(e)}$, we define $\beta : \overline{d} \to \underline{B}$ by $\beta(r) = \int \eta(s)(r)\nu(ds)$, in other words the integral of a path of paths is defined pointwise. For each $r \in \overline{d}$ we have

$$\begin{aligned}
\|\beta(r)\| = \left\|\int \eta(s)(r)\nu(ds)\right\| &\leq \|\eta(s)\|\,\|\nu\| \quad \text{by Lemma 5.2} \\
&\leq \|\eta\|\,\|\nu\|
\end{aligned}$$

so β is a bounded function. This function is a measurable path by Lemma 5.6 so β ranges in $\underline{\mathsf{Path}(d,B)}$. Let $p \in \mathcal{M}_0^{\mathsf{Path}(d,B)}$ so that $p = r \triangleright m$ for some $r \in \overline{d}$ and $m \in \mathcal{M}_0^B$, we have

$$\begin{aligned} p(\beta) &= m(\beta(r)) \\ &= m(\int \eta(s)(r)\nu(ds)) \\ &= \int m(\eta(s)(r))\nu(ds) \\ &= \int p(\eta(s))\nu(ds) \quad \text{by definition of } p \end{aligned}$$

and hence $\beta = \int \eta(r)\mu(dr)$. □

Theorem 5.12. *The operation* Path*, extended to morphisms by pre and post-composition, is a functor* $\mathbf{ar}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$*. In other words, given* $f \in \mathbf{ICones}(B,C)$ *and* $\varphi \in \mathbf{ar}(e,d)$*, we have*

$$\mathsf{Path}(\varphi, f) = \boldsymbol{\lambda}\beta \in \underline{\mathsf{Path}(d,B)} \cdot (f \circ \beta \circ \varphi) \in \mathbf{ICones}(\mathsf{Path}(d,B), \mathsf{Path}(e,C))\,.$$

Proof. Indeed, functoriality is obvious. We check first measurability of $\mathsf{Path}(\varphi,f)$ so let $e' \in \mathbf{ar}$ and let $\eta \in \underline{\mathsf{Path}(e', \mathsf{Path}(d,B))}$, we must check that $\mathsf{Path}(\varphi,f) \circ \eta \in \underline{\mathsf{Path}(e', \mathsf{Path}(e,C))}$. Let $e'' \in \mathbf{ar}$ and $p \in \mathcal{M}_{e''}^{\mathsf{Path}(e,C)}$, we check that $\psi = \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot p(s'', \mathsf{Path}(\varphi,f)(\eta(s')))$ is measurable. So let $\rho \in \mathbf{ar}(e'',e)$ and $m \in \mathcal{M}_{e''}^C$ be such that $p = \rho \triangleright m$, we have

$$\begin{aligned} \psi &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot m(s'', \mathsf{Path}(\varphi,f)(\eta(s'))(\rho(s''))) \\ &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot m(s'', f(\eta(s')(\varphi(\rho(s''))))) \\ &= \boldsymbol{\lambda}(s'',s') \in \overline{e''+e'} \cdot m(s'', f(\mathsf{fl}(\eta)(s', \varphi(\rho(s''))))) \end{aligned}$$

and this map is measurable by Lemma 4.2 because $f \circ \mathsf{fl}(\eta) \in \underline{\mathsf{Path}(e'+d,B)}$ and $\varphi \circ \rho$ is measurable. We prove that $\mathsf{Path}(\varphi,f)$ preserves integrals. Let $e' \in \mathbf{ar}$, $\eta \in \underline{\mathsf{Path}(e', \mathsf{Path}(d,B))}$, $\nu' \in \underline{\mathsf{Meas}(e')}$ and let $s \in \overline{e}$. We have

$$\begin{aligned} \mathsf{Path}(\varphi,f)(\int \eta(s')\nu'(ds'))(s) &= f(\int \eta(s')\nu'(ds'))(\varphi(s)) \\ &= f(\int \eta(s')(\varphi(s))\nu'(ds')) \quad \text{by def. of integration in } \mathsf{Path}(d,B) \\ &= \int f(\eta(s')(\varphi(s)))\nu'(ds') \quad \text{since } f \text{ preserves integrals} \\ &= (\int \mathsf{Path}(\varphi,f)(\eta(s'))\nu'(ds'))(s) \end{aligned}$$

which proves our contention. □

Lemma 5.13. *The bijection* $\mathsf{fl}_{d,e}$ *defined in Lemma 4.2, as well as its inverse, preserve integrals and hence*

$$\mathsf{fl}_{d,e} \in \mathbf{ICones}(\mathsf{Path}(d, \mathsf{Path}(e,B)), \mathsf{Path}(d+e,B))$$

is an iso in $\mathbf{ICones}$.

Proof. Results straightforwardly from the "pointwise" definition of integration in the cones of paths. □

Theorem 5.14 (Fubini). *Let $d, e \in \mathbf{ar}$, $\eta \in \underline{\mathsf{Path}(d, \mathsf{Path}(e, B))}$, $\mu \in \underline{\mathsf{Meas}(d)}$ and $\nu \in \underline{\mathsf{Meas}(e)}$. We have*

$$\int_e (\int_d \eta(r)\mu(dr))(s)\nu(ds) = \int_{d+e} \mathsf{fl}(\eta)(t)(\mu \times \nu)(dt)$$

Proof. Denoting by x_1 and x_2 these two elements of $\underline{B}$ it suffices to prove that for any $m \in \mathcal{M}_0^B$ one has $m(x_1) = m(x_2)$. Setting $\eta' = \mathsf{fl}(\eta)$ we have

$$x_1 = \int_e (\int_d \eta'(r, s)\mu(dr))\nu(ds) \qquad\qquad x_2 = \int_{d+e} \eta'(t)(\mu \times \nu)(dt)$$

and the equation follows by application of the usual Fubini theorem to the bounded non-negative measurable function $m \circ \eta'$ and to the finite measures μ and ν. Notice that in the expression of x_2 the variable t ranges over pairs. □

5.2. The category of integrable cones.

Theorem 5.15. *The category* **ICones** *is complete.*

Proof. We prove first that **ICones** has all small products. We use implicitly Theorem 2.7 at several places. Let $(C_i)_{i\in I}$ be a collection of integrable cones and let $P = \&_{i\in I} \underline{C_i}$ which is the product of the $\underline{C_i}$'s in **Cones**. Given $d \in \mathbf{ar}$, $i \in I$ and $m \in \mathcal{M}_d^{C_i}$ we define $\mathsf{in}_i(m) : \overline{d} \times P \to \mathbb{R}_{\geq 0}$ by $\mathsf{in}_i(m)(d, \vec{x}) = m(d, x_i)$. We set $\mathcal{M} = (\mathcal{M}_d)_{d\in\mathbf{ar}}$ where $\mathcal{M}_d = \{\mathsf{in}_i(m) \mid i \in I \text{ and } m \in \mathcal{M}_d^{C_i}\}$. With the notations above, given $\vec{x} \in P$ the function $\boldsymbol{\lambda} r \in \overline{d} \cdot \mathsf{in}_i(m)(r, \vec{x}) = \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, x_i)$ is measurable since $\mathcal{M}^{C_i}$ satisfies (**Msmeas**).

Let $\varphi \in \mathbf{ar}(e, d)$, we have $\mathsf{in}_i(m) \circ (\varphi \times P) = \mathsf{in}_i(m \circ (\varphi \times \underline{C_i})) \in \mathcal{M}_e$ since $m \circ (\varphi \times \underline{C_i}) \in \mathcal{M}_e^{C_i}$ by (**Mscomp**) in C_i.

Let $\vec{x}(1), \vec{x}(2) \in P$ be such that $\forall p \in \mathcal{M}_0\ p(\vec{x}(1) = p(\vec{x}(2))$. Then for each $i \in I$ we have $x(1)_i = x(2)_i$ by (**Mssep**) holding in C_i and hence $\vec{x}(1) = \vec{x}(2)$.

Let $\vec{x} \in P \setminus \{0\}$ and $\varepsilon > 0$. Since $\|\vec{x}\| = \sup_{i\in I} \|x_i\|$ there is $i \in I$ such that $\|\vec{x}\| \leq \|x_i\| + \varepsilon/2$ and $x_i \neq 0$. We can find $m \in \mathcal{M}_0^{C_i} \setminus \{0\}$ such that $\|x_i\| \leq m(x_i)/\|m\| + \varepsilon/2$. Let $p = \mathsf{in}_i(m) \in \mathcal{M}_0$, notice that $\|p\| = \|m_i\|$ since for any $x \in \mathcal{B}\underline{C_i}$ the family $\vec{y}$ defined by $y_i = x$ and $y_j = 0$ if $j \neq i$ satisfies $\vec{y} \in \mathcal{B}P$. So we have $\|\vec{x}\| \leq p(\vec{x})/\|p\| + \varepsilon$ thus showing that $\mathcal{M}$ satisfies (**Msnorm**).

So the pair $(P, \mathcal{M})$ is a measurable cone $C = \&_{i\in I} C_i$, we prove that it is integrable. An element of $\underline{\mathsf{Path}(d, C)}$ is a family $(\gamma_i \in \underline{\mathsf{Path}(d, C_i)})_{i\in I}$ such that $(\|\gamma_i\|)_{i\in I}$ is bounded and, given $\mu \in \underline{\mathsf{Meas}(d)}$, the family

$$\vec{x} = (\int \gamma_i(r)\mu(dr))_{i\in I}$$

is in P by Lemma 5.2 and is the integral of γ over μ by definition of $\mathcal{M}^C$.

With the same notations as above, for each $i \in I$, the map $\mathsf{pr}_i \circ \gamma$ is a measurable path since, given $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^{C_i}$ one has $\boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot \mathsf{in}_i(m)(s, \gamma(r)) = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \mathsf{pr}_i(\gamma(r)))$. The fact that $\mathsf{pr}_i \in \mathbf{ICones}(C, C_i)$ results from the definition of integration in C.

Let $(f_i \in \mathbf{ICones}(D, C_i))_{i\in I}$, then we know that $f = \langle f_i \rangle_{i\in I} \in \mathbf{Cones}(\underline{D}, \underline{C})$. Let $\delta \in \underline{\mathsf{Path}(d, D)}$, we prove that $f \circ \delta \in \underline{\mathsf{Path}(d, C)}$ so let $i \in I$ and $m \in \mathcal{M}_e^{C_i}$. We have $\boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot \mathsf{in}_i(m)(s, f(\delta(r))) = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, f_i(\delta(r)))$ and this latter map is

measurable for each $i \in I$ thus proving that $f \circ \delta$ is measurable. Using the same notations, let furthermore $\mu \in \underline{\mathsf{Meas}(d)}$, we have

$$\begin{aligned} f(\int \delta(r)\mu(dr)) &= (f_i(\int \delta(r)\mu(dr)))_{i\in I} \\ &= (\int f_i(\delta(r))\mu(dr)))_{i\in I} \quad \text{since each } f_i \text{ preserves integrals} \\ &= \int f(\delta(r))\mu(dr) \end{aligned}$$

which shows that $f \in \mathbf{ICones}(D, C)$ as required. This proves that **ICones** has all small products.

We prove now that **ICones** has equalizers, so let $f, g \in \mathbf{ICones}(C, D)$. Let P be the set of all $x \in \underline{C}$ such that $f(x) = g(x)$ which is clearly a precone (with algebraic operations induced by those of $\underline{C}$) by linearity of f and g. Notice that if $x, y \in P$ satisfy $x \leq_{\underline{C}} y$ then $y - x \in P$ since $f(y - x) = f(y) - f(x) = g(y) - g(x) = g(y - x)$ and hence $x \leq_P y$. We define $\|\_\|_P : P \to \mathbb{R}_{\geq 0}$ by $\|x\|_P = \|x\|_{\underline{C}}$ and one easily checks that (**Normh**), (**Normz**), (**Normt**) and (**Normp**) hold. We check (**Normc**) so let $(x_n)_{n\in\mathbb{N}}$ be a monotone sequence of elements of $\mathcal{B}P$, so that it is also a monotone sequence in $\mathcal{B}\underline{C}$ and hence has a lub x in $\mathcal{B}\underline{C}$. Since f, g are continuous we have $f(x) = g(x)$ and hence $x \in P$. For each $n \in \mathbb{N}$ we have $x_n \leq_{\underline{C}} x$ and hence $x_n \leq_P x$ since $x_n, x \in P$. Let $y \in P$ be such that $\forall n \in \mathbb{N}\ x_n \leq_P y$, we have $\forall n \in \mathbb{N}\ x_n \leq_{\underline{C}} y$ and hence $x \leq_{\underline{C}} y$ and therefore, since $x, y \in P$, we have $x \leq_P y$ which shows that x is the lub of the x_n's in P. Moreover $\|x\|_P = \|x\|_{\underline{C}} \leq 1$ and hence $x \in \mathcal{B}P$ which concludes the proof that (**Normc**) holds in P, and hence P is a cone.

We define $\mathcal{M}_d$ as the set of all $p : \overline{d} \times P \to \mathbb{R}_{\geq 0}$ such that there is $m \in \mathcal{M}_d^C$ satisfying $\forall x \in P\, \forall r \in \overline{d}\ p(r, x) = m(r, x)$. Then it is clear that $p \in (P')^{\overline{d}}$ and we actually identify $\mathcal{M}_d$ with $\mathcal{M}_d^C$ although several elements of the latter can induce the same element of the former. We prove that $(\mathcal{M}_d)_{d\in\mathbf{ar}}$ defines a measurability structure on P, the only non trivial property being (**Msnorm**). Let $x \in P \setminus \{0\}$ and $\varepsilon > 0$. Let $\varepsilon' > 0$ be such that $\varepsilon' \leq \varepsilon$ and $\varepsilon' < \|x\|$. Applying (**Msnorm**) in C we can find $m \in \mathcal{M}_0^C \setminus \{0\}$ such that $\|x\|_C \leq m(x)/\|m\|^{\underline{C}} + \varepsilon'$ where we have added the superscript to $\|m\|$ to insist on the fact that it is computed in C, that is $\|m\|^{\underline{C}} = \sup_{y\in\mathcal{B}\underline{C}} m(y)$. By our assumption that $\varepsilon' < \|x\|$ we must have $m(x) \neq 0$. By definition of $\|\_\|_P$ we have $\mathcal{B}P \subseteq \mathcal{B}C$ and hence $\|m\|^P \leq \|m\|^{\underline{C}}$ (and $\|m\|^P \neq 0$ since $m(x) \neq 0$ and $x \in P$) and hence

$$\|x\|_P = \|x\|_C \leq \frac{m(x)}{\|m\|^{\underline{C}}} + \varepsilon' \leq \frac{m(x)}{\|m\|^P} + \varepsilon$$

and P satisfies (**Msnorm**).

So we have defined a measurable cone $E = (P, \mathcal{M})$, we check that it is integrable. Let $d \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(d, E)}$ and $\mu \in \underline{\mathsf{Meas}(d)}$, we have

$$f(\int \beta(r)\mu(dr)) = \int f(\beta(r))\mu(dr) = \int g(\beta(r))\mu(dr) = g(\int \beta(r)\mu(dr))$$

since β ranges in $\underline{E}$ and f and g preserve integrals. Hence $\int \beta(r)\mu(dr) \in \underline{E}$ and this element of $\underline{E}$ is the integral of β over μ by definition of $\mathcal{M}^E$.

We check now that (E, l) is the equalizer of f, g where $l : \underline{E} \to \underline{C}$ is the obvious inclusion map. Observe first that l is linear and continuous. It is measurable $E \to C$ by definition of

the measurability structure of E which is essentially the same as that of C and preserves integrals because the integral in E is defined as in C.

By definition of E, $f\,l = g\,l$. Let H be a measurable cone and $h \in \mathbf{ICones}(H, C)$ be such that $f\,h = g\,h$. So we have $\forall z \in \underline{H}\ h(z) \in \underline{E}$. Since $h \in \mathbf{Cones}(\underline{H}, \underline{C})$, it is also linear and continuous $\underline{H} \to \underline{E}$, let us denote by h_0 the corresponding element of $\mathbf{Cones}(\underline{H}, \underline{E})$. Let $d \in \mathbf{ar}$ and $\gamma : \underline{\mathsf{Path}(d, H)}$ be a measurable path of H. Let $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^E$ so that actually $m \in \mathcal{M}_e^C$. We have $\boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, h_0(\gamma(r))) = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, h(\gamma(r)))$ which is measurable since h is. With the same notation, taking also μ in $\underline{\mathsf{Meas}(d)}$, we have

$$
\begin{aligned}
h_0(\int^H \gamma(r)\mu(dr)) &= h(\int^H \gamma(r)\mu(dr)) \quad \text{by definition of } h_0 \\
&= \int^C h(\gamma(r))\mu(dr) \quad \text{since } h \text{ preserves integrals} \\
&= \int^E h_0(\gamma(r))\mu(dr)
\end{aligned}
$$

and hence $h_0 \in \mathbf{ICones}(H, E)$. Last observe that $h = l\,h_0$ and that h_0 is unique with this property by definition of l.

This shows that $\mathbf{ICones}$ has all small limits. $\square$

Theorem 5.16. *In the category* $\mathbf{ICones}$ *the object* 1 *is cogenerating and generating and* $\mathbf{ICones}$ *is well powered.*

Proof. Let $f \neq g \in \mathbf{ICones}(C, D)$ and let $x \in \underline{C}$ be such that $f(x) \neq g(x)$. By (**Mssep**) there is $m \in \mathcal{M}_0^C$ such that $m(f(x)) \neq m(f(y))$ and since $m \in \mathbf{ICones}(C, 1)$ (using the definition of integrals) this shows that 1 is cogenerating.

Given $x \in \underline{C}$ we check that the function $\widehat{x} : \mathbb{R}_{\geq 0} \to \underline{C}$ defined by $\widehat{x}(\lambda) = \lambda x$ belongs to $\mathbf{ICones}(1, C)$. It is clearly linear and continuous. Let $\beta \in \underline{\mathsf{Path}(d, 1)}$ for some $d \in \mathbf{ar}$. This simply means that β is a measurable and bounded function $\overline{d} \to \mathbb{R}_{\geq 0}$, we must check that $\widehat{x} \circ \beta \in \underline{\mathsf{Path}(d, C)}$ so let $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^C$, we have

$$
\begin{aligned}
\boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \widehat{x}(\beta(r))) &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \beta(r)x) \\
&= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot \beta(r)m(s, x)
\end{aligned}
$$

which is measurable by measurability of multiplication. With the same notations and using moreover some $\mu \in \underline{\mathsf{Meas}(d)}$ we must prove that

$$
\widehat{x}(\int^1 \beta(r)\mu(dr)) = \int^C \widehat{x}(\beta(r))\mu(dr)\,.
$$

Remember that the second member of this equation is well defined since we have shown that $\widehat{x} \circ \beta \in \underline{\mathsf{Path}(d, C)}$. To check the equation, let $m \in \mathcal{M}_0^C$, we have

$$\begin{aligned} m(\widehat{x}(\int^1 \beta(r)\mu(dr))) &= m((\int^1 \beta(r)\mu(dr))x) \\ &= (\int^1 \beta(r)\mu(dr))m(x) \\ &= \int^1 \beta(r)m(x)\mu(dr) \\ &= \int^1 m(\widehat{x}(\beta(r)))\mu(dr) \\ &= m(\int^C \widehat{x}(\beta(r))\mu(dr))\,. \end{aligned}$$

So $\widehat{x} \in \mathbf{ICones}(1, C)$ as contended. Since $\widehat{x}(1) = x$ this shows that 1 is generating.

Let D be a subobject of C; more precisely let $h \in \mathbf{ICones}(D, C)$ be a mono. This implies that h is injective because 1 is generating. By transporting the whole structure of C through the injection h one defines an integrable cone D_0 such that $\underline{D_0} \subseteq \underline{C}$ which is isomorphic to D in **ICones**. The class of all integrable cones D_0 such that $\underline{D_0} \subseteq \underline{C}$ is a set because **ar** is a set[10]. This shows that **ICones** is well-powered. □

Theorem 5.17. *If $\mathcal{C}$ is a locally small category and R :* **ICones** $\to \mathcal{C}$ *is a functor which preserves all limits, then R has a left adjoint.*

Proof. Apply the special adjoint functor theorem. □

5.3. **Colimits and coproducts.**

Theorem 5.18. *The category* **ICones** *has all small colimits.*

Proof. Let I be a small category, we use $\mathbf{ICones}^I$ for the category whose objects are the functors $I \to \mathbf{ICones}$ and the morphisms are the natural transformations, which is locally small since I is small. Then we have a "diagonal" functor $\Delta : \mathbf{ICones} \to \mathbf{ICones}^I$ which maps any object of **ICones** to the corresponding constant functor and any morphism to the identity natural transformation. It is easily checked that Δ preserves all limits and hence it has a left adjoint by Theorem 5.17. By definition of an adjunction, this functor maps any functor $I \to \mathbf{ICones}$ to its colimit which shows that **ICones** is cocomplete. □

This theorem does not give any insight on the structure of these colimits[11], so it is reasonable to have at least a closer look at coproducts.

[10] It is only here that we use this assumption but it is essential.

[11] In particular it would be interesting to have a more explicit description of coequalizers.

Coproducts of cones. Let I be a set, without any restrictions on its cardinality for the time being. Let $(P_i)_{i\in I}$ be a family of cones. Let P be the set of all families $\vec{x} = (x_i)_{i\in I} \in \prod_{i\in I} P_i$ such that $\sum_{i\in I} \|x_i\| < \infty$. Notice that for such a family $\vec{x}$, the set $\{i \in I \mid x_i \neq 0\}$ is countable. We turn P into a cone by defining the operations componentwise and by setting $\|\vec{x}\| = \sum_{i\in I} \|x_i\|$. The induced algebraic order relation on P is the pointwise order and Scott-completeness is easily proven (by commutations of lubs with sums in $\overline{\mathbb{R}_{\geq 0}}$). In **Cones**, this cone P is the coproduct of the P_i's with obvious injections $\mathsf{in}_i \in \mathbf{Cones}(P_i, P)$ mapping x to the family $\vec{x}$ such that $x_i = x$ and $x_j = 0$ for $j \neq i$. Given a family $(f_i \in \mathbf{Cones}(P_i, Q))_{i\in I}$ the unique map $[f_i]_{i\in I} \in \mathbf{Cones}(P, Q)$ such that $\forall j \in I\ \ [f_i]_{i\in I}\ \mathsf{in}_j = f_j$ is given by

$$[f_i]_{i\in I}(\vec{x}) = \sum_{i\in I} f_i(x_i)\,.$$

This sum converges because for any finite $J \subseteq I$ one has

$$\left\|\sum_{i\in J} f_i(x_i)\right\| \leq \sum_{i\in J} \|f_i(x_i)\| \leq \sum_{i\in J} \|x_i\| = \|x\|$$

and this map $[f_i]_{i\in I}$ is easily seen to be linear and continuous. We use $\oplus_{i\in I} P_i$ for the cone P defined in that way.

Lemma 5.19. *For any cone Q the cones $(\oplus_{i\in I} P_i) \multimap Q$ and $\&_{i\in I}(P_i \multimap Q)$ are isomorphic in* **Cones***.*

Proof. The fact that $\oplus_{i\in I} P_i$ is the coproduct of the P_i's means that the function

$$\mathcal{B}((\bigoplus_{i\in I} P_i) \multimap Q) \to \mathcal{B}(\mathop{\&}_{i\in I}(P_i \multimap Q))$$

which maps f to $(f\ \mathsf{in}_i)_{i\in I}$ is a bijection. It is linear and continuous by linearity and continuity of composition of morphisms. So this bijection is an isomorphism. $\square$

In particular $(\oplus_{i\in I} P_i)' \simeq \&_{i\in I} P_i'$. Given $\vec{x'} \in \&_{i\in I} P_i'$ the associated linear and continuous form $\mathsf{fun}(\vec{x'})$ on $\oplus_{i\in I} P_i$ is given by

$$\mathsf{fun}(\vec{x'})(\vec{x}) = \langle \vec{x}, \vec{x'}\rangle = \sum_{i\in I} \langle x_i, x_i'\rangle \leq \|\vec{x}\| \left\|\vec{x'}\right\|\,.$$

Coproduct of measurable cones. Let $(C_i)_{i\in I}$ be a family of measurable cones. Let $P = \oplus_{i\in I} \underline{C_i}$. Let $\mathcal{M} = (\mathcal{M}_d)_{d\in\mathbf{ar}}$ where $\mathcal{M}_d$ is the set of all $p \in (P')^{\overline{d}}$ such that there is a family of coefficients $(\lambda_i \geq 1)_{i\in I}$ with $\boldsymbol{\lambda} r \in \overline{d} \cdot \lambda_i p(r)_i \in \mathcal{M}_d^{C_i}$, identifying P' with $\&_{i\in I} \underline{C_i}'$ as explained above. In other words $p \in \mathcal{M}_d$ means that there are families $\vec{m} = (m_i \in \mathcal{M}_d^{C_i})_{i\in I}$, $\vec{\lambda} = (\lambda_i \geq 1)_{i\in I}$ such that, for all $r \in \overline{d}$, the family $(\lambda_i \|m_i(r)\|)_{i\in I}$ is bounded by 1, and we have $p(r) = \mathsf{fun}(\vec{\lambda}\vec{m}(r))$ (where $\vec{\lambda}\vec{x} = (\lambda_i x_i)_{i\in I}$).

We prove that $\mathcal{M}$ is a measurability structure on P. Given $p \in \mathcal{M}_d$ and $\vec{x} \in P$ the map $\boldsymbol{\lambda} r \in \overline{d} \cdot p(r)(\vec{x})$ is measurable by the monotone convergence theorem because the set $\{i \in I \mid x_i \neq 0\}$ is countable so the condition (**Msmeas**) holds. The conditions (**Mscomp**) and (**Mssep**) obviously hold, let us check (**Msnorm**). Let $\vec{x} \in P \setminus \{0\}$ and let $\varepsilon > 0$. Let $J = \{i \in I \mid x_i \neq 0\}$ which is countable and let $(i(n))_{n\in\mathbb{N}}$ be an enumeration of this

set (assuming that it is infinite, the case where it is finite is simpler). For each $n \in \mathbb{N}$ let $m_n \in \mathcal{M}_d^{C_{i(n)}}$ be such that $m_n \neq 0$ and

$$\|x_{i(n)}\|_{C_{i(n)}} \leq \frac{m_n(x_{i(n)})}{\|m_n\|} + \frac{\varepsilon}{2^{n+1}}$$

Let $p \in \mathcal{M}_0$ be given by $p(\vec{y}) = \sum_{n\in\mathbb{N}} \frac{m_n(y_{i(n)})}{\|m_n\|}$. Then $\|p\| = 1$ and we have

$$\|\vec{x}\| = \sum_{n\in\mathbb{N}} \|x_{i(n)}\| \leq \sum_{n\in\mathbb{N}} \frac{m_n(x_{i(n)})}{\|m_n\|} + \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^{n+1}} = p(\vec{x}) + \varepsilon$$

proving our contention. We have shown that $C = (P, \mathcal{M})$ is a measurable cone that we denote as $\oplus_{i\in I} C_i$.

Theorem 5.20. *For each $j \in I$ one has $(\mathsf{in}_j \in \mathbf{MCones}(C_j, \oplus_{i\in I} C_i))_{i\in I}$.*

If I is countable then $(\oplus_{i\in I} C_i, (\mathsf{in}_i)_{i\in I})$ is the coproduct of the C_i's in **MCones**. *If moreover the C_i's are integrable then so is $\oplus_{i\in I} C_i$, the in_i's preserve integrals and $(\oplus_{i\in I} C_i, (\mathsf{in}_i)_{i\in I}$ is the coproduct of the C_i's in* **ICones**.

Proof. The measurability of the in_i's is easy to prove.

We assume that I is countable. Let $(f_i \in \mathbf{MCones}(C_i, D))_{i\in I}$, we have already defined $f = [f_i]_{i\in I} \in \mathbf{Cones}(\oplus_{i\in I} \underline{C_i}, \underline{D})$ and we must prove that this function is measurable. Let $d \in \mathbf{ar}$ and $\gamma \in \underline{\mathsf{Path}(d, \oplus_{i\in I} C_i)}$, we prove that $f \circ \gamma \in \underline{\mathsf{Path}(d, D)}$ so let $e \in \mathbf{ar}$ and $q \in \mathcal{M}_e^D$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot q(s, f(\gamma(r))) &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot q(s, \sum_{i\in I} f_i(\gamma(r)_i)) \\ &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \sum_{i\in I} q(s, f_i(\gamma(r)_i))\end{aligned}$$

which is measurable by the monotone convergence theorem since I is countable.

Assume that the C_i's are integrable and let $\mu \in \underline{\mathsf{Meas}(d)}$. For each $i \in I$ we have $\boldsymbol{\lambda} r \in \overline{d} \cdot \gamma(r)_i \in \underline{\mathsf{Path}(d, C_i)}$ because, for each $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^{C_i}$ we know that $\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \gamma(r))$ is measurable, where $p = \mathsf{fun}(\vec{m})$ with $m_j = m$ is $i = j$ and $m_j = 0$ otherwise[12]. Therefore we can define $\vec{x} \in \prod_{i\in I} \underline{C_i}$ by $x_i = \int \gamma(r)_i \mu(dr)$.

[12]It is harmless to assume that $0 \in \mathcal{M}_d^B$ for any measurable cone B and $d \in \mathbf{ar}$: if B is a measurable cone and $C = (\underline{B}, \mathcal{M})$ where $\mathcal{M}_d = \mathcal{M}_d^B \cup \{0\}$ then C is a measurable cone and B and C are isomorphic in **MCones**, see Remark 3.11. And similarly in the category of integrable cones.

Given $p = \mathsf{fun}(\vec{\lambda}\vec{m}) \in \mathcal{M}_0^C$, the map $p \circ \gamma : \overline{d} \to \mathbb{R}_{\geq 0}$ is bounded and measurable, and we have

$$\begin{aligned}
\int p(\gamma(r))\mu(dr) &= \int (\sum_{i \in I} \lambda_i m_i(\gamma(r)_i))\mu(dr) \\
&= \sum_{i \in I} \int \lambda_i m_i(\gamma(r)_i)\mu(dr) \quad \text{by the monotone convergence theorem,} \\
&\qquad \text{since } I \text{ is countable} \\
&= \sum_{i \in I} \lambda_i m_i(\int \gamma(r)_i \mu(dr)) \quad \text{by definition of integrals} \\
&= p(\vec{x})\,.
\end{aligned}$$

By (**Msnorm**) holding in C as shown above, this proves that $\|\vec{x}\| < \infty$ and hence $\vec{x} \in \underline{C}$ and the computation above shows also that $\vec{x} = \int \gamma(r)\mu(dr)$ and hence the cone C is integrable. The proof that it is the coproduct of the C_i's in **ICones** is routine. □

Even if I is not countable we know that $(C_i)_{i \in I}$ has a coproduct in **ICones** by Theorem 5.18, but we don't know yet how to describe it.

6. Internal linear hom and the tensor product

6.1. **The cone of linear morphisms.** Let C, D be objects of **ICones** and let P be the set of all $f : \underline{C} \to \underline{D}$ such that, for some $\varepsilon > 0$, one has $\varepsilon f \in \mathbf{ICones}(C, D)$, equipped with the same algebraic structure as $\underline{C} \multimap \underline{D}$ (see Lemma 2.6). This makes sense since the algebraic laws of the cone $\underline{C} \multimap \underline{D}$ preserve measurability and since integration is linear. Moreover given a monotone sequence $(f_n)_{n \in \mathbb{N}}$ of measurable and integral preserving elements of $\underline{C} \multimap \underline{D}$ such that $\|f_n\| \leq 1$ (remember that $\|f\| = \sup_{x \in \mathcal{B}\underline{C}} \|f(x)\|$), the linear and continuous map $f = \sup_{n \in \mathbb{N}} f_n$ is measurable and preserves integrals by the monotone convergence theorem.

Let indeed $\gamma : \underline{\mathsf{Path}(d, C)}$ be a measurable path and let $m \in \mathcal{M}_e^D$ for some $e \in \mathbf{ar}$. The function $\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, f(\gamma(r)) : \overline{e + d} \to [0, 1]$ satisfies $\varphi(s, r) = \sup_{n \in \mathbb{N}} \varphi_n$ where $(\varphi_n = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, f_n(\gamma(r)))_{n \in \mathbb{N}}$ is a monotone sequence of measurable functions by measurability of the f_n's and linearity and continuity of m in its second parameter, so φ is measurable which shows that f is measurable. Next, with the same notation and taking

moreover some $\mu \in \underline{\mathsf{Meas}(d)}$ we have, for any $m \in \mathcal{M}_0^D$,

$$\begin{aligned}
m(f(\int^C \gamma(r)\mu(dr)))) &= m(\sup_{n\in\mathbb{N}} f_n(\int^C \gamma(r)\mu(dr))) \\
&= m(\sup_{n\in\mathbb{N}} \int^D f_n(\gamma(r))\mu(dr)) \\
&= \sup_{n\in\mathbb{N}} m(\int^D f_n(\gamma(r))\mu(dr)) \quad \text{by cont. of } m \\
&= \sup_{n\in\mathbb{N}} \int^1 m(f_n(\gamma(r)))\mu(dr) \quad \text{by def. of integration in } D \\
&= \int^1 \sup_{n\in\mathbb{N}} m(f_n(\gamma(r)))\mu(dr) \quad \text{by the monotone conv. th.} \\
&= \int^1 m(f(\gamma(r)))\mu(dr) \quad \text{by continuity of } m \\
&= m(\int^D f(\gamma(r)))\mu(dr) \quad \text{by def. of integration in } D
\end{aligned}$$

and hence f preserves integrals by (**Mssep**).

Given $\gamma : \underline{\mathsf{Path}(d, C)}$ and $m \in \mathcal{M}_d^D$ we define $\gamma \triangleright m = \boldsymbol{\lambda}(r, f) \in \overline{d} \times P \cdot m(r, f(\gamma(r))) : \overline{d} \times P \to \mathbb{R}_{\geq 0}$. For each $r \in \overline{d}$ the function $l = (\gamma \triangleright m)(r, \_) : \underline{C \multimap D} \to \mathbb{R}_{\geq 0}$ is linear and continuous by linearity and continuity of m in its second argument. We define $\mathcal{M}_d = \{\gamma \triangleright m \mid \gamma \in \underline{\mathsf{Path}(d, C)}$ and $m \in \mathcal{M}_d^D\}$. We check that the family $(\mathcal{M}_d)$ is a measurability structure on $\overline{P}$.

▶ (**Msmeas**) Let $f \in \mathcal{B}P$, $\gamma \in \underline{\mathsf{Path}(d, C)}$ and $m \in \mathcal{M}_d^D$, then the map $\varphi = \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, f(\gamma(r)))$ is measurable because $f \circ \gamma \in \underline{\mathsf{Path}(d, D)}$ by measurability of f and hence $\boldsymbol{\lambda}(s, r) \in \overline{d + d} \cdot m(s, f(\gamma(r))) : \overline{d + d} \to [0, 1]$ is measurable from which follows the measurability of φ. The fact that φ ranges in $[0, 1]$ results from the assumption that $\|f\| \leq 1$.

▶ (**Mscomp**) Let $\gamma \in \underline{\mathsf{Path}(d, C)}$ and $m \in \mathcal{M}_d^D$, and let $\varphi \in \mathbf{ar}(e, d)$ for some $e \in \mathbf{ar}$. Then we have $(\gamma \triangleright m) \circ (\varphi \times \overline{P}) = \boldsymbol{\lambda}(s, f) \in \overline{e} \times P \cdot m(\varphi(s), f(\gamma(\varphi(s))))) = (\gamma \circ \varphi) \triangleright (m \circ (\varphi \times \underline{D}))$ and since $\gamma \circ \varphi \in \underline{\mathsf{Path}(e, C)}$ by Lemma 3.7 and $m \circ (\varphi \times \underline{D}) \in \mathcal{M}_e^D$ by property (**Mscomp**) satisfied in D, we have $(\gamma \triangleright m) \circ (\varphi \times P) \in \mathcal{M}_e$.

▶ (**Mssep**) Let $f_1, f_2 \in P$ and assume that for all $x \in \underline{C}$ and any $m \in \mathcal{M}_0^D$ one has $(x \triangleright m)(f_1) = (x \triangleright m)(f_2)$, that is $m(f_1(x)) = m(f_2(x))$. By (**Mssep**) in D we have $f_1(x) = f_2(x)$, and since this holds for all $x \in \underline{C}$ we have $f_1 = f_2$.

▶ (**Msnorm**) Let $f \in P \setminus \{0\}$. Let $\varepsilon > 0$, we can assume without loss of generality that $\varepsilon < 2\|f\|$. By definition of $\|f\|$ there is $x \in \mathcal{B}\underline{C}$ such that $\|f\| \leq \|f(x)\| + \varepsilon/3$ and hence $\|f\| < \|f(x)\| + \varepsilon/2$. This implies in particular that $f(x) \neq 0$ by our assumption that $\varepsilon < 2\|f\|$. By (**Msnorm**) in D there is $m \in \mathcal{M}_0^D \setminus \{0\}$ such that

$$\|f(x)\| \leq m(f(x))/\|m\| + \min(\varepsilon/2, \|f\| - \varepsilon/2)\,.$$

If $m(f(x)) = 0$ we have $\|f(x)\| \leq \|f\| - \varepsilon/2$ which is not possible since $\|f\| < \|f(x)\| + \varepsilon/2$, so $(x \triangleright m)(f) = m(f(x)) \neq 0$. This implies in particular that $\|x \triangleright m\| \neq 0$. We have

$\|x \triangleright m\| = \sup_{g\in\mathcal{B}P} m(g(x)) \leq \|m\| \sup_{g\in\mathcal{B}P} \|g(x)\| \leq \|m\|$ and hence

$$\|f\| \leq \|f(x)\| + \varepsilon/2 \leq \frac{m(f(x))}{\|m\|} + \varepsilon = \frac{(x \triangleright m)(f)}{\|m\|} + \varepsilon \leq \frac{(x \triangleright m)(f)}{\|x \triangleright m\|} + \varepsilon\,.$$

So we have defined a measurable cone that we denote as $C \multimap D$.

Lemma 6.1. *There is an isomorphism* $\mathsf{sw} \in \mathbf{MCones}(C \multimap \mathsf{Path}(d, D), \mathsf{Path}(d, C \multimap D))$ *which maps* f *to* $\boldsymbol{\lambda} r \in \overline{d} \cdot \boldsymbol{\lambda} x \in \underline{C} \cdot f(x)(r)$.

This iso preserves integrals, but this is not required for what follows.

Proof. Let $f \in \underline{C \multimap \mathsf{Path}(d, D)}$. If $r \in \overline{d}$, the map $g = \boldsymbol{\lambda} x \in \underline{C} \cdot f(x)(r) : \underline{C} \to \underline{D}$ is linear and continuous because f is, and the algebraic operations and the lubs are computed pointwise in $\underline{\mathsf{Path}(d, D)}$, we prove that g is measurable. Let $e, e' \in \mathbf{ar}$, $\gamma \in \underline{\mathsf{Path}(e, C)}$ and $m \in \mathcal{M}^D_{e'}$, we set $\varphi = \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', g(\gamma(s))) = \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', f(\gamma(s))(r))$. Notice that $r \triangleright m \in \mathcal{M}^{\mathsf{Path}(d,D)}_{e'}$" and $\varphi = \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot (r \triangleright m)(s', f(\gamma(s)))$ which is measurable because $f \circ \gamma \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, D))}$ since $f \in \underline{C \multimap \mathsf{Path}(d, D)}$. This shows that g is measurable, we prove that g preserves integrals so let moreover $\nu \in \underline{\mathsf{Meas}(d)}$, we have

$$\begin{aligned}
g(\int^C \gamma(s)\nu(ds)) &= f(\int^C \gamma(s)\nu(ds))(r) \\
&= (\int^{\mathsf{Path}(d,D)} f(\gamma(s))\nu(ds))(r) \quad \text{since } f \text{ preserves integrals.} \\
&= \int^D f(\gamma(s))(r)\nu(ds) \quad \text{by def. of integrals in } \mathsf{Path}(d, D) \\
&= \int^D g(\gamma(s))\nu(ds)\,.
\end{aligned}$$

This shows that $g = \mathsf{sw}(f)(r) \in \underline{C \multimap D}$ for all $r \in \overline{d}$. We prove next that $\eta = \mathsf{sw}(f)$ belongs to $\underline{\mathsf{Path}(d, C \multimap D)}$ so let $e \in \mathbf{ar}$ and $p \in \mathcal{M}^{C\multimap D}_e$. Let $\gamma \in \underline{\mathsf{Path}(e, C)}$ and $m \in \mathcal{M}^D_e$ be such that $p = \gamma \triangleright m$. The function $\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot p(s, \eta(r))$ satisfies

$$\begin{aligned}
\varphi &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \eta(r)(\gamma(s))) \\
&= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, f(\gamma(s))(r))\,.
\end{aligned}$$

We know that $\delta = f \circ \gamma \circ \mathsf{pr}_1 \in \underline{\mathsf{Path}(e + d, \mathsf{Path}(d, D))}$ because $\gamma \circ \mathsf{pr}_1 \in \underline{\mathsf{Path}(e + d, C)}$ and $f \in \mathbf{MCones}(C, \mathsf{Path}(d, D))$. Let $m' \in \mathcal{M}^D_{e+d}$ be defined by $m'(s, r, y) = m(s, y)$, we have $\mathsf{pr}_2 \triangleright m' \in \mathcal{M}^{\mathsf{Path}(d,D)}_{e+d}$ and hence $\varphi' = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot (\mathsf{pr}_2 \triangleright m')(s, r, \delta(s, r))$ is measurable. But

$$\begin{aligned}
\varphi' &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m'(s, r, \delta(s, r)(\mathsf{pr}_2(s, r))) \\
&= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, f(\gamma(s))(r)) = \varphi
\end{aligned}$$

so that φ is measurable, this shows that $\mathsf{sw}(f) \in \underline{\mathsf{Path}(d, C \multimap D)}$. The linearity and continuity of sw are obvious (the algebraic operations and lubs are defined pointwise) as well as the fact that $\|\mathsf{sw}\| \leq 1$. Its measurability relies on the obvious bijection between $\mathcal{M}^{C\multimap\mathsf{Path}(d,D)}_{e'}$ and $\mathcal{M}^{\mathsf{Path}(d,C\multimap D)}_{e'}$ which maps $\gamma \triangleright \varphi \triangleright m$ to $\varphi \triangleright \gamma \triangleright m$ for all $e' \in \mathbf{ar}$ (with $\gamma \in \underline{\mathsf{Path}(e', C)}$, $\varphi \in \mathbf{ar}(e', d)$ and $m \in \mathcal{M}^D_{e'}$). We have proven that sw is a morphism in **MCones**.

Conversely given $\eta \in \underline{\mathsf{Path}(d, C \multimap D)}$ we define $f = \mathsf{sw}'(\eta) = \boldsymbol{\lambda} x \in \underline{C} \cdot \boldsymbol{\lambda} r \in \overline{d} \cdot \eta(r)(x)$ and prove first that $f \in \underline{C \multimap \mathsf{Path}(d, D)}$. Let $x \in \underline{C}$ and $\delta = f(x) : \overline{d} \to \underline{D}$. If $r \in \overline{d}$ we have $\eta(r) \leq \|\eta\|$ and hence $\|\delta(r)\| = \|\eta(r)(x)\| \leq \|\eta\| \, \|x\|$ which shows that the function δ is bounded. Let $e \in \mathbf{ar}$ and $m \in \mathcal{M}_e^D$, we set $\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \delta(r))$. We have

$$\begin{aligned}\varphi &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \delta(r)) \\ &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \eta(r)(x)) \\ &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot (x \triangleright m)(s, \eta(r))\end{aligned}$$

where we consider x as an element of $\mathcal{M}_e^C$ so that $x \triangleright m \in \mathcal{M}_e^{C \multimap D}$. It follows that φ is measurable and hence $\delta \in \underline{\mathsf{Path}(d, D)}$.

Linearity of f is obvious and continuity results from the fact that lubs in $\underline{\mathsf{Path}(d, D)}$ are computed pointwise. Let $\gamma \in \underline{\mathsf{Path}(e, C)}$ for some $e \in \mathbf{ar}$, we must prove next that $f \circ \gamma \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, D))}$. Let $e' \in \mathbf{ar}$ and $p \in \mathcal{M}_{e'}^{\mathsf{Path}(d,D)}$, we must prove that $\psi = \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot p(s', f(\gamma(s)))$ is measurable. Let $\varphi \in \mathbf{ar}(e', d)$ and $m \in \mathcal{M}_{e'}^D$ be such that $p = \varphi \triangleright m$. We have

$$\begin{aligned}\psi &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', f(\gamma(s))(\varphi(s'))) \\ &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', \eta(\varphi(s'))(\gamma(s))) \\ &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot ((\gamma \circ \mathsf{pr}_2) \triangleright (m \circ (\mathsf{pr}_1 \times \underline{D})))(s', s, \eta \circ \varphi)\end{aligned}$$

and hence ψ is measurable since $\eta \circ \varphi$ is a measurable path. The proof that sw' is a morphism in **MCones** follows the same pattern as for sw. $\square$

Lemma 6.2. *The measurable cone $C \multimap D$ is integrable.*

Proof. Let $d \in \mathbf{ar}$, $\eta \in \mathsf{Path}(d, C \multimap D)$ and $\mu \in \underline{\mathsf{Meas}(d)}$. Let

$$f = \boldsymbol{\lambda} x \in \underline{C} \cdot \int^D \eta(r)(x)\mu(dr) = \boldsymbol{\lambda} x \in \underline{C} \cdot \int^D \mathsf{sw}(\eta)(x)(r)\mu(dr)\,.$$

This function is well defined since for each $x \in \underline{C}$ one has $\mathsf{sw}(\eta)(x) \in \underline{\mathsf{Path}(d, D)}$ by Lemma 6.1 so the integral $\int \mathsf{sw}(\eta)(x)(r)\mu(dr) \in \underline{D}$ is well defined. The fact that $f : \underline{C} \to \underline{D}$ is linear and continuous results from the linearity of integration and from the monotone convergence theorem. Let us check that f is measurable so let $e \in \mathbf{ar}$ and let $\gamma \in \underline{\mathsf{Path}(e, C)}$, we must prove that

$$\boldsymbol{\lambda} s \in \overline{e} \cdot \int^D \mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr) \in \underline{\mathsf{Path}(e, D)}$$

so let $e' \in \mathbf{ar}$ and $m \in \mathcal{M}_{e'}^D$, we must check that the function

$$\begin{aligned}\psi &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', \int^D \mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr)) \\ &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot \int m(s', \mathsf{sw}(\eta)(\gamma(s))(r))\mu(dr)\end{aligned}$$

is measurable. We know that the function $\boldsymbol{\lambda}(s', s, r) \in \overline{e' + e + d} \cdot m(s', \mathsf{sw}(\eta)(\gamma(s))(r))$ is measurable and bounded because $\mathsf{sw}(\eta) \circ \gamma \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, D))}$ by Lemma 6.1 and we get the announced measurability by Lemma 5.6 (in the special case where κ is the kernel

constantly equal to μ). Next we prove that f preserves integrals, so let moreover $\nu \in \underline{\mathsf{Meas}(e)}$, we have

$$
\begin{aligned}
f(\int_e^C \gamma(s)\nu(ds)) &= \int_d^D \eta(r)(\int_e^C \gamma(s)\nu(ds))\mu(dr) \\
&= \int_d^D (\int_e^D \eta(r)(\gamma(s))\nu(ds))\mu(dr) \quad \text{since } \eta(r) \in \underline{C \multimap D} \\
&= \int_d^D (\int_e^D \mathsf{sw}(\eta)(\gamma(s))(r)\nu(ds))\mu(dr) \\
&= \int_e^D (\int_d^D \mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr))\nu(ds) \\
&\qquad \text{by Th. 5.14, since } \mathsf{sw}(\eta) \circ \gamma \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, D))} \\
&= \int_e^D f(\gamma(s))\nu(ds)\,.
\end{aligned}
$$

This completes the proof that $f \in \underline{C \multimap D}$ as contended.

Let $p \in \mathcal{M}_0^{C \multimap D}$. Let $x \in \underline{C}$ and $m \in \mathcal{M}_0^D$ be such that $p = x \triangleright m$, we have

$$
\begin{aligned}
p(f) &= m(\int^D \eta(r)(x)\mu(dr)) \\
&= \int^D m(\eta(r)(x))\mu(dr) \\
&= \int p(\eta(r))\mu(dr)\,,
\end{aligned}
$$

so $f = \int^{C \multimap D} \eta(r)\mu(dr)$. □

6.2. **Bilinear maps.** Let C_1, C_2, D be integrable cones, we define formally

$$C_1, C_2 \multimap D = C_1 \multimap (C_2 \multimap D)$$

and call this integrable cone the cone of integrable bilinear and continuous maps $C_1, C_2 \to D$. Indeed an element of $\underline{C_1, C_2 \multimap D}$ can be seen as a function $f : \underline{C_1} \times \underline{C_2} \to \underline{D}$ which is separately linear and (Scott) continuous. Notice that separate Scott continuity is equivalent to Scott continuity from the cone $C_1 \,\&\, C_2$ to the cone D, as usual in domain theory. Measurability of f is expressed equivalently by saying that given $(d_i \in \mathbf{ar})_{i=1,2}$ and $(\gamma_i \in \underline{\mathsf{Path}(d_i, C_i)})_{i=1,2}$ the map $\boldsymbol{\lambda}(r_1, r_2) \in \overline{d_1 + d_2} \cdot f(\gamma_1(r_1), \gamma_2(r_2)) : \overline{d_1 + d_2} \to \underline{D}$ is a measurable path or that, given $d \in \mathbf{ar}$ and $(\gamma_i \in \underline{\mathsf{Path}(d, C_i)})_{i=1,2}$, the map $\boldsymbol{\lambda} r \in \overline{d} \cdot f(\gamma_1(r), \gamma_2(r))$ is a measurable path. Preservation of integrals means that, given moreover $(\mu_i \in \underline{\mathsf{Meas}(d_i)})_{i=1,2}$, we have

$$f(\int^{C_1} \gamma_1(r_1)\mu_1(dr_1), \int^{C_2} \gamma_2(r_2)\mu_1(dr_2)) = \iint^D f(\gamma_1(r_1), \gamma_2(r_2))\mu_1(dr_1)\mu_2(dr_2)$$

where we can use the double integral symbol by Theorem 5.14.

Continuing to spell out the definition above of the integrable cone $C_1, C_2 \multimap D$, we see that, given $d \in \mathbf{ar}$, an element of $\mathcal{M}_d^{C_1, C_2 \multimap D}$ is a

$$\gamma_1, \gamma_2 \triangleright m = \boldsymbol{\lambda}(r, f) \in \overline{d} \times \underline{(C_1, C_2 \multimap D)} \cdot m(r, f(\gamma_1(r), \gamma_2(r))$$

where $(\gamma_i \in \underline{\mathsf{Path}(d, C_i)})_{i=1,2}$ and $m \in \mathcal{M}_d^D$. Last the integral of a measurable path $\eta \in \underline{\mathsf{Path}(d, C_1, C_2 \multimap D)}$ over $\mu \in \underline{\mathsf{Meas}(d)}$ is characterized by

$$(\int \eta(r)\mu(dr))(x_1, x_2) = \int \eta(r)(x_1, x_2)\mu(dr)\,.$$

6.3. **The linear hom functor.** Let $g \in \mathbf{ICones}(D_1, D_2)$ and $h \in \mathbf{ICones}(C_2, C_1)$, we define a function

$$\begin{aligned} h \multimap g : \underline{C_1 \multimap D_1} &\to \underline{C_2 \multimap D_2} \\ f &\mapsto g\, f\, h \end{aligned}$$

The linearity and continuity of $h \multimap g$ result from the same properties satisfied by g, h. The fact that $\|h \multimap g\| \leq 1$ results from the fact that $\|g\|, \|h\| \leq 1$, so let us check that $h \multimap g$ is measurable. Let $\eta_1 \in \underline{\mathsf{Path}(d, C_1 \multimap D_1)}$ for some $d \in \mathbf{ar}$. We must prove that $(h \multimap g) \circ \eta_1 \in \underline{\mathsf{Path}(d, C_2 \multimap D_2)}$ so let $p \in \mathcal{M}_e^{C_2 \multimap D_2}$ for some $e \in \mathbf{ar}$, we must prove that

$$\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot p(s, (h \multimap g)(\eta_1(r)))$$

is measurable. Let $\gamma \in \underline{\mathsf{Path}(e, C_2)}$ and $m \in \mathcal{M}_e^{D_2}$ be such that $p = \gamma \triangleright m$. We have

$$\begin{aligned} \varphi &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, g(\eta_1(r)(h(\gamma(s))))) \\ &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, g(\delta_1(s)(r))) \\ &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, g(\mathsf{fl}(\delta_1)(s, r))) \end{aligned}$$

where $\delta_1 = \mathsf{sw}(\eta_1) \circ h \circ \gamma \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, D_1))}$ and hence $g \circ \mathsf{fl}(\delta_1) \in \underline{\mathsf{Path}(e + d, D_2)}$ so that φ is measurable. We need last to prove that $h \multimap g$ preserves integrals so let moreover $\mu \in \underline{\mathsf{Meas}(d)}$, we have

$$\begin{aligned} (h \multimap g)(\int^{C_1 \multimap D_1} \eta_1(r)\mu(dr)) &= \boldsymbol{\lambda} x \in \underline{C_2} \cdot g((\int^{C_1 \multimap D_1} \eta_1(r)\mu(dr))(h(x))) \\ &= \boldsymbol{\lambda} x \in \underline{C_2} \cdot g(\int^{D_1} \eta_1(r)(h(x))\mu(dr)) \\ &= \boldsymbol{\lambda} x \in \underline{C_2} \cdot \int^{D_2} g(\eta_1(r)(h(x)))\mu(dr) \\ &= \boldsymbol{\lambda} x \in \underline{C_2} \cdot \int^{D_2} (h \multimap g)(\eta_1(r))(x)\mu(dr) \\ &= \int^{C_2 \multimap D_2} (h \multimap g)(\eta_1(r))\mu(dr)\,. \end{aligned}$$

We have proven that $h \multimap g \in \mathbf{ICones}(C_1 \multimap D_1, C_2 \multimap D_2)$.

So we have defined a functor $\_ \multimap \_ : \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$. We identify $1 \multimap \_$ with the identity functor: we make no distinction between $x \in \underline{C}$ and the function $\widehat{x} \in \underline{1 \multimap C}$.

Theorem 6.3. *For each integrable cone C, the functor $C \multimap \_$ has a left adjoint.*

Proof. By Theorem 5.17 it suffices to prove that $C \multimap \_$ preserves all limits.

▶ Products. Let $(D_i)_{i\in I}$ be a family of measurable cones and let $D = \&_{i\in I} D_i$ as described in the proof of Theorem 5.15. We have a morphism

$$k = \langle C \multimap \mathsf{pr}_i \rangle_{i\in I} \in \mathbf{ICones}(C \multimap D, \mathop{\&}_{i\in I} (C \multimap D_i))$$

and we must prove that it is an iso. It is clearly injective, to prove surjectivity, let $\vec{f} = (f_i \in \underline{C \multimap D_i})_{i\in I} \in \underline{\& (C \multimap D_i)}$ so that $(\|f_i\|)_{i\in I}$ is bounded in $\mathbb{R}_{\geq 0}$ and hence for each $x \in \underline{C}$ the family $(\|f_i(x)\|)_{i\in I}$ is bounded. So we can define a function $f : \underline{C} \to \underline{D}$ by $f(x) = (f_i(x))_{i\in I}$. This function is clearly linear and continuous. To prove measurability, take $\gamma \in \underline{\mathsf{Path}(d, C)}$ for some $d \in \mathbf{ar}$ and $p \in \mathcal{M}^D_e$ for some $e \in \mathbf{ar}$. This means that $p = \mathsf{in}_i(m)$ for some $i \in I$ and $m \in \mathcal{M}^{D_i}_e$. Then $\boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot p(s, f(\gamma(r))) = \boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot m(s, f_i(\gamma(r)))$ is measurable because f_i is. Last let moreover $\mu \in \underline{\mathsf{Meas}(d)}$, we have $f(\int^C \gamma(r)\mu(dr)) = \int^D f(\gamma(r))\mu(dr)$ by definition of f and of integration in D. This shows that $f \in \underline{C \multimap D}$ and hence that k is a bijection since $f_i = \mathsf{pr}_i\, f$ for each $i \in I$ and hence $\vec{f} = k(f)$.

We prove that $k^{-1} \in \mathbf{ICones}(\&_{i\in I} (C \multimap D_i), C \multimap D)$. Linearity and continuity follow from the fact that all operations are defined componentwise in $\underline{\&_{i\in I}} (C \multimap D_i)$. Next given $\vec{f} \in \mathcal{B}(\underline{\&_{i\in I}} (C \multimap D_i))$ we have

$$\begin{aligned}
\left\| k^{-1}(\vec{f}) \right\| &= \sup_{x\in\mathcal{B}\underline{C}} \left\| k^{-1}(\vec{f})(x) \right\| \\
&= \sup_{x\in\mathcal{B}\underline{C}} \sup_{i\in I} \|f_i(x)\| \\
&= \sup_{i\in I} \sup_{x\in\mathcal{B}\underline{C}} \|f_i(x)\| \\
&= \sup_{i\in I} \|f_i\| \leq 1\,.
\end{aligned}$$

Next we prove that k^{-1} is measurable so let $\eta \in \underline{\mathsf{Path}(d, \&_{i\in I} (C \multimap D_i))}$ for some $d \in \mathbf{ar}$, we must prove that $\eta' = k^{-1} \circ \eta \in \underline{\mathsf{Path}(d, C \multimap D)}$. Notice that for all $r \in \overline{d}$ we can write $\eta(r) = (\eta_i(r))_{i\in I}$ where $\eta_i = \mathsf{pr}_i \circ \eta \in \underline{\mathsf{Path}(d, C \multimap D_i)}$ for each $i \in I$. Let $e \in \mathbf{ar}$, $\gamma \in \underline{\mathsf{Path}(e, C)}$ and $p \in \mathcal{M}^D_e$, so that $p = \mathsf{in}_i(m)$ for some $i \in I$ and $m \in \mathcal{M}^{D_i}_e$. We have $\boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot (\gamma \rhd p)(s, \eta'(r)) = \boldsymbol{\lambda}(s, r) \in \overline{e+d} \cdot m_i(s, \eta_i(r))$ which is measurable since each η_i is a measurable path. Last let moreover $\mu \in \underline{\mathsf{Meas}(d)}$, we must prove that $g_1 = k^{-1}(\int^{\&_{i\in I}(C\multimap D_i)} \eta(r)\mu(dr))$ and $g_2 = \int^{C\multimap D} k^{-1}(\eta(r))\mu(dr)$ are the same function. Let $x \in \underline{C}$, we have $g_1(x) = (\int \eta_i(r)\mu(dr))_{i\in I} = g_2(x)$. This ends the proof that k is an iso in $\mathbf{ICones}$ and hence that $C \multimap \_$ preserves all products.

▶ Equalizers. Let $f, g \in \mathbf{ICones}(D_1, D_2)$ and let (E, e) be the corresponding equalizer in $\mathbf{ICones}$, as described in the proof of Theorem 5.15. Then we have $(C \multimap f)\ (C \multimap e) = (C \multimap g)\ (C \multimap e)$ by functoriality of $C \multimap \_$ and it will be sufficient to prove that $(C \multimap E, C \multimap e)$ has the universal property of an equalizer. Let H be an integrable cone and $h \in \mathbf{ICones}(H, C \multimap D_1)$ be such that $(C \multimap f)\ h = (C \multimap g)\ h$. Identifying h with its "uncurryfied" version $h' \in \underline{H, C \multimap D_1}$, the integrable bilinear and continuous map (see Section 6.2) given by $h'(z, x) = h(z)(x)$, we have $f \circ h' = g \circ h'$. In other words h' ranges in $\underline{E} \subseteq \underline{D_1}$, allowing to define $h'_0 \in \underline{H, C \multimap E}$ which is the same function as h' and is bilinear continuous and integrable by definition of E (which inherits the norm, the measurability and integrability structure of C). We use h_0 for the corresponding element of $\mathbf{ICones}(H, C \multimap E)$, so that $h = (C \multimap e)h_0$. The fact that h_0 is unique with this property

results from the fact that $C \multimap e$ is a mono (it is actually the inclusion of $\underline{C \multimap E}$ into $\underline{C \multimap D_1}$ resulting from the inclusion e of $\underline{E}$ into $\underline{D_1}$). This shows that $(C \multimap E, C \multimap e)$ is the equalizer of $C \multimap f$ and $C \multimap g$ and ends the proof that $C \multimap \_$ preserves all limits. $\square$

Lemma 6.4. *Let $d \in \mathbf{ar}$ and let B, C, D be measurable cones. Let f be an element of $\mathbf{ICones}(B, \mathsf{Path}(d, C \multimap D))$. Then $f' = \boldsymbol{\lambda}(y, d, x) \in \underline{C} \times \overline{d} \times \underline{B} \cdot f(x, d, y)$ belongs to $\mathbf{ICones}(C, \mathsf{Path}(d, B \multimap D))$.*

Proof. This results from the following sequence of isos in **ICones**:

$$\begin{aligned} & B \multimap \mathsf{Path}(d, C \multimap D) \\ & \qquad \simeq B \multimap (C \multimap \mathsf{Path}(d, D)) \quad \text{by Lemma 6.1 and functoriality of } C \multimap \_ \\ & \qquad = B, C \multimap \mathsf{Path}(d, D) \\ & \qquad \simeq C, B \multimap \mathsf{Path}(d, D) \quad \text{by Section 6.2.} \end{aligned}$$

$\square$

6.4. **The tensor product of integrable cones.** Let C be an integrable cone. We denote by $\_ \otimes C$ the left adjoint of the functor $C \multimap \_$, see Theorem 6.3. Because $\_ \multimap \_$ is a functor $\mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ (see Section 6.3), we know by the adjunction with a parameter theorem ([Mac71], Chapter IV, Section 7, Theorem 3), that the so defined operation $\otimes$ can uniquely be extended into a bifunctor $\otimes : \mathbf{ICones}^2 \to \mathbf{ICones}$ in such a way that the bijection

$$\Phi_{B,C,D} : \mathbf{ICones}(B \otimes C, D) \to \mathbf{ICones}(B, C \multimap D)$$

given by the adjunction for each C is natural in B, C, D. We define

$$\tau_{B,C} = \Phi_{B,C,B \otimes C}(\mathsf{Id}_{B \otimes C}) \in \mathbf{ICones}(B, C \multimap B \otimes C) = \mathcal{B}(\underline{B, C \multimap B \otimes C})$$

and, for $x \in \underline{B}$ and $y \in \underline{C}$ we use the notation $x \otimes y = \tau_{B,C}(x, y)$. By naturality of Φ we have that, for any $f \in \underline{B \otimes C \multimap D}$,

$$\Phi_{B,C,D}(f) = f \circ \tau_{B,C}. \tag{6.1}$$

The next lemma is the key observation for proving that the above bijection is a cone isomorphism.

Lemma 6.5. *Let $d \in \mathbf{ar}$ and B, C be integrable cones. Let $\eta : \overline{d} \to \underline{B \otimes C \multimap 1}$ be a function. One has $\eta \in \underline{\mathsf{Path}(d, B \otimes C \multimap 1)}$ as soon as*

- *$\eta(\overline{d}) \subseteq \underline{B \otimes C \multimap 1}$ is bounded*
- *and for all $e \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(e, B)}$ and $\gamma \in \underline{\mathsf{Path}(e, C)}$, the function $\boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot \eta(r)(\beta(s) \otimes \gamma(s)) : \overline{e + d} \to \mathbb{R}_{\geq 0}$ is measurable.*

Proof. Let $\eta' : \underline{B} \times \underline{C} \times \overline{d} \to \mathbb{R}_{\geq 0}$ be defined by $\eta'(x, y, r) = \eta(r)(x \otimes y)$. We have $\eta' \in \mathbf{ICones}(B, C \multimap \mathsf{Path}(d, 1))$ by our assumptions. Let

$$\eta'' = \Phi^{-1}_{B,C,\mathsf{Path}(d,1)}(\eta') \in \mathbf{ICones}(B \otimes C, \mathsf{Path}(d, 1)) = \mathbf{ICones}(B \otimes C, \mathsf{Path}(d, 1 \multimap 1))$$

up to a trivial **ICones** iso and so by Lemma 6.4 there is a $h \in \mathbf{ICones}(1, \mathsf{Path}(d, B \otimes C \multimap 1))$ such that $h(1)(r)(z) = \eta''(z)(r)$ for all $z \in \underline{B \otimes C}$ and $r \in \overline{d}$. So we have $h(1)(r)(x \otimes y) = \eta(r)(x \otimes y)$ and hence $\eta(r) = h(1)(r)$ since both are elements of $\underline{B \otimes C \multimap 1}$. Since this holds for all $r \in \overline{d}$ we have proven that $\eta = h(1)$ and hence $\eta \in \mathsf{Path}(d, B \otimes C \multimap 1)$ as contended. $\square$

Theorem 6.6. *For any integrable cones B, C, D, the function $\Phi_{B,C,D}$ is an isomorphism of integrable cones from $B \otimes C \multimap D$ to $B \multimap (C \multimap D) = (B, C \multimap D)$.*

Proof. By linearity and Scott continuity of composition on the left, the function $\Phi_{B,C,D}$ — characterized by (6.1) — is a linear and continuous map $\Phi_{B,C,D} : \underline{B \otimes C \multimap D} \to \underline{B, C \multimap D}$ and which satisfies $\|\Phi_{B,C,D}(f)\| \leq \|f\|$ for all $f \in \underline{B \otimes C \multimap D}$. This latter property is due to the fact that if $f \in \underline{B \otimes C \multimap D}$ satisfies $\|f\| \leq 1$ then $f \in \mathbf{ICones}(B \otimes C, D)$ and hence $\Phi_{B,C,D}(f) \in \mathbf{ICones}(B, C \multimap D)$ so that $\|\Phi_{B,C,D}(f)\| \leq 1$ and hence for an arbitrary $f \in \underline{B \otimes C \multimap D}$ such that $f \neq 0$ we have $\|(1/\|f\|)f\| \leq 1$ and hence $\|\Phi_{B,C,D}((1/\|f\|)f)\| \leq 1$ which is exactly our contention, which trivially holds when $f = 0$.

Let us prove that $\Phi_{B,C,D}$ is measurable, so let $d \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(d, B \otimes C \multimap D)}$, we must prove that $\Phi_{B,C,D} \circ \eta \in \underline{\mathsf{Path}(d, (B, C \multimap D))}$. So let $e \in \mathbf{ar}$ and $p \in \mathcal{M}_e^{B, C \multimap D}$, which means that $p = \beta, \gamma \triangleright m$ for some $\beta \in \underline{\mathsf{Path}(e, B)}$, $\gamma \in \underline{\mathsf{Path}(e, C)}$ and $m \in \mathcal{M}_e^D$. We have

$$\begin{aligned}\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \Phi_{B,C,D}(\eta(r))) &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \eta(r) \circ \tau_{B,C}) \\ &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, \eta(r)(\beta(s) \otimes \gamma(s)))\end{aligned}$$

which is measurable because $\beta \otimes \gamma \in \underline{\mathsf{Path}(e, B \otimes C)}$ (defining $\beta \otimes \gamma$ by $(\beta \otimes \gamma)(s) = \beta(s) \otimes \gamma(s)$) by measurability of τ, and by our assumption that η is a measurable path. Altogether we have proven that

$$\Phi_{B,C,D} \in \mathbf{MCones}((B \otimes C \multimap D), (B, C \multimap D))$$

and we prove now that this morphism preserves integrals, so let moreover $\mu \in \underline{\mathsf{Meas}(d)}$, we have

$$\begin{aligned}\Phi_{B,C,D}(\int \eta(r)\mu(dr)) &= \boldsymbol{\lambda}(x,y) \in \underline{B} \times \underline{C} \cdot (\int \eta(r)\mu(dr))(x \otimes y) \\ &= \boldsymbol{\lambda}(x,y) \in \underline{B} \times \underline{C} \cdot (\int \eta(r)(x \otimes y)\mu(dr)) \\ &= \boldsymbol{\lambda}(x,y) \in \underline{B} \times \underline{C} \cdot (\int \Phi_{B,C,D}(\eta(r))(x,y)\mu(dr)) \\ &= \int \Phi_{B,C,D}(\eta(r))\mu(dr)\,.\end{aligned}$$

This shows that $\Phi_{B,C,D} \in \mathbf{ICones}((B \otimes C \multimap D), (B, C \multimap D))$ and we show now that this morphism is an iso.

We know that this function is bijective, let us use $\Psi_{B,C,D}$ for its inverse, which is linear and continuous by Lemma 2.2. Since $\Psi_{B,C,D} : \mathbf{ICones}(B, C \multimap D) \to \mathbf{ICones}(B \otimes C, D)$, we have $\|\Psi_{B,C,D}(g)\| \leq \|g\|$ for all $g \in \underline{B, C \multimap D}$, using also the linearity of $\Psi_{B,C,D}$. We prove that $\Psi_{B,C,D}$ is measurable. Let $d \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(d, (B, C \multimap D))}$, we must prove that $\Psi_{B,C,D} \circ \eta \in \underline{\mathsf{Path}(d, (B \otimes C \multimap D))}$. Without loss of generality we assume that $\|\eta\| \leq 1$. Let $e \in \mathbf{ar}$ and $p \in \mathcal{M}_e^{B \otimes C \multimap D}$, we must check that $\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \Psi_{B,C,D}(\eta(r)))$ is measurable. There is $\theta \in \mathsf{Path}(e, B \otimes C)$ and $m \in \mathcal{M}_e^D$ such that $p = \theta \triangleright m$, and we must check that $\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, \Psi_{B,C,D}(\eta(r))(\theta(s)))$ is measurable. For this, since $\mathbf{ar}$ is cartesian, it suffices to prove that

$$\boldsymbol{\lambda}(s,s',r) \in \overline{e+e+d} \cdot m(s, \Psi_{B,C,D}(\eta(r))(\theta(s')))$$

is measurable. Since $\theta \in \underline{\mathsf{Path}(e, B \otimes C)}$ it suffices to prove that

$$\eta' = \boldsymbol{\lambda}(s, r, z) \in \overline{e + d} \times \underline{B \otimes C} \cdot m(s, \Psi_{B,C,D}(\eta(r))(z)) \in \underline{\mathsf{Path}(e + d, B \otimes C \multimap 1)}$$

and to this end we apply Lemma 6.5. The boundedness assumption is satisfied because $\|\eta\| \leq 1$ and hence $\|\Psi_{B,C,D}(\eta(r))\| \leq 1$ for each $r \in \overline{d}$. So let $e' \in \mathbf{ar}$, $\beta \in \mathsf{Path}(e', B)$ and $\gamma \in \mathsf{Path}(e', C)$. We have

$$\begin{aligned}\boldsymbol{\lambda}(s', s, r) \in \overline{e' + e + d} \cdot \eta'(s, r)(\beta(s') \otimes \gamma(s')) \\ &= \boldsymbol{\lambda}(s', s, r) \in \overline{e' + e + d} \cdot m(s, \Psi_{B,C,D}(\eta(r))(\beta(s') \otimes \gamma(s'))) \\ &= \boldsymbol{\lambda}(s', s, r) \in \overline{e' + e + d} \cdot m(s, \eta(r)(\beta(s'), \gamma(s')))\end{aligned}$$

which is measurable by our assumption about η. Last we must prove that $\Psi_{B,C,D}$ preserves integrals. Using the same path η let furthermore $\mu \in \underline{\mathsf{Meas}(d)}$, we must prove that

$$g_1 = \Psi_{B,C,D}(\int \eta(r)\mu(dr)) \in \underline{B \otimes C \multimap D}$$

$$\text{and } g_2 = \int \Psi_{B,C,D}(\eta(r))\mu(dr) \in \underline{B \otimes C \multimap D}$$

are equal. Since $\Phi_{B,C,D}$ preserves integrals we have $\Phi_{B,C,D}(g_1) = \Phi_{B,C,D}(g_2)$ and the required property follows from the injectivity of $\Phi_{B,C,D}$. □

Theorem 6.7. *For any $x \in \underline{B}$ and $y \in \underline{C}$ we have $\|x \otimes y\| = \|x\| \, \|y\|$.*

Proof. Since $\tau_{B,C} \in \mathbf{ICones}(B, C \multimap B \otimes C)$ we have $\|x \otimes y\| \leq \|x\| \, \|y\|$, we just have to prove the converse. If $x = 0$ or $y = 0$ our contention trivially holds so we can assume without loss of generality that $\|x\| = \|y\| = 1$ and let $\varepsilon > 0$. By Theorem 3.8 there is $x' \in \mathcal{B}B'$ and $y' \in \mathcal{B}C'$ such that $\langle x, x' \rangle \geq 1 - \varepsilon/2$ and $\langle y, y' \rangle \geq 1 - \varepsilon/2$. Let $g : \underline{B} \times \underline{C} \to \mathbb{R}_{\geq 0}$ be defined by $g(x_0, y_0) = \langle x_0, x' \rangle \langle y_0, y' \rangle$. Then $g \in \underline{B, C \multimap 1}$ and moreover $\|g\| \leq 1$. Let $z' = \Psi(g) \in \mathcal{B}(B \otimes C)'$, we have

$$\begin{aligned}\|x \otimes y\| &\geq \langle x \otimes y, z' \rangle \\ &= \langle x, x' \rangle \langle y, y' \rangle \\ &\geq \big(1 - \frac{\varepsilon}{2}\big)^2 > 1 - \varepsilon\end{aligned}$$

so that $\|x \otimes y\| \geq 1$. □

Theorem 6.8. *The category* $\mathbf{ICones}$*, equipped with the bifunctor* $\otimes$ *and unit* 1 *as well as suitably defined natural isomorphisms* $\alpha_{B_1,B_2,B_3} \in \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, B_1 \otimes (B_2 \otimes B_3))$ etc. *is a symmetric monoidal closed category.*

Proof. This is routine. Using the iso Φ and the functor $\_ \multimap \_$ we build natural bijections

$$\mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, C) \to \mathbf{ICones}(B_1, B_2 \multimap (B_3 \multimap C))$$
$$\mathbf{ICones}(B_1 \otimes (B_2 \otimes B_3), C) \to \mathbf{ICones}(B_1, B_2 \multimap (B_3 \multimap C))$$

and we apply Lemma 1.1 to define the associativity isos. The other ones are built similarly, see [EK65] for details. □

7. Categorical properties of integration

In Lemma 5.10 we have defined the functor $\mathsf{Meas} : \mathbf{ar} \to \mathbf{ICones}$ which maps any $d \in \mathbf{ar}$ to the integrable cone $\mathsf{Meas}(d)$ of finite non-negative measures on $\overline{d}$ and acts on measurable functions by the standard push-forward operation, $\mathsf{Meas}(\varphi) = \varphi_*$.

Notice that for each $d \in \mathbf{ar}$ we have a specific element $\boldsymbol{\delta}^d \in \underline{\mathsf{Path}(d, \mathsf{Meas}(d))}$ such that $\boldsymbol{\delta}^d(r)$ is the Dirac mass at $r \in \overline{d}$, the measure defined by

$$\boldsymbol{\delta}^d(U) = \begin{cases} 1 & \text{if } r \in U \\ 0 & \text{otherwise.} \end{cases}$$

The boundedness of $\boldsymbol{\delta}^d$ is obvious and its measurability results from the observation that if $m = \widetilde{U}$ (for some $U \in \sigma_d$) we have $m \circ \boldsymbol{\delta}^d = \chi_U$ (the characteristic function of U) which is measurable.

Theorem 7.1. *For any $d \in \mathbf{ar}$ and integrable cone B, one has*

$$\mathcal{I}_d^B \in \mathbf{ICones}(\mathsf{Path}(d, B), \mathsf{Meas}(d) \multimap B)$$

and $\mathcal{I}_d^B$ is an isomorphism which is natural in d and in B (between functors $\mathbf{ar}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$).

This means that $\mathcal{I}_B^d$ is a bilinear continuous and measurable, and preserves integrals on both sides, and that, considered as a linear morphism acting on $\mathsf{Path}(d, B)$, it is an iso in $\mathbf{ICones}$.

Proof. For the first statement we just have to prove preservation of integrals in both arguments since bilinearity, continuity and measurability have already been proven in Lemma 5.6. So let $e \in \mathbf{ar}$, $\nu \in \underline{\mathsf{Meas}(e)}$, $\eta \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, B))}$ and $\mu \in \underline{\mathsf{Meas}(d)}$, we have

$$\begin{aligned}
\mathcal{I}_d^B(\int_e^{\mathsf{Path}(d,B)} \eta(s)\nu(ds), \mu) &= \int_d^B (\int_e^{\mathsf{Path}(d,B)} \eta(s)\nu(ds))(r)\mu(dr) \\
&= \int_d^B (\int_e^B \eta(s)(r)\nu(ds))\mu(dr) \\
&= \int_e^B (\int_d^B \eta(s)(r)\mu(dr))\nu(ds) \quad \text{by Theorem 5.14} \\
&= \int \mathcal{I}_d^B(\eta(s), \mu)\nu(ds)\,.
\end{aligned}$$

Next let $\beta \in \underline{\mathsf{Path}(d, B)}$ and $\kappa \in \underline{\mathsf{Path}(e, \mathsf{Meas}(d))}$, we have

$$\mathcal{I}_d^B(\beta, \int_e^{\mathsf{Meas}(d)} \kappa(s)\nu(ds)) = \int_d^B \beta(r)(\int_e^{\mathsf{Meas}(d)} \kappa(s)\nu(ds))(dr)$$

where one should remember that the value of the integral $\int \kappa(s)\nu(ds)$ is the finite measure on $\overline{d}$ which maps $U \in \sigma_d$ to $\int \kappa(s, U)\nu(ds) \in \mathbb{R}_{\geq 0}$. We claim that $x_1 = x_2$ where

$$x_1 = \int_d^B \beta(r)(\int_e^{\mathsf{Meas}(d)} \kappa(s)\nu(ds))(dr) \qquad x_2 = \int_e^B (\int_d^B \beta(r)\kappa(s, dr))\nu(ds)\,.$$

Upon applying to both members an element of $\mathcal{M}_0^B$ and using (**Mssep**) for B we can assume that $B = 1$. By the monotone convergence theorem and the fact that any measurable

function is the lub of a monotone sequence of simple measurable functions, we can assume that β is simple, and by linearity of integrals we can assume that $\beta = \chi_U$ for some $U \in \sigma_d$. Then we have $x_1 = \int \kappa(s, U)\nu(ds) = x_2$.

Now we define a function $\mathcal{K}_d^B : \underline{\mathsf{Meas}(d) \multimap B} \to \underline{\mathsf{Path}(d, B)}$ by setting

$$\mathcal{K}_d^B(f) = f \circ \boldsymbol{\delta}^d \,,$$

which is a bounded measurable path because $\boldsymbol{\delta}^d$ is a bounded measurable path. Linearity and continuity of $\mathcal{K}_d^B$ result from linearity and continuity of composition. We prove measurability so let $e \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(e, \mathsf{Meas}(d) \multimap B)}$, we contend that $\mathcal{K}_d^B \circ \eta \in \underline{\mathsf{Path}(e, \mathsf{Path}(d, B))}$. So let $e' \in \mathbf{ar}$ and let $p \in \mathcal{M}_{e'}^{\mathsf{Path}(d,B)}$, we must prove that

$$\psi = \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot p(s', \mathcal{K}_d^B(\eta(s)))$$

is measurable. Let $\varphi \in \mathbf{ar}(e', d)$ and $m \in \mathcal{M}_{e'}^B$ be such that $p = \varphi \rhd m$, we have

$$\begin{aligned}
\psi &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', \mathcal{K}_d^B(\eta(s))(\varphi(s'))) \\
&= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', \eta(s)(\boldsymbol{\delta}^d(\varphi(s')))) \\
&= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot (((\boldsymbol{\delta}^d \circ \varphi) \rhd m)(s', \eta(s))
\end{aligned}$$

which is measurable because $(\boldsymbol{\delta}^d \circ \varphi) \rhd m \in \mathcal{M}_{e'}^{\mathsf{Meas}(d) \multimap B}$ and $\eta \in \underline{\mathsf{Path}(e, \mathsf{Meas}(d) \multimap B)}$.

We prove that $\mathcal{K}_d^B$ preserves integrals so let moreover $\nu \in \underline{\mathsf{Meas}(e)}$, we have

$$\begin{aligned}
\mathcal{K}_d^B\Big(\int_e^{\mathsf{Meas}(d) \multimap B} \eta(s)\nu(ds)\Big) &= \boldsymbol{\lambda} r \in \overline{d} \cdot \int_e^B \eta(s)(\boldsymbol{\delta}^d(r))\nu(ds) \\
&= \boldsymbol{\lambda} r \in \overline{d} \cdot \int_e^B \mathcal{K}_d^B(\eta(s))(r)\nu(ds) \\
&= \int_e^{\mathsf{Path}(d,B)} \mathcal{K}_d^B(\eta(s))\nu(ds)
\end{aligned}$$

so that $\mathcal{K}_d^B \in \mathbf{ICones}(\mathsf{Meas}(d) \multimap B, \mathsf{Path}(d, B))$.

Let $f \in \underline{\mathsf{Meas}(d) \multimap B}$, we have

$$\begin{aligned}
\mathcal{I}_d^B(\mathcal{K}_d^B(f)) &= \boldsymbol{\lambda}\mu \in \underline{\mathsf{Meas}(d)} \cdot \int_d^B f(\boldsymbol{\delta}^d(r))\mu(dr) \\
&= \boldsymbol{\lambda}\mu \in \underline{\mathsf{Meas}(d)} \cdot f\Big(\int_d^{\mathsf{Meas}(d)} \boldsymbol{\delta}^d(r)\mu(dr)\Big) \quad \text{since } f \text{ preserves integrals} \\
&= \boldsymbol{\lambda}\mu \in \underline{\mathsf{Meas}(d)} \cdot f(\mu) = f
\end{aligned}$$

and let $\beta \in \underline{\mathsf{Path}(d, B)}$, we have

$$\mathcal{K}_d^B(\mathcal{I}_d^B(\beta)) = \boldsymbol{\lambda} r \in \overline{d} \cdot \Big(\int_d^B \beta(r')\boldsymbol{\delta}^d(r, dr')\Big) = \beta \,.$$

Checking naturality is routine. $\square$

Theorem 7.2. *Let $d \in \mathbf{ar}$ and B be an object of* $\mathbf{ICones}$. *Let $f_1, f_2 \in \mathbf{ICones}(\mathsf{Meas}(d), B)$. If, for all $r \in \overline{d}$, one has $f_1(\boldsymbol{\delta}^d(r)) = f_2(\boldsymbol{\delta}^d(r))$ then $f_1 = f_2$.*

Proof. This results from the fact that $\mathcal{K}_d^B$ defined above is a bijection. $\square$

Remark 7.3. In other words the Dirac measures $\boldsymbol{\delta}^d(r)$ are dense in the integrable cone $\mathsf{Meas}(d)$ of all finite measures on the measurable space $\overline{d}$. This property is one of the main benefits of integrability of cones: as explained in [Ehr20], it does not hold in **MCones** because the function $\mathsf{disc}(d)$ of Remark 8.27 is a morphisms of **MCones**.

It is easy to check that for any $d \in \mathbf{ar}$ the functor $\mathsf{Path}(d, \_)$ preserves all limits. It follows by the special adjoint functor theorem that it has a left adjoint. We provide an explicit description of this adjoint.

Lemma 7.4. *Let $d \in \mathbf{ar}$ and B, C be measurable cones. The function* sw *introduced in Lemma 6.1, which maps $\eta \in \underline{\mathsf{Path}(d, B \multimap C)}$ to $\boldsymbol{\lambda} x \in \underline{B} \cdot \boldsymbol{\lambda} r \in \overline{d} \cdot \eta(r)(x)$ is an* **ICones** *iso from $\mathsf{Path}(d, B \multimap C)$ to $B \multimap \mathsf{Path}(d, C)$ which is natural in d, B and C.*

Proof. By Lemma 6.1 we only need to prove that sw preserves integrals which results easily from the fact that integrals in $\mathsf{Path}(d, C)$ and $B \multimap C$ are defined pointwise, and this also implies that sw^{-1} preserves integrals. Naturality is obvious. $\square$

We can define the functor $\mathsf{Meas}^{\otimes} : \mathbf{ar} \times \mathbf{ICones} \to \mathbf{ICones}$ by $\mathsf{Meas}^{\otimes}(d, B) = \mathsf{Meas}(d) \otimes B$.

Theorem 7.5. *For each $d \in \mathbf{ar}$ we have* $\mathsf{Meas}^{\otimes}(d, \_) \dashv \mathsf{Path}(d, \_)$

Proof. We have the following sequence of natural bijections:

$$\begin{aligned}
\mathbf{ICones}(\mathsf{Meas}(d) \otimes B, C) &\simeq \mathcal{B}(\underline{\mathsf{Meas}(d) \otimes B \multimap C}) \quad \text{by def. of } _ \multimap _ \\
&\simeq \mathcal{B}(\underline{\mathsf{Meas}(d) \multimap (B \multimap C)}) \quad \text{by Theorem 6.8} \\
&\simeq \mathcal{B}(\underline{\mathsf{Path}(d, B \multimap C)}) \quad \text{by Theorem 7.1} \\
&\simeq \mathcal{B}(B \multimap \mathsf{Path}(d, C)) \quad \text{by Lemma 7.4} \\
&\simeq \mathbf{ICones}(B, \mathsf{Path}(d, C))\,.
\end{aligned}$$

$\square$

Remark 7.6. This adjunction induces a monad M_d on **ICones** such that $\mathsf{M}_d(B) = \mathsf{Path}(d, \mathsf{Meas}(d) \otimes B)$. What can we say about this monad? Is it a kind of Giry monad? The comonad $B \mapsto \mathsf{Meas}(d) \otimes \mathsf{Path}(d, B)$, whose counit is integration, is perhaps easier to understand.

7.1. **The category of substochastic kernels as a full subcategory of ICones.** If $d, e \in \mathbf{ar}$, a substochastic kernel from d to e is an element of $\mathbf{Skern}(d, e) = \mathcal{B}\underline{\mathsf{Path}(d, \mathsf{Meas}(e))}$: this is an equivalent characterization of this standard measure theory and probability notion. Then **Skern** is a category:

- the identity at d is $\boldsymbol{\delta}^d \in \mathbf{Skern}(d, d)$
- and given $\kappa_1 \in \mathbf{Skern}(d_1, d_2)$ and $\kappa_2 \in \mathbf{Skern}(d_2, d_3)$, their composite $\kappa = \kappa_2\, \kappa_1$ is given by

$$\kappa(r_1)(U_3) = \int_{d_2}^{1} \kappa_2(r_2, U_3) \kappa_1(r_1, dr_2)$$

for $U_3 \in \sigma_{d_3}$, that is $\kappa(r_1) = \mathcal{I}_{d_2}^{\mathsf{Meas}(d_3)}(\kappa_2)(\kappa_1(r_1))$: this formula is a continuous generalization of the product of substochastic matrices.

As is well known this category **Skern** can also be presented as the Kleisli category of the Giry monad, but this point of view is not central in this work[13].

If $\kappa \in \mathbf{Skern}(d,e)$, we set $\mathsf{Klin}(\kappa) = \mathcal{I}_d^{\mathsf{Meas}(e)}(\kappa) \in \mathbf{ICones}(\mathsf{Meas}(d), \mathsf{Meas}(e))$ defining a functor $\mathsf{Klin} : \mathbf{Skern} \to \mathbf{ICones}$ which maps $d \in \mathbf{ar}$ to $\mathsf{Meas}(d)$. Remember that we use Meas for the functor $\mathbf{ar} \to \mathbf{ICones}$ defined on morphisms by $\mathsf{Meas}(\varphi) = \varphi_* = \mathsf{Klin}(\boldsymbol{\delta}^e \circ \varphi) \in \mathbf{ICones}(\mathsf{Meas}(d), \mathsf{Meas}(e))$ for $\varphi \in \mathbf{ar}(d,e)$.

Theorem 7.7. *The functor* $\mathsf{Klin} : \mathbf{Skern} \to \mathbf{ICones}$ *is full and faithful.*

Proof. By Theorem 7.1. □

Remark 7.8. So we can consider the category of measurable spaces (at least those sorted out by **ar**) and substochastic kernels as a full subcategory of **ICones** and again, this is a major consequence of the assumption that linear morphisms must preserve integrals. This has to be compared with QBSs which form a cartesian closed category which contain the category of measurable spaces and measurable functions (or a full subcategory thereof such as our **ar**) as a full subcategory through the Yoneda embedding. See also Remark 10.9.

Theorem 7.9. *If* $d, e \in \mathbf{ar}$*, then there is an iso in* $\mathbf{ICones}(\mathsf{Meas}(d+e), \mathsf{Meas}(d) \otimes \mathsf{Meas}(e))$ *which is natural in* d *and* e *on the category* **ar**.

Proof. Given an object B of **ICones** we have the following sequence of natural bijections

$$\begin{aligned}
\mathbf{ICones}(\mathsf{Meas}(d+e), B) &\simeq \mathcal{B}\underline{\mathsf{Meas}(d+e) \multimap B} \\
&\simeq \mathcal{B}\underline{\mathsf{Path}(d+e, B)} \quad \text{by Theorem 7.1} \\
&\simeq \mathcal{B}\underline{\mathsf{Path}(d, (\mathsf{Path}(e, B)))} \quad \text{by Lemma 4.2} \\
&\simeq \mathcal{B}\underline{\mathsf{Meas}(d) \multimap (\mathsf{Meas}(e) \multimap B)} \quad \text{by Theorem 7.1} \\
&\simeq \mathbf{ICones}(\mathsf{Meas}(d), (\mathsf{Meas}(e) \multimap B)) \\
&\simeq \mathbf{ICones}(\mathsf{Meas}(d) \otimes \mathsf{Meas}(e), B) \quad \text{because } \mathbf{ICones} \text{ is a SMCC}
\end{aligned}$$

and we conclude the proof using Lemma 1.1. □

8. Stable and measurable functions

We start studying the non-linear maps between integrable cones. The first notion we consider was introduced in [EPT18].

8.1. **The local cone.**

Let B be an integrable cone and $x \in \mathcal{B}\underline{B}$. Let

$$P = \{u \in \underline{B} \mid \exists \varepsilon > 0 \; x + \varepsilon u \in \mathcal{B}B\}\,.$$

Then P is a precone. Indeed if $u_1, u_2 \in P$ and $\varepsilon > 0$ is such that $x + \varepsilon u_i \in \mathcal{B}\underline{B}$ and hence

$$x + \frac{\varepsilon}{2}(u_1 + u_2) = \frac{1}{2}(x + \varepsilon u_1) + \frac{1}{2}(x + \varepsilon u_2) \in \mathcal{B}\underline{B}$$

so that $u_1 + u_2 \in P$. The fact that $u \in P \Rightarrow \forall \lambda \in \mathbb{R}_{\geq 0} \; \lambda u \in P$ is clear. The condition (**Simpl**) and (**Pos**) result easily from the fact that they hold in $\underline{B}$.

We can equip P with a map (sometimes called a gauge) $N : P \to \mathbb{R}_{\geq 0}$ defined by

$$N(u) = (\sup\{\lambda > 0 \mid x + \lambda u \in \mathcal{B}\underline{B}\})^{-1} = \inf\{\lambda^{-1} \mid \lambda > 0 \text{ and } x + \lambda u \in \mathcal{B}\underline{B}\}$$

[13]Especially because we have no reason to assume that **ar** is closed under the action of this monad.

Lemma 8.1. *The function N is a norm on P and, equipped with this norm, P is a cone.*

Proof. Assume that $N(u) = 0$, this means that $\forall\lambda \in \mathbb{R}_{\geq 0}$ $\|x + \lambda u\| \leq 1$ and hence $\forall\lambda \in \mathbb{R}_{\geq 0}$ $\lambda\,\|u\| \leq 1$ so that $u = 0$. Let $u_1, u_2 \in P$ and let $\varepsilon > 0$. We can find $\lambda_1, \lambda_2 > 0$ such that $x + \lambda_i u_i \in \mathcal{B}\underline{B}$ and $\lambda_i^{-1} \leq N(u_i) + \varepsilon/2$ for $i = 1, 2$. We have

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}(x + \lambda_2 u_2) + \frac{\lambda_2}{\lambda_1 + \lambda_2}(x + \lambda_1 u_1) \in \mathcal{B}\underline{B}$$

so that

$$x + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}(u_1 + u_2) \in \mathcal{B}\underline{B}$$

so that $N(u_1 + u_2) \leq (\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2})^{-1} = \lambda_1^{-1} + \lambda_2^{-1} \leq N(u_1) + N(u_2) + \varepsilon$. Since this holds for all $\varepsilon > 0$ we have $N(u_1 + u_2) \leq N(u_1) + N(u_2)$. The property (**Normp**) is easy, let us prove (**Normc**). Observe first that, for $u, v \in P$, we have $u \leq_P v$ iff $u \leq_B v$. Let $(u_n)_{n\in\mathbb{N}}$ be a monotone sequence in P such that $\forall n \in \mathbb{N}$ $N(u_n) \leq 1$. Then we have $\forall n \in \mathbb{N}$ $x + u_n \in \mathcal{B}\underline{B}$ and hence the sequence $(x + u_n)_{n\in\mathbb{N}}$ is a monotone sequence in $\mathcal{B}\underline{B}$ and so it has a lub in $\mathcal{B}\underline{B}$ which coincides with $x + u$ where u is the lub of $(u_n)_{n\in\mathbb{N}}$ in $\underline{B}$. Thus we have $u \in P$ and $N(u) \leq 1$. Last observe that u is *a fortiori* the lub of the u_n's in P. $\square$

We use now the standard notation $\|\_\|$ for that norm. Notice that

$$\mathcal{B}P = \{u \in \underline{B} \mid x + u \in \mathcal{B}\underline{B}\}$$

and also that $\|u\|_B \leq \|u\|$, for all $u \in P$.

For each $d \in \mathbf{ar}$ we define $\mathcal{M}_d$ as the set of all test functions $\boldsymbol{\lambda} r \in \mathbf{ar}(d) \cdot \boldsymbol{\lambda} u \in P \cdot m(r, u)$ for the elements m of $\mathcal{M}_d^B$ (such an element of $\mathcal{M}_d$ will still be denoted by m even if two different elements of $\mathcal{M}_d^B$ possibly induce the same test). The fact that $\forall r \in \overline{d}$, $u \in \mathcal{B}P$ $m(r, u) \leq 1$ whenever $m \in \mathcal{M}_d$ results from $\mathcal{B}P \subseteq \mathcal{B}B$.

Then it is straightforward to check that $(P, (\mathcal{M}_d)_{d\in\mathbf{ar}})$ is a measurable cone, that we denote as B_x and call *the local subcone of B at x* and it is also clear that this measurable cone is integrable, the integral of a path in B_x being defined exactly as in B.

It is important to keep in mind the meaning of the norm in this local cone, which is most usefully described as follows.

Lemma 8.2. *Let $x \in \mathcal{B}\underline{B}$ and $u \in \underline{B_x} \setminus \{0\}$. Then we have $x + \|u\|_{B_x}^{-1} u \in \mathcal{B}B$ and $x + \lambda u \notin \mathcal{B}B$ for all $\lambda > \|u\|_{B_x}^{-1}$.*

Proof. By definition of the norm in B_x and by the Scott-closedness of $\mathcal{B}\underline{B}$. $\square$

Example 8.3. Let $B = 1$ so that $\underline{1} = \mathbb{R}_{\geq 0}$ and $\mathcal{B}\underline{1} = [0, 1]$. If $x \in [0, 1]$ we have two cases: if $x = 1$ then $B_x = 0$ and if $x < 1$ we have $\underline{B_x} = \mathbb{R}_{\geq 0}$. If $u \in \underline{B_x} \setminus \{0\}$ (which implies $x < 1$) then the largest $\lambda > 0$ such that $x + \lambda u \in [0, 1]$ is $\frac{1-x}{u}$ and hence $\|u\|_{B_x} = \frac{u}{1-x} = \frac{1}{1-x}\|u\|_B$ so the local cone B_x can be seen as an homothetic image of B by a factor $\frac{1}{1-x}$ which goes to ∞ when x gets closer to the border 1 of the unit ball $[0, 1]$. This very simple example gives an intuition of what happens in general, with the difference that B_x has no reason to be always homothetic to B, think for instance of the case where $B = 1 \,\&\, 1$ and $x = (1, 0)$: then B_x is isomorphic to 1.

8.2. **The integrable cone of stable and measurable functions.** Given $n \in \mathbb{N}$, we use $\mathcal{P}^-(n)$ (resp. $\mathcal{P}^+(n)$) for the set of all subsets I of $\{1, \dots, n\}$ such that $n - \# I$ is even (resp. odd).

Lemma 8.4. *Let $n \in \mathbb{N}$, $j \in \{1, \dots, n+1\}$ and $\varepsilon \in \{-, +\}$. Given $I \in \mathcal{P}^\varepsilon(n)$, the set*

$$\mathsf{inj}_j(I) = \{i \in I \mid i < j\} \cup \{j\} \cup \{i + 1 \mid i \in I \text{ and } i \geq j\}$$

belongs to $\mathcal{P}^\varepsilon(n+1)$ and inj_j defines a bijection between $\mathcal{P}^\varepsilon(n)$ and the set of all $J \subseteq \{1, \dots, n+1\}$ such that $J \in \mathcal{P}^\varepsilon(n+1)$ and $j \in J$.

The proof is straightforward.

Definition 8.5. Given cones P, Q, a function $f : \mathcal{B}P \to Q$ is *totally monotone* if for any $n \in \mathbb{N}$ and any $x, u_1, \dots, u_n \in P$ such that $x + \sum_{i=1}^n u_i \in \mathcal{B}P$ one has

$$\sum_{I \in \mathcal{P}^-(n)} f(x + \sum_{i \in I} u_i) \leq \sum_{I \in \mathcal{P}^+(n)} f(x + \sum_{i \in I} u_i)\,.$$

For $n = 1$ this condition means that f is monotone. For $n = 2$ it means

$$f(x + u_1) + f(x + u_2) \leq f(x + u_1 + u_2) + f(x)$$

that is, if the condition also holds for $n = 1$,

$$f(x + u_1) - f(x) \leq f(x + u_1 + u_2) - f(x + u_2)$$

in other words, the map $x \mapsto f(x + u_1) - f(x)$ is monotone (where it is defined). For $n = 3$:

$$\begin{aligned} &f(x + u_1 + u_2) + f(x + u_2 + u_3) + f(x + u_1 + u_3) + f(x) \\ &\qquad \leq f(x + u_1 + u_2 + u_3) + f(x + u_1) + f(x + u_2) + f(x + u_3)\,. \end{aligned}$$

Definition 8.6. A function $f : \mathcal{B}P \to Q$ is *stable* if f is totally monotone, bounded and Scott continuous (see Definition 2.1).

Definition 8.7. Let C, D be measurable cones. A stable function $f : \mathcal{B}\underline{C} \to \underline{D}$ is measurable if for any $d \in \mathbf{ar}$ and $\gamma \in \mathcal{B}\underline{\mathsf{Path}(d, C)}$ one has $f \circ \gamma \in \underline{\mathsf{Path}(d, D)}$.

Stable and measurable functions, with algebraic operations defined pointwise, form a precone P.

Lemma 8.8. *Let $f, g \in P$. One has $f \leq g$ iff for any $n \in \mathbb{N}$ and any $x, u_1, \dots, u_n \in \mathcal{B}\underline{C}$ such that $x + \sum_{i=1}^n u_i \in \mathcal{B}\underline{C}$ one has*

$$\begin{aligned} &\sum_{I \in \mathcal{P}^-(n)} g(x + \sum_{i \in I} u_i) + \sum_{I \in \mathcal{P}^+(n)} f(x + \sum_{i \in I} u_i) \\ &\qquad \leq \sum_{I \in \mathcal{P}^+(n)} g(x + \sum_{i \in I} u_i) + \sum_{I \in \mathcal{P}^-(n)} f(x + \sum_{i \in I} u_i)\,. \end{aligned}$$

Proof. Assume first that $f \leq g$ and let $h = g - f$. Notice that since addition is defined pointwise in P we have $h(x) = g(x) - f(x)$ for all $x \in \mathcal{B}\underline{C}$. Given $n \in \mathbb{N}$ and $x, u_1, \dots, u_n \in \mathcal{B}\underline{C}$ such that $x + \sum_{i=1}^n u_i \in \mathcal{B}\underline{C}$ we have

$$\sum_{I \in \mathcal{P}^-(n)} h(x + \sum_{i \in I} u_i) \leq \sum_{I \in \mathcal{P}^+(n)} h(x + \sum_{i \in I} u_i)$$

and the announced inequality follows. Conversely if the property expressed in the lemma holds we have in particular $\forall x \in \mathcal{B}P\ f(x) \leq g(x)$ (take $n = 0$) and so we can define a function $h : \mathcal{B}\underline{C} \to \underline{D}$ by $h(x) = g(x) - f(x)$ and this function is totally monotone. This function is Scott-continuous by Lemma 2.3 and is measurable because subtraction is measurable on $\mathbb{R}^2$. □

Remark 8.9. Notice that if $f \leq g$ then $f(x) \leq g(x)$ for all $x \in \mathcal{B}\underline{B}$, but the converse is not true. As an example take $f(x) = x$ and $g(x) = 1$, defining two stable functions (for $B = C = 1$) which do not satisfy $f \leq g$ but such that $f(x) \leq g(x)$ holds for all $x \in [0, 1]$. It is natural to call this order relation on stable functions the *stable order* in reference to [Ber78, Gir86] where the stable order behaves in a very similar way.

We equip this precone P with the norm $\|f\| = \sup_{x \in \mathcal{B}\underline{C}} \|f(x)\|$ which is well defined by our assumptions that stable functions are bounded.

Lemma 8.10. *Let $(f_n \in \mathcal{B}P)_{n \in \mathbb{N}}$ be a monotone sequence (for the stable order). Then $f : \mathcal{B}\underline{C} \to \underline{D}$ defined by $f(x) = \sup_{n \in \mathbb{N}} f_n(x)$ is bounded, totally monotone, Scott continuous and measurable. This map f is the lub of the f_n's in $\underline{B \Rightarrow_{\mathsf{s}} C}$.*

Proof sketch. Total monotonicity follows from Scott-continuity of addition, Scott-continuity is straightforward and measurability results from the monotone convergence theorem. The fact that f is the lub of the f_n's results from the fact that it is defined as a pointwise lub. □

So P is a cone that we equip with a measurability structure $\mathcal{M}$ defined as in $C \multimap D$: a $p \in \mathcal{M}_d$ is a function $p = \gamma \triangleright m$ where $\gamma \in \underline{\mathsf{Path}(d, C)}$ and $m \in \mathcal{M}^D_e$, given by

$$\gamma \triangleright m = \boldsymbol{\lambda}(r, f) \in \overline{d} \times P \cdot m(r, f(\gamma(r)))\,.$$

Then we check that $\mathcal{M}$ satisfies the required conditions exactly as we did for $C \multimap D$ in Section 6.1. We have defined a measurable cone $C \Rightarrow_{\mathsf{s}} D$ that we prove now to be integrable.

Let $d \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(d, C \Rightarrow_{\mathsf{s}} D)}$, and let $\mu \in \underline{\mathsf{Meas}(d)}$. We define a function $f : \mathcal{B}\underline{C} \to \underline{D}$ by

$$f(x) = \int \eta(r)(x)\mu(dr)\,.$$

This integral is well defined because, for any $x \in \mathcal{B}B$, the function $\boldsymbol{\lambda} r \in \overline{d} \cdot \eta(r)(x)$ is measurable and bounded since η is a measurable path. The function f is totally monotone by bilinearity of integration, Scott-continuous by the monotone convergence theorem, we check that it is measurable. Let $e \in \mathbf{ar}$ and $\gamma \in \underline{\mathsf{Path}(e, C)}$, we have

$$f \circ \gamma = \boldsymbol{\lambda} s \in \overline{e} \cdot \int \eta(r)(\gamma(s))\mu(dr)$$

and we must prove that $f \circ \gamma \in \underline{\mathsf{Path}(e, D)}$, so let $e' \in \mathbf{ar}$ and $m \in \mathcal{M}^D_{e'}$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', (f \circ \gamma)(s)) &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot m(s', \int \eta(r)(\gamma(s))\mu(dr)) \\ &= \boldsymbol{\lambda}(s', s) \in \overline{e' + e} \cdot \int m(s', \eta(r)(\gamma(s)))\mu(dr)\end{aligned}$$

which is measurable because $\boldsymbol{\lambda}(s', s, r) \in \overline{e' + e + d} \cdot m(s', \eta(r)(\gamma(s)))$ is measurable by our assumption about η and by Lemma 5.5. This shows that $f \in \underline{C \Rightarrow_{\mathsf{s}} D}$. Let $p \in \mathcal{M}^{C \Rightarrow D}_0$ so

that $p = x \triangleright m$ for some $x \in \underline{C}$ and $m \in \mathcal{M}_0^D$, we have

$$\begin{aligned} p(f) &= m(\int \eta(r)(x)\mu(dr)) \\ &= \int m(\eta(r)(x))\mu(dr) \\ &= \int p(\eta(r))\mu(dr) \end{aligned}$$

so that f is the integral of η over μ, this shows that $C \Rightarrow_{\mathsf{s}} D$ is an integrable cone.

8.3. **Finite differences.** We introduce a natural difference operator on totally monotone which provides an inductive characterization of total monotonicity, and that we will use to prove two basic properties of totally monotone functions, Lemmas 8.18 and 8.19, that we will use to prove a property which is not completely obvious: the composition of two stable functions is stable,

Given $\vec{u} \in \underline{B}^n$ such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$ we use $B_{\vec{u}}$ for the local cone $B_{\sum_{i=1}^n u_i}$.

Let B, C be cones, $f : \mathcal{B}\underline{B} \to \underline{C}$ be a function, $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \underline{B}$ such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$ we define

$$\begin{aligned} \Delta^\varepsilon f(\vec{u}) : \mathcal{B}\underline{B_{\vec{u}}} &\to \underline{C} \\ x &\mapsto \sum_{I \in \mathcal{P}^\varepsilon(n)} h(x + \sum_{i \in I} u_i) \end{aligned}$$

for $\varepsilon \in \{-, +\}$ and if f is totally monotone we set

$$\Delta f(\vec{u}) = \Delta^+ f(\vec{u}) - \Delta^- f(\vec{u}) : \mathcal{B}\underline{B_{\vec{u}}} \to \underline{C}\,,$$

the difference being computed pointwise. Notice that for $n = 0$ (so that $\vec{u} = (\,)$) we have $\Delta f((\,)) = f$ since $\mathcal{P}^+(0) = \{\emptyset\}$ and $\mathcal{P}^-(0) = \emptyset$.

Definition 8.11. Let $f \in \mathcal{B}\underline{B} \to \underline{C}$ be a function and let $n \in \mathbb{N}$. We say that f is *n-monotone from B to C* if

- $n = 0$ and f is monotone
- or $n > 0$, f is monotone and, for all $u \in \mathcal{B}\underline{B}$ the function $\Delta f(u) : \mathcal{B}\underline{B_u} \to \underline{C}$ is $(n-1)$-monotone from B_u to C.

Theorem 8.12. *A function $f \in \mathcal{B}\underline{B} \to \underline{C}$ is totally monotone iff it is n-monotone for all $n \in \mathbb{N}$.*

Proof. Boils down to the observation that, when this makes sense, we have

$$\Delta f(u, \vec{u}) = \Delta(\Delta f(\vec{u}))(u)$$

which is proven by induction on the length of $\vec{u}$. $\square$

Lemma 8.13. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotone and $\vec{u} \in \underline{B}^n$ be such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$. Then the functions*

$$\Delta^+ f(\vec{u}), \Delta^- f(\vec{u}), \Delta f(\vec{u}) : \mathcal{B}\underline{B_{\vec{u}}} \to \underline{C}$$

are totally monotone.

Proof. The total monotonicity of $\Delta^\varepsilon f(\vec{u})$ results from the easy observation that if $g : \mathcal{B}\underline{B} \to \mathcal{B}C$ and $u \in \mathcal{B}\underline{B}$ then the map $g_u : \mathcal{B}\underline{B_u} \to \underline{C}$ defined by $g_u(x) = g(x+u)$ is also totally monotone, and any sum of totally monotone functions is totally monotone.

The total monotonicity of $\Delta f(\vec{u})$ results from Theorem 8.12. $\square$

Lemma 8.14. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotone. Then for any $\vec{u} = (u_1, \dots, u_n) \in \underline{B}^n$ such that $\sum_{i=1}^n u_i \in \mathcal{B}B$ and $x \in \underline{B_{\vec{u}}}$ we have*

$$\Delta f(\vec{u})(x) \leq f(x + \sum_{i=1}^n u_i)\,.$$

Proof. By induction on n. The base case $n = 0$ is trivial since then $\Delta f(\vec{u})(x) = f(x)$. For the inductive case we have let $(u, \vec{u}) \in \underline{B}^{n+1}$ with $u + \sum_{i=1}^n u_i \in \mathcal{B}B$ and $x \in \underline{B_{u,\vec{u}}}$, that is $x + u \in \underline{B_{\vec{u}}}$. We have

$$\begin{aligned}\Delta f(u, \vec{u})(x) &= \Delta f(\vec{u})(x+u) - \Delta f(\vec{u})(x)\\ &\leq \Delta f(\vec{u})(x+u)\\ &\leq f(x + u + \sum_{i=1}^n u_i)\end{aligned}$$

by inductive hypothesis. $\square$

Lemma 8.15. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be a pre-stable function from B to C. Let $n \in \mathbb{N}$, $u, v \in \mathcal{B}\underline{B}$ and $\vec{u} \in \mathcal{B}\underline{B}^n$, and assume that $u + v + \sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$. Then for each $x \in \mathcal{B}\underline{B_{\vec{u}}}$ we have*

$$\begin{aligned}\Delta f(\vec{u})(x+u) &= \Delta f(\vec{u})(x) + \Delta f(u, \vec{u})(x)\\ \Delta f(u+v, \vec{u})(x) &= \Delta f(u, \vec{u})(x) + \Delta f(v, \vec{u})(x+u)\,.\end{aligned}$$

Proof. Simple computations. $\square$

Lemma 8.16. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotone. Let $n \in \mathbb{N}$, $u \in \underline{B}$ and $\vec{u}, \vec{v} \in \underline{B}^n$, and assume that $u + \sum_{i=1}^n (u_i + v_i) \in \mathcal{B}P$. Then for each $x \in \mathcal{B}\underline{B_{u,\vec{u},\vec{v}}}$ we have*

$$\begin{aligned}\Delta f(\vec{u} + \vec{v})(x+u) = {} & \Delta f(\vec{u})(x) + \Delta f(u, \vec{u} + \vec{v})(x) + \Delta f(v_1, u_2 + v_2, \dots, u_n + v_n)(x + u_1)\\ &+ \Delta f(u_1, v_2, u_3 + v_3, \dots, u_n + v_n)(x + u_2) + \cdots\\ &+ \Delta f(u_1, \dots, u_{n-1}, v_n)(x + u_n)\,.\end{aligned}$$

Proof. Simple computations using Lemma 8.15. $\square$

Let $\mathsf{S}^n B$ be the cone defined by $\underline{\mathsf{S}^n B} = \underline{B}^{n+1}$ with operations defined pointwise and norm defined by $\|(x, \vec{u})\|_{\mathsf{S}^n B} = \|x + \sum_{i=1}^n u_i\|_B$. It is easy to check that one actually defines a cone in that way.

Lemma 8.17. *If $f : \mathcal{B}\underline{B} \to \underline{C}$ is totally monotone, the map $(x, \vec{u}) \to \Delta f(\vec{u})(x)$ is monotone $\mathcal{B}\underline{\mathsf{S}^n B} \to \underline{C}$.*

Lemma 8.18. *Let $n \in \mathbb{N}$, $f, h_1, \dots, h_n : \mathcal{B}\underline{B} \to \underline{C}$ and $g : \mathcal{B}\underline{C} \to \underline{D}$ be totally monotone functions such that $\forall x \in \mathcal{B}\underline{B}$ $f(x) + \sum_{i=1}^n h_i(x) \in \mathcal{B}\underline{C}$. Then the function $k : \mathcal{B}\underline{B} \to \underline{D}$ defined by $k(x) = \Delta g(h_1(x), \dots, h_n(x))(f(x))$ is totally monotone.*

Proof. Observe that our hypotheses imply that, for all $x \in \mathcal{B}\underline{B}$, one has $f(x) + h_1(x) + \cdots + h_n(x) \in \mathcal{B}\underline{C}$.

With the notations and conventions of the statement, we prove by induction on p that, for all $p \in \mathbb{N}$, for all $n \in \mathbb{N}$, for all $f, h_1, \dots, h_n, g$ which are totally monotone and satisfy $\forall x \in \mathcal{B}\underline{B}\ f(x) + \sum_{i=1}^{n} h_i(x) \in \mathcal{B}\underline{C}$, the function k is p-monotone.

For $p = 0$, the property results from Lemma 8.17.

We assume the property for p and prove it for $p+1$. Let $u \in \mathcal{B}\underline{B}$ we have to prove that the function $\Delta k(u)$ is p-monotone from B_u to D. Let $x \in \mathcal{B}\underline{B_u}$, we have

$$\begin{aligned}
\Delta k(u)(x) &= \Delta g(h_1(x+u), \dots, h_n(x+u))(f(x+u)) - \Delta g(h_1(x), \dots, h_n(x))(f(x)) \\
&= \Delta g(h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x) + \Delta f(u)(x)) \\
&\quad - \Delta g(h_1(x), \dots, h_n(x))(f(x)) \quad \text{by definition of } \Delta h_i(u) \\
&= \Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x)) \\
&\quad + \Delta g(\Delta h_1(u)(x), h_2(x) + \Delta h_2(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x) + h_1(x)) \\
&\quad + \Delta g(h_1(x), \Delta h_2(u)(x), h_3(x) + \Delta h_3(u)(x), \dots, \\
&\qquad\qquad\qquad\qquad h_n(x) + \Delta h_n(u)(x))(f(x) + h_2(x)) \\
&\quad + \cdots + \Delta g(h_1(x), \dots, h_{n-1}(x), \Delta h_n(u)(x))(f(x) + h_n(x))
\end{aligned}$$

by Lemma 8.16, observing that the first term of the sum which appears in that lemma is annihilated by the subtraction above.

We can apply the inductive hypothesis to each of the terms of this sum. Let us consider for instance the first of these expressions:

$$\Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x))$$

We know that the functions $h'_1, \dots, h'_{n+1}$ defined by $h'_1(x) = \Delta f(u)(x)$, $h'_2(x) = h_1(x) + \Delta h_1(u)(x) = h_1(x+u)$,…, $h'_{n+1}(x) = h_n(x) + \Delta h_n(u)(x) = h_n(x+u)$ are totally monotone from $\underline{B_u}$ to $\underline{C}$: this results from Lemma 8.13. Moreover we have

$$\forall x \in \mathcal{B}\underline{B}\ f(x) + \sum_{i=1}^{n+1} h'_i(x) = f(x+u) + \sum_{i=1}^{n} h_i(x+u) \in \mathcal{B}\underline{C}\,.$$

Therefore the inductive hypothesis applies and we know that the function

$$x \mapsto \Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_p(x) + \Delta h_p(u)(x))(f(x))$$

is p-monotone. The same reasoning applies to all terms and hence the function $\Delta k(u)$ is p-monotone from $\mathcal{B}\underline{B_u}$ to $\underline{C}$, as contended. □

Lemma 8.19. *Let $f : \underline{B} \times \mathcal{B}\underline{C} \to \underline{D}$ be linear in its first argument and totally monotone in its second argument. Then, when restricted to $\mathcal{B}\underline{B} \times \mathcal{B}\underline{C}$, the function f is totally monotone.*

Proof. Let $n \in \mathbb{N}$, $(x, y), (u_1, v_1), \dots, (u_n, v_n) \in \underline{B} \times \underline{C}$ be such that $(x, y) + \sum_{i=1}^{n} (u_i, v_i) \in \mathcal{B}\underline{B} \times \mathcal{B}\underline{C}$. For $\varepsilon \in \{+, -\}$, we have

$$\begin{aligned}
\Delta^\varepsilon f((u_1, v_1), \dots, (u_n, v_n))(x, y) &= \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x + \sum_{i \in I} u_i, y + \sum_{i \in I} v_i) \\
&= \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x, y + \sum_{i \in I} v_i) + \sum_{I \in \mathcal{P}^\varepsilon(n)} \sum_{j \in I} f(u_j, y + \sum_{i \in I} v_i)
\end{aligned}$$

by linearity of f in its first argument. By total monotonicity of f in its second argument we have

$$\sum_{I\in\mathcal{P}^+(n)} f(x, y+\sum_{i\in I} v_i) \geq \sum_{I\in\mathcal{P}^-(n)} f(x, y+\sum_{i\in I} v_i)\,.$$

Next, assuming that $n>0$, we have

$$\begin{aligned}
\sum_{I\in\mathcal{P}^\varepsilon(n)}\sum_{j\in I} f(u_j, y+\sum_{i\in I} v_i) &= \sum_{j=1}^{n} \sum_{\substack{I\in\mathcal{P}^\varepsilon(n)\\ j\in I}} f(u_j, y+\sum_{i\in I} v_i)\\
&= \sum_{j=1}^{n} \sum_{I\in\mathcal{P}^\varepsilon(n-1)} f(u_j, y+\sum_{i\in \mathsf{inj}_j(I)} v_i) \text{ by Lemma 8.4}\\
&= \sum_{j=1}^{n} \sum_{I\in\mathcal{P}^\varepsilon(n-1)} f(u_j, y+v_j+\sum_{i\in I} v(j)_i)
\end{aligned}$$

where $(v(j)_i)_{i=1}^{n-1}$ is defined by

$$v(j)_i = \begin{cases} v_i & \text{if } i<j\\ v_{i+1} & \text{if } i\geq j\,. \end{cases}$$

By our assumption that f is totally monotone in its second argument we have, for each $j=1,\dots,n$,

$$\sum_{I\in\mathcal{P}^+(n-1)} f(u_j, y+v_j+\sum_{i\in I} v(j)_i) \geq \sum_{I\in\mathcal{P}^-(n-1)} f(u_j, y+v_j+\sum_{i\in I} v(j)_i)$$

from which it follows that

$$\sum_{I\in\mathcal{P}^+(n)}\sum_{j\in I} f(u_j, y+\sum_{i\in I} v_i) \geq \sum_{I\in\mathcal{P}^-(n)}\sum_{j\in I} f(u_j, y+\sum_{i\in I} v_i)$$

and hence

$$\Delta^+ f((u_1,v_1),\dots,(u_n,v_n))(x,y) \geq \Delta^- f((u_1,v_1),\dots,(u_n,v_n))(x,y)$$

for $n>0$. This inequation also holds trivially for $n=0$. □

8.4. **The category of integrable cones and stable functions.** Let $\mathbf{SCones}(B,C)$ be the set of all stable functions from B to C whose norm is ≤ 1.

Theorem 8.20. *If $f\in\mathbf{SCones}(B,C)$ and $g\in\mathbf{SCones}(C,D)$ then $g\circ f\in\mathbf{SCones}(B,D)$.*

Proof. The only non-obvious fact is that $g\circ f$ is totally monotone, which is obtained by Lemma 8.18 (applied with $n=0$). □

So we have defined a category **SCones** whose objects are the integrable cones, and the morphisms are the measurable and stable functions.

Lemma 8.21. $\mathbf{ICones}(B,C)\subseteq\mathbf{SCones}(B,C)$.

Proof. Indeed linearity clearly implies total monotonicity. □

So we have a functor $\mathsf{Der} : \mathbf{ICones} \to \mathbf{SCones}$ which acts as the identity on objects and morphisms. We can consider this functor as a forgetful functor since it forgets linearity. It is obviously faithful but of course not full.

Theorem 8.22. *The category* $\mathbf{SCones}$ *has all products and is cartesian closed.*

Notice however that $\mathbf{SCones}$ is not complete.

Proof. If $(B_i)_{i\in I}$ is a family of integrable cones, we have already defined $B = \&_{i\in I} B_i$ which is the cartesian product of the B_i's in $\mathbf{ICones}$ (when equipped with the projections $\mathsf{pr}_i \in \mathbf{ICones}(B, B_i)$). So $\mathsf{Der}(\mathsf{pr}_i) \in \mathbf{SCones}(B, B_i)$ for each $i \in I$. Let $(f_i \in \mathbf{SCones}(C, B_i))_{i\in I}$, we define $f : \mathcal{B}\underline{C} \to \mathcal{B}\underline{B}$ by $f(x) = (f_i(x))_{i\in I}$ which is well defined by our assumption that $\forall i \in I\ \ \|f_i\| \leq 1$. Then f is easily seen to be stable because all the operations of B, as well as its canonical order relations, are defined componentwise. Measurability of f is proven as in the proof of Theorem 5.15. This shows that B is the cartesian product of the B_i's in $\mathbf{SCones}$.

Let B and C be integrable cones. We have defined in Section 8.2 the integrable cone $B \Rightarrow_{\mathsf{s}} C$ of stable functions $B \to C$, we show that it is the internal hom of B and C in $\mathbf{SCones}$. We define $\mathsf{Ev} : \mathcal{B}\underline{((B \Rightarrow_{\mathsf{s}} C) \ \&\ B)} \to \underline{C}$ by $\mathsf{Ev}(f, x) = f(x)$. The total monotonicity of Ev results from Lemma 8.19. We have

$$\mathcal{B}\underline{((B \Rightarrow_{\mathsf{s}} C) \ \&\ B)} = \mathcal{B}\underline{(B \Rightarrow_{\mathsf{s}} C)} \times \mathcal{B}\underline{B}$$

by definition of the norm in the cartesian product. It follows that $\|\mathsf{Ev}\| \leq 1$. We prove that Ev is measurable so let $d \in \mathbf{ar}$ and $\delta \in \mathcal{B}\underline{\mathsf{Path}(d, ((B \Rightarrow_{\mathsf{s}} C) \ \&\ B))}$ which means that $\delta = \langle \eta, \beta\rangle$ with $\eta \in \mathcal{B}\underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{s}} C)}$ and $\beta \in \mathcal{B}\underline{\mathsf{Path}(d, B)}$, we must prove that $\mathsf{Ev} \circ \delta \in \underline{\mathsf{Path}(d, C)}$ so let $e \in \mathbf{ar}$ and $m \in \mathcal{M}^C_e$. We must prove that

$$\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \mathsf{Ev}(\delta(r))) = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \eta(r)(\beta(r)))$$

is measurable. We have $p = (\beta \circ \mathsf{pr}_2) \triangleright (m \circ \mathsf{pr}_1) \in \mathcal{M}^{B \Rightarrow_{\mathsf{s}} C}_{e+d}$ and by our assumption about η we know that

$$\boldsymbol{\lambda}(s, r, r') \in \overline{d + e + d} \cdot p(s, r, \eta(r')) = \boldsymbol{\lambda}(s, r, r') \in \overline{d + e + d} \cdot m(s, \eta(r')(\beta(r)))$$

is measurable and hence so is φ and we have shown that $\mathsf{Ev} \in \mathbf{SCones}((B \Rightarrow_{\mathsf{s}} C) \ \&\ B, C)$, we prove that $(B \Rightarrow_{\mathsf{s}} C, \mathsf{Ev})$ is the internal hom of B, C in the cartesian category $\mathbf{SCones}$.

So let $f \in \mathbf{SCones}(D \ \&\ B, C)$. For each given $z \in \mathcal{B}\underline{D}$ we see easily that $g = \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(z, x) \in \mathcal{B}\underline{B \Rightarrow_{\mathsf{s}} C}$, it remains to check that $g \in \mathbf{SCones}(D, B \Rightarrow_{\mathsf{s}} C)$. Let us first check that g is totally monotone so let $n \in \mathbb{N}$ and $z, w_1, \dots, w_n \in \underline{D}$ be such that $z + \sum_{i=1}^n w_i \in \mathcal{B}\underline{D}$. We must prove that

$$h^- = \sum_{I \in \mathcal{P}^-(n)} g(z + \sum_{i\in I} w_i) \leq \sum_{I \in \mathcal{P}^+(n)} g(z + \sum_{i\in I} w_i) = h^+$$

in $\underline{B \Rightarrow_{\mathsf{s}} C}$. We use the characterization of the algebraic order in that cone given by Lemma 8.8. So let $k \in \mathbb{N}$ and let $x, u_1, \dots, u_k \in \underline{B}$ be such that $x + \sum_{j=1}^k u_j \in \mathcal{B}B$. We

must prove that

$$y^- = \sum_{J\in\mathcal{P}^-(k)} h^+(x+\sum_{i\in J} u_j) + \sum_{J\in\mathcal{P}^+(k)} h^-(x+\sum_{i\in J} u_j)$$
$$\leq \sum_{J\in\mathcal{P}^+(k)} h^+(x+\sum_{i\in J} u_j) + \sum_{J\in\mathcal{P}^-(k)} h^-(x+\sum_{i\in J} u_j) = y^+$$

in $\underline{C}$. We have

$$y^- = \sum_{\substack{I\in\mathcal{P}^+(n)\\ J\in\mathcal{P}^-(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j) + \sum_{\substack{I\in\mathcal{P}^-(n)\\ J\in\mathcal{P}^+(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j)$$
$$y^+ = \sum_{\substack{I\in\mathcal{P}^+(n)\\ J\in\mathcal{P}^+(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j) + \sum_{\substack{I\in\mathcal{P}^-(n)\\ J\in\mathcal{P}^-(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j)$$

Notice that $(\mathcal{P}^+(n)\times\mathcal{P}^-(k))\cap(\mathcal{P}^-(n)\times\mathcal{P}^+(k))=\emptyset$ and that there is a bijection

$$(\mathcal{P}^+(n)\times\mathcal{P}^-(k))\cup(\mathcal{P}^-(n)\times\mathcal{P}^+(k))\to\mathcal{P}^-(n+k)$$
$$(I,J)\mapsto I\cup(J+n)$$

and similarly that $(\mathcal{P}^+(n)\times\mathcal{P}^+(k))\cap(\mathcal{P}^-(n)\times\mathcal{P}^-(k))=\emptyset$ and that there is a bijection

$$(\mathcal{P}^+(n)\times\mathcal{P}^+(k))\cup(\mathcal{P}^-(n)\times\mathcal{P}^-(k))\to\mathcal{P}^+(n+k)$$
$$(I,J)\mapsto I\cup(J+n)\,.$$

We define a sequence $(w'_l,u'_l)_{l=1}^{n+k}$ of elements of $\underline{D}\times\underline{B}$ as follows:

$$(w'_l,u'_l)=\begin{cases}(w_l,0) & \text{if } l\in\{1,\dots,n\}\\ (0,u_{l-n}) & \text{if } l\in\{n+1,\dots,n+k\}\,.\end{cases}$$

so that $(z,x)+\sum_{l=1}^{n+k}(w'_l,u'_l)\in\mathcal{B}\underline{D}\times\mathcal{B}\underline{B}$. With these notations, we have

$$y^- = \sum_{K\in\mathcal{P}^-(n+k)} g((z,x)+\sum_{l\in K}(w'_l,u'_l))$$
$$y^+ = \sum_{K\in\mathcal{P}^+(n+k)} g((z,x)+\sum_{l\in K}(w'_l,u'_l))$$

and hence $y^-\leq y^+$ since g is totally monotone.

Scott-continuity of g results from Lemma 8.10. We prove that g is measurable so let $d\in\mathbf{ar}$ and $\delta\in\mathcal{B}\underline{\mathsf{Path}(d,D)}$, we must prove that $g\circ\delta\in\underline{\mathsf{Path}(d,B\Rightarrow_{\mathsf{s}}C)}$ so let $e\in\mathbf{ar}$ and $p\in\mathcal{M}_e^{B\Rightarrow_{\mathsf{s}}C}$. Let $\beta\in\mathcal{B}\underline{\mathsf{Path}(e,B)}$ and $m\in\mathcal{M}_e^C$ be such that $p=\beta\triangleright m$, we have

$$\boldsymbol{\lambda}(s,r)\in\overline{e+d}\cdot p(s,g(\delta(r))) = \boldsymbol{\lambda}(s,r)\in\overline{e+d}\cdot m(s,g(\delta(r))(\beta(s)))$$
$$= \boldsymbol{\lambda}(s,r)\in\overline{e+d}\cdot m(s,f(\delta(r),\beta(s)))$$

and this map is measurable because $\langle\delta\circ\mathsf{pr}_2,\beta\circ\mathsf{pr}_1\rangle\in\mathcal{B}\underline{\mathsf{Path}(e+d,D\mathbin{\&}B)}$ and by measurability of f. $\square$

So we have a functor $\_\Rightarrow_{\mathsf{s}}\_:\mathbf{SCones}^{\mathsf{op}}\times\mathbf{SCones}\to\mathbf{SCones}$ mapping (B,C) to $B\Rightarrow_{\mathsf{s}}C$ and $f\in\mathbf{SCones}(B',B), g\in\mathbf{SCones}(C,C'))$ to $f\Rightarrow_{\mathsf{s}}g\in\mathbf{SCones}(B\Rightarrow_{\mathsf{s}}C,B'\Rightarrow_{\mathsf{s}}C')$ which is given by $(f\Rightarrow_{\mathsf{s}}g)(h)=g\circ h\circ f$. Observe that if $g\in\mathbf{ICones}(C,C')$ we have

$f \Rightarrow_{\mathsf{s}} g \in \mathbf{ICones}(B \Rightarrow_{\mathsf{s}} C, B' \Rightarrow_{\mathsf{s}} C')$ so that in the sequel we consider only $\_ \Rightarrow_{\mathsf{s}} \_$ as a functor $\mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ (using implicitly a precomposition with Der).

8.5. **The linear/non-linear adjunction, in the stable case.**

Theorem 8.23. *The functor* $\mathsf{Der} : \mathbf{ICones} \to \mathbf{SCones}$ *has a left adjoint.*

Proof. By Theorem 5.17 it suffices to prove that Der preserves all products and all equalizers. Since products are defined in the same way in both categories, the first property is obvious, let us check the second one.

Let B, C be objects of $\mathbf{ICones}$ and $f, g \in \mathbf{ICones}(B, C)$, we have already defined the equalizer $(E, e \in \mathbf{ICones}(E, B))$ of f, g in the proof of Theorem 5.15. We just have to check that (E, e) is the equalizer of f, g in $\mathbf{SCones}$ as well. So let H be an integrable cone and $h \in \mathbf{SCones}(H, B)$ be such that $f \circ h = g \circ h$. This simply means that $h(\mathcal{B}\underline{H}) \subseteq \mathcal{B}\underline{E}$ from which it follows that $h \in \mathbf{SCones}(H, E)$ because the canonical order relation of E is the restriction of that of B to $\underline{E} \subseteq \underline{B}$ (and similarly for the measurability structure). Let us call h' this version of h ranging in $\mathcal{B}\underline{E}$ instead of $\mathcal{B}\underline{B}$, so that $h = e \circ h'$. It is obvious that h' is the only morphism in $\mathbf{SCones}$ having this property. $\square$

Let $\mathsf{E} : \mathbf{SCones} \to \mathbf{ICones}$ be the left adjoint of Der, and let us introduce the notation $\Theta^{\mathsf{s}}_{B,C} : \mathbf{ICones}(\mathsf{E}\, B, C) \to \mathbf{SCones}(B, \mathsf{Der}\, C) = \mathbf{SCones}(B, C)$ for the associated natural bijection (remember that $\mathsf{Der}\, C = C$).

We use $(!^{\mathsf{s}}, \mathsf{der}^{\mathsf{s}}, \mathsf{dig}^{\mathsf{s}})$ for the induced comonad on $\mathbf{ICones}$ whose Kleisli category is (equivalent to) $\mathbf{SCones}$ since

$$\begin{aligned}\mathbf{ICones}_{!^{\mathsf{s}}}(B, C) &= \mathbf{ICones}(!^{\mathsf{s}} B, C) \\ &= \mathbf{ICones}(\mathsf{E}\ \mathsf{Der}\, B, C) \\ &\simeq \mathbf{SCones}(\mathsf{Der}\, B, \mathsf{Der}\, C) \\ &= \mathbf{SCones}(B, C)\,.\end{aligned}$$

Notice that actually $!^{\mathsf{s}} B = \mathsf{E}\, B$. Let $\mathsf{st}_B = \Theta^{\mathsf{s}}_{B, \Theta^{\mathsf{s}} B}(\mathsf{Id}_{\mathsf{E}\, B}) \in \mathbf{SCones}(B, !^{\mathsf{s}} B)$ be the unit of the adjunction, which is the "universal stable map" on B in the sense that for any integrable cone C and any $f \in \mathbf{SCones}(B, C)$ one has $f = \varphi \circ \mathsf{st}_B$ for a unique $\varphi \in \mathbf{ICones}(!^{\mathsf{s}} B, C)$, namely $\varphi = (\Theta^{\mathsf{s}}_{B,C})^{-1}(f)$ (dropping the Der symbol since this functor acts as the identity on objects and morphisms considered as functions). So that for $h \in \mathbf{ICones}(!^{\mathsf{s}} B, C)$, one has $\Theta^{\mathsf{s}}_{B,C}(h) = h \circ \mathsf{st}_B$. For any $x \in \mathcal{B}\underline{B}$ we set $x^{!_{\mathsf{s}}} = \mathsf{st}_B(x) \in \mathcal{B}\underline{\mathsf{E}\, B}$ so that, for $f \in \mathbf{SCones}(B, C)$ we have $f(x) = (\Theta^{\mathsf{s}}_{B,C})^{-1}(f)(x^{!_{\mathsf{s}}})$.

The counit $\mathsf{der}^{\mathsf{s}}_B \in \mathbf{ICones}(!^{\mathsf{s}} B, B)$ of the comonad $!^{\mathsf{s}}\_$ is also the counit of the adjunction. It satisfies therefore

$$\forall x \in \mathcal{B}\underline{B} \quad \mathsf{der}^{\mathsf{s}}_B(x^{!_{\mathsf{s}}}) = x\,.$$

The comultiplication $\mathsf{dig}^{\mathsf{s}}_B \in \mathbf{ICones}(!^{\mathsf{s}} B, !^{\mathsf{s}}!^{\mathsf{s}} B) = \mathbf{ICones}(\mathsf{E}\,\mathsf{Der}\, B, \mathsf{E}\,\mathsf{Der}\,\mathsf{E}\,\mathsf{Der}\, B)$ is defined by $\mathsf{dig}^{\mathsf{s}}_B = \mathsf{E}\,\mathsf{st}_B$ so that we have

$$\forall x \in \mathcal{B}B \quad \mathsf{dig}^{\mathsf{s}}_B(x^{!_{\mathsf{s}}}) = x^{!_{\mathsf{s}}!_{\mathsf{s}}}\,.$$

since by naturality of st_B we have $\mathsf{Der}\,\mathsf{E}\,\mathsf{st}_B \circ \mathsf{st}_B = \mathsf{st}_{\mathsf{Der}\,\mathsf{E}\, B} \circ \mathsf{st}_B$ in $\mathbf{SCones}$.

Lemma 8.24. *Let* $f \in \mathbf{ICones}(B, C)$ *and* $x \in \mathcal{B}\underline{B}$. *We have* $(!^{\mathsf{s}} f)(x^{!_{\mathsf{s}}}) = f(x)^{!_{\mathsf{s}}}$.

Proof. We have $(!^{\mathsf{s}} f)(x^{!_{\mathsf{s}}}) = (\mathsf{E}\,\mathsf{Der}\, f)(x^{!_{\mathsf{s}}}) = ((\mathsf{Der}\,\mathsf{E}\,\mathsf{Der}\, f) \circ \mathsf{st}_{\mathsf{Der}\, B})(x)$ where the composition is taken in **SCones**. By naturality we get $(!^{\mathsf{s}} f)(x^{!_{\mathsf{s}}}) = (\mathsf{st}_{\mathsf{Der}\, C} \circ \mathsf{Der}\, f)(x) = f(x)^{!_{\mathsf{s}}}$. $\square$

Consider the two functors $L, R : \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ defined on objects by $L(B, C, D) = (B \Rightarrow_{\mathsf{s}} (C \multimap D))$ and $R(B, C, D) = (C \multimap (B \Rightarrow_{\mathsf{s}} D))$.

Lemma 8.25. *Let B, C, D be integrable cones. There is an isomorphism in* **ICones** *from $L(B, C, D) = (B \Rightarrow_{\mathsf{s}} (C \multimap D))$ to $R(B, C, D) = (C \multimap (B \Rightarrow_{\mathsf{s}} D))$ which is natural in B, C and D.*

Proof. This is straightforward, the natural isomorphism maps $f \in \underline{B \Rightarrow_{\mathsf{s}} (C \multimap D)}$ to $\boldsymbol{\lambda} y \in \underline{C} \cdot \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(x, y)$. $\square$

Then we have

$$\begin{aligned}
\mathbf{ICones}(!^{\mathsf{s}}(B_1 \,\&\, B_2), C) &\simeq \mathbf{SCones}(B_1 \,\&\, B_2, C) \\
&\simeq \mathbf{SCones}(B_1, B_2 \Rightarrow_{\mathsf{s}} C) \\
&\simeq \mathbf{ICones}(!^{\mathsf{s}} B_1, B_2 \Rightarrow_{\mathsf{s}} C) \\
&\simeq \mathbf{ICones}(1, !^{\mathsf{s}} B_1 \multimap (B_2 \Rightarrow_{\mathsf{s}} C)) \\
&\simeq \mathbf{ICones}(1, B_2 \Rightarrow_{\mathsf{s}} (!^{\mathsf{s}} B_1 \multimap C)) \text{ by Lemma 8.25} \\
&\simeq \mathbf{SCones}(\top, B_2 \Rightarrow_{\mathsf{s}} (!^{\mathsf{s}} B_1 \multimap C)) \\
&\simeq \mathbf{SCones}(B_2, (!^{\mathsf{s}} B_1 \multimap C)) \\
&\simeq \mathbf{ICones}(!^{\mathsf{s}} B_2, (!^{\mathsf{s}} B_1 \multimap C)) \\
&\simeq \mathbf{ICones}(!^{\mathsf{s}} B_2 \otimes !^{\mathsf{s}} B_1, C) \\
&\simeq \mathbf{ICones}(!^{\mathsf{s}} B_1 \otimes !^{\mathsf{s}} B_2, C)
\end{aligned}$$

by a sequence of natural bijections and hence by Lemma 1.1 we have a natural isomorphism $\mathsf{m}^2_{B_1, B_2}$ in

$$\mathbf{ICones}(!^{\mathsf{s}} B_1 \otimes !^{\mathsf{s}} B_2, !^{\mathsf{s}}(B_1 \,\&\, B_2)) .$$

Similarly we define an iso $\mathsf{m}^0 \in \mathbf{ICones}(1, !^{\mathsf{s}}\top)$. Then one can prove using Lemma 1.1 again that $!^{\mathsf{s}}$ is a strong monoidal comonad.

Theorem 8.26. *Equipped with the strong monoidal monad $!^{\mathsf{s}}$, the category* **ICones** *is a Seely category in the sense of* [Mel09].

Remark 8.27. Assume that in $\overline{d}$ all singletons are measurable and consider the map $\mathsf{disc}_d : \underline{\mathsf{Meas}(d)} \to \underline{\mathsf{Meas}(d)}$ defined by

$$\mathsf{disc}_d(\mu)(U) = \sum_{r \in U} \mu(\{r\}) \leq \mu(\overline{d})$$

where $U \in \sigma_{\overline{d}}$, observing that the sum above can have only a countable number of terms $\neq 0$. Intuitively, this map extracts the discrete component of a measure μ, so that the measure $\mu - \mathsf{disc}_d(\mu)$ has no atoms.

It is easy to see that disc_d is linear, Scott-continuous and measurable. Assume that $\overline{d} = [0, 1]$ with the usual Lebesgue σ-algebra and let λ be the Lebesgue measure. We have $\mathsf{disc}_d(\lambda) = 0$. On the other hand, remember that $\boldsymbol{\delta}^d \in \underline{\mathsf{Path}(d, \mathsf{Meas}(d))}$ and let

$$\mu = \int \mathsf{disc}_d(\boldsymbol{\delta}^d(r)) \lambda(dr) \in \underline{\mathsf{Meas}(d)} .$$

If disc_d were integrable we would have $\mu = \mathsf{disc}_d(\int \boldsymbol{\delta}^d(r)\lambda(dr)) = \mathsf{disc}_d(\lambda) = 0$. But given a measurable subset U of $[0,1]$ we have

$$\mu(U) = \int \mathsf{disc}_d(\boldsymbol{\delta}^d(r))(U)\lambda(dr) = \int \chi_U(r)\lambda(dr) = \lambda(U)$$

which shows that the function disc_d is not a morphism in **ICones**. Nevertheless we have $\mathsf{disc}_d \in \mathbf{SCones}(\mathsf{Meas}(d), \mathsf{Meas}(d))$ and so there is $f \in \mathbf{ICones}(!^{\mathsf{s}}\mathsf{Meas}(d), \mathsf{Meas}(d))$ such that $\mathsf{disc}_d(\mu) = f(\mu^{!_{\mathsf{s}}})$. It would be interesting to understand how this function f works to get some insight on the internal structure of the stable exponential, which is defined in a rather implicit way (by the special adjoint functor theorem).

8.6. **The coalgebra structure of $\mathsf{Meas}(d)$.** Let $d \in \mathbf{ar}$. In Section 7 we defined the Dirac path $\boldsymbol{\delta}^d \in \mathcal{B}\underline{\mathsf{Path}(d, \mathsf{Meas}(d))}$ which maps $r \in \overline{d}$ to $\boldsymbol{\delta}^d(r)$, the Dirac measure at r. In Section 8.5 we have introduced the universal stable map $\mathsf{st}_{\mathsf{Meas}(d)} \in \mathbf{SCones}(\mathsf{Meas}(d), !^{\mathsf{s}}\mathsf{Meas}(d))$. Since morphisms in **SCones** are measurable we have

$$\mathsf{st}_{\mathsf{Meas}(d)} \circ \boldsymbol{\delta}^d \in \mathcal{B}\underline{\mathsf{Path}(d, !^{\mathsf{s}}\mathsf{Meas}(d))}$$

and we define

$$\mathsf{h}_d = \mathcal{I}_d^{!^{\mathsf{s}}\mathsf{Meas}(d)}(\mathsf{st}_{\mathsf{Meas}(d)} \circ \boldsymbol{\delta}^d) \in \mathbf{ICones}(\mathsf{Meas}(d), !^{\mathsf{s}}\mathsf{Meas}(d))$$

using Theorem 7.1. In other words h_d is defined by

$$\mathsf{h}_d(\mu) = \int \boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}}\mu(dr)$$

and satisfies $\mathsf{h}_d(\boldsymbol{\delta}^d(r)) = \boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}}$.

Theorem 8.28. *Equipped with h_d, the object $\mathsf{Meas}(d)$ of **ICones** is a coalgebra of the comonad $!^{\mathsf{s}}\_$.*

Proof. We must first prove that $\mathsf{der}^{\mathsf{s}}_{\mathsf{Meas}(d)}\,\mathsf{h}_d = \mathsf{Id}_{\mathsf{Meas}(d)} \in \mathbf{ICones}(\mathsf{Meas}(d), \mathsf{Meas}(d))$. By Theorem 7.2 this results from the fact that for all $r \in \overline{d}$ one has $(\mathsf{der}^{\mathsf{s}}_{\mathsf{Meas}(d)}\,\mathsf{h}_d)(\boldsymbol{\delta}^d(r)) = \mathsf{der}^{\mathsf{s}}_{\mathsf{Meas}(d)}(\boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}}) = \boldsymbol{\delta}^d(r)$.

Next we must prove that $f_1 = f_2 \in \mathbf{ICones}(\mathsf{Meas}(d), !^{\mathsf{s}}!^{\mathsf{s}}(\mathsf{Meas}(d)))$ where

$$f_1 = \mathsf{dig}^{\mathsf{s}}_{\mathsf{Meas}(d)}\,\mathsf{h}_d \quad \text{and} \quad f_2 = !^{\mathsf{s}}\mathsf{h}_d\,\mathsf{h}_d\,.$$

Let $r \in \overline{d}$, we have $f_1(\boldsymbol{\delta}^d(r)) = \mathsf{dig}^{\mathsf{s}}_{!^{\mathsf{s}}\mathsf{Meas}(d)}(\boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}}) = \boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}!_{\mathsf{s}}}$ and $f_2(\boldsymbol{\delta}^d(r)) = !^{\mathsf{s}}\mathsf{h}_d(\boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}}) = (\mathsf{h}_d(\boldsymbol{\delta}^d(r)))^{!_{\mathsf{s}}}$ by Lemma 8.24. And hence $f_2(\boldsymbol{\delta}^d(r)) = \boldsymbol{\delta}^d(r)^{!_{\mathsf{s}}!_{\mathsf{s}}}$ so that $f_1 = f_2$ by Theorem 7.2. □

Remark 8.29. One of the main goals of introducing integrable cones was precisely to get this additional structure for each cone $\mathsf{Meas}(d)$. It means more specifically that for any $f \in \mathbf{SCones}(\mathsf{Meas}(d), B) = \mathbf{ICones}(!^{\mathsf{s}}\mathsf{Meas}(d), B)$ we can define $g = f\,\mathsf{h}_d \in \mathbf{ICones}(\mathsf{Meas}(d), B)$ such that

$$\forall \mu \in \underline{\mathsf{Meas}(d)} \quad g(\mu) = \int f(\boldsymbol{\delta}^d(r))\mu(dr)$$

which is a "linearization" of f allowing to interpret the sampling operation of probabilistic programming languages: one samples a real number r according to the distribution μ and feeds the program f with the value r represented as the Dirac measure $\boldsymbol{\delta}^d(r)$.

From the viewpoint of Linear Logic this means that each $d \in \mathbf{ar}$ can be seen as a positive type, that is, a type equipped with structural rules allowing to erase and duplicate its values, see for instance [Gir91, LR03, ET19]. Another way to understand h_d is to see it as a *storage operator* in the sense of [Kri94], that is, $\mathsf{Meas}(d)$ is a *data-type*.

9. Analytic and integrable functions on cones

Our goal now is to define another exponential on **ICones** based on a notion of morphisms which are analytic in the sense that they are limits of polynomial functions. The main difference wrt. stable functions is that the definition of analytic functions is based on a notion of multilinear maps (as in [KT18] but without the support of complex analysis) in **ICones** which preserve integrals so that analytic functions have an implicit "integral preservation" property[14] whereas stable functions don't.

9.1. **The cone of multilinear and symmetric functions.** Let B, C be integrable cones and $n \in \mathbb{N}$. A function $f : \underline{B}^n \to \underline{C}$ is multilinear and continuous if it is linear and continuous, separately, with respect to each of its n arguments. When this holds, f is bounded (use for instance Lemma 2.4 and monoidal closedness in a proof by induction on n). One says that f is symmetric if, for all $\sigma \in \mathfrak{S}_n$ (the group of permutations on $\{1, \dots, n\}$), one has

$$\forall x_1, \dots, x_n \in \underline{B} \quad f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})\,.$$

One says that f is measurable if for all $d \in \mathbf{ar}$ and $(\beta_i \in \underline{\mathsf{Path}(d, B)})_{i=1}^n$, one has $f \circ \langle \beta_1, \dots, \beta_n \rangle \in \underline{\mathsf{Path}(d, C)}$. Last f preserves integrals if it does so separately with respect to each of its arguments.

The multilinear, continuous, measurable and integrable functions $\underline{B}^n \to \underline{C}$ are easily seen to form a cone $\mathbf{Sym}_n(B, C)$ with operations defined pointwise and norm defined by

$$\|f\| = \sup\{\|f(x_1, \dots, x_n)\| \mid x_1, \dots, x_n \in \mathcal{B}\underline{B}\}\,.$$

We equip this cone with a measurability structure $(\mathcal{M}_d^{\mathbf{Sym}_n(B,C)})_{d \in \mathbf{ar}}$ where $\mathcal{M}_d^{\mathbf{Sym}_n(B,C)}$ is the set of all $p = \vec{\beta} \triangleright m$ where $\vec{\beta} = (\beta_i \in \underline{\mathsf{Path}(d, B)})_{i=1}^n$ and $m \in \mathcal{M}_d^C$, given by $p(f) = \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, f(\beta_1(r), \dots, \beta_n(r)))$. The order relation in this cone is the pointwise order because multilinear symmetric functions are closed under subtraction.

Notice last that this cone is integrable. Let indeed $d \in \mathbf{ar}$, $\mu \in \underline{\mathsf{Meas}(d)}$ and $\eta \in \underline{\mathsf{Path}(d, \mathbf{Sym}_n(B, C))}$ and let us define a function $f : \underline{B}^n \to \underline{C}$ by

$$f(x_1, \dots, x_n) = \int \eta(r)(x_1, \dots, x_n)\mu(dr)$$

then it is easy to check as usual that f is well defined, $f \in \underline{\mathbf{Sym}_n(B, C)}$ and that $f = \int \eta(r)\mu(dr)$.

Remark 9.1. This integrable cone is a subcone of the integrable cone $B \otimes \cdots \otimes B \multimap C$, but we have not developed the notion of subcone in the present paper.

[14] A property that we don't really know yet how to express simply and directly in terms of the functions; of course it is not plain integral preservation which cannot be expected from non-linear maps. We know that it is a property of the analytic functions themselves because the symmetric multilinear functions of their Taylor expansion at 0 are associated with analytic functions in a unambiguous way by means of standard polarization formulas as we shall see.

9.2. **The cone of homogeneous polynomial functions.**

Definition 9.2. An n-homogeneous polynomial function from B to C is a function $f : \underline{B} \to \underline{C}$ such that there exists $h \in \underline{\mathbf{Sym}_n(B,C)}$ satisfying

$$\forall x \in \underline{B} \quad f(x) = h(x, \ldots, x)\,.$$

Then h is called a *linearization* of f. We use $\underline{\mathbf{Hpol}}_n(B,C)$ for the set of n-homogeneous polynomial functions from B to C and set $\mathsf{M}_n(h) = f$.

Notice that we would define exactly the same class of functions without requiring h to be symmetric. We make this choice only to reduce the number of notions at hand.

Lemma 9.3. *An n-homogeneous polynomial function is totally monotone.*

Proof. By induction on n. For $n = 0$, this is obvious, so assume that the property holds for n, let $f \in \underline{\mathbf{Hpol}}_{n+1}(B,C)$ and let h be a linearization of f. Let $k \in \mathbb{N}$, $\varepsilon \in \{+,-\}$ and $x, u_1, \ldots, u_k \in \underline{B}$. We have

$$\begin{aligned}\sum_{I\in\mathcal{P}^\varepsilon(k)} f(x+\sum_{i\in I} u_i) &= \sum_{I\in\mathcal{P}^\varepsilon(k)} h(\overline{x+\sum_{i\in I} u_i}^{\,n+1}) \\ &= \sum_{I\in\mathcal{P}^\varepsilon(k)} \left(h(\overline{x+\sum_{i\in I} u_i}^{\,n}, x) + \sum_{j\in I} h(\overline{x+\sum_{i\in I} u_i}^{\,n}, u_j)\right).\end{aligned}$$

by linearity in the last argument. Observe that

$$\sum_{I\in\mathcal{P}^+(k)} h(\overline{x+\sum_{i\in I} u_i}^{\,n}, x) \geq \sum_{I\in\mathcal{P}^-(k)} h(\overline{x+\sum_{i\in I} u_i}^{\,n}, x) \tag{9.1}$$

by inductive hypothesis. Next, assuming that $k > 0$,

$$\begin{aligned}\sum_{I\in\mathcal{P}^\varepsilon(k)} \sum_{j\in I} h(\overline{x+\sum_{i\in I} u_i}^{\,n}, u_j) &= \sum_{j=1}^{k} \sum_{\substack{I\in\mathcal{P}^\varepsilon(k)\\ j\in I}} h(\overline{x+\sum_{i\in I} u_i}^{\,n}, u_j) \\ &= \sum_{j=1}^{k} \sum_{I\in\mathcal{P}^\varepsilon(k-1)} h(\overline{x+\sum_{i\in \mathsf{inj}_j(I)} u_i}^{\,n}, u_j) \quad \text{by Lemma 8.4} \\ &= \sum_{j=1}^{k} \sum_{I\in\mathcal{P}^\varepsilon(k-1)} h(\overline{x+u_j+\sum_{i\in I} u(j)_i}^{\,n}, u_j)\end{aligned}$$

where $(u(j)_i)_{i=1}^{k-1}$ is defined by

$$u(j)_i = \begin{cases} u_i & \text{if } i < j \\ u_{i+1} & \text{if } i \geq j\,. \end{cases}$$

For each $j \in \{1, \ldots, k\}$ we have, by inductive hypothesis:

$$\sum_{I\in\mathcal{P}^+(k-1)} h(\overline{x+u_j+\sum_{i\in I} u(j)_i}^{\,n}, u_j) \geq \sum_{I\in\mathcal{P}^-(k-1)} h(\overline{x+u_j+\sum_{i\in I} u(j)_i}^{\,n}, u_j)$$

and hence

$$\sum_{j=1}^{k}\sum_{I\in\mathcal{P}^+(k-1)} h(\overline{x+u_j+\sum_{i\in I}u(j)_i}^{\,n}, u_j) \geq \sum_{j=1}^{k}\sum_{I\in\mathcal{P}^-(k-1)} h(\overline{x+u_j+\sum_{i\in I}u(j)_i}^{\,n}, u_j)$$

and observe that this holds also (trivially) when $k=0$. Summing this inequation with (9.1) we get

$$\sum_{I\in\mathcal{P}^+(k)} f(x+\sum_{i\in I}u_i) \geq \sum_{I\in\mathcal{P}^-(k)} f(x+\sum_{i\in I}u_i)$$

as expected. □

Lemma 9.4. *An n-homogeneous polynomial function f has exactly one linearization $\mathsf{L}_n f$. Moreover*

$$\|\mathsf{L}_n f\| \leq \frac{n^n}{n!}\|f\|$$

where $\|f\| = \sup_{x\in\mathcal{B}\underline{B}}\|f(x)\|$.

Proof. Let $f:\underline{B}\to\underline{C}$ be an n-homogeneous polynomial and let $h\in\underline{\mathbf{Sym}_n(B,C)}$ be a linearization of f. By Lemma 9.3 we can define a function $h':\underline{B}^n\to\underline{C}$ by

$$h'(x_1,\dots,x_n) = \frac{1}{n!}\left(\sum_{I\in\mathcal{P}^+(n)} f(\sum_{i\in I}x_i) - \sum_{I\in\mathcal{P}^-(n)} f(\sum_{i\in I}x_i)\right) = \Delta h(x_1,\dots,x_n)(0)$$

and the usual proof of the polarization theorem shows that necessarily $h'=h$. So h is completely determined by f, proving our contention.

Next, given $x_1,\dots,x_n\in\mathcal{B}\underline{B}$, we have

$$\begin{aligned}
\|h(x_1,\dots,x_n)\| &\leq \frac{1}{n!}\left\|f(\sum_{i=1}^n x_i)\right\| \quad \text{by Lemma 8.14}\\
&= \frac{n^n}{n!}\left\|h(\frac{1}{n}\sum_{i=1}^n x_i,\dots,\frac{1}{n}\sum_{i=1}^n x_i)\right\|\\
&= \frac{n^n}{n!}\left\|f(\frac{1}{n}\sum_{i=1}^n x_i)\right\|\\
&\leq \frac{n^n}{n!}\|f\|
\end{aligned}$$

since $\left\|\frac{1}{n}\sum_{i=1}^n x_i\right\|\leq 1$. □

The set $\underline{\mathbf{Hpol}}_n(B,C)$ is canonically a precone. Indeed if $f,g\in\underline{\mathbf{Hpol}}_n(B,C)$ then $f+g$ (defined pointwise) belongs to $\underline{\mathbf{Hpol}}_n(B,C)$ because clearly $f+g=\mathsf{M}_n(\mathsf{L}_n f+\mathsf{L}_n g)$ and we know that $\mathsf{L}_n f+\mathsf{L}_n g\in\mathbf{Sym}_n(B,C)$. Multiplication by a scalar in $\mathbb{R}_{\geq 0}$ is dealt with similarly. Notice that this reasoning also shows that the maps $\mathsf{L}_n:\underline{\mathbf{Hpol}}_n(B,C)\to\underline{\mathbf{Sym}_n(B,C)}$ and $\mathsf{M}_n:\underline{\mathbf{Sym}_n(B,C)}\to\underline{\mathbf{Hpol}}_n(B,C)$ are linear.

We define $\|\text{-}\|_{\underline{\mathbf{Hpol}}_n(B,C)}$ as usual by $\|f\|_{\underline{\mathbf{Hpol}}_n(B,C)}=\sup_{x\in\mathcal{B}\underline{B}}\|f(x)\|_C$ so that clearly $\|f\|_{\underline{\mathbf{Hpol}}_n(B,C)}\leq\|\mathsf{L}_n f\|_{\mathbf{Sym}_n(B,C)}$. Equipped with this norm $\underline{\mathbf{Hpol}}_n(B,C)$ is a cone: the only non obvious property is completeness so let $(f_k\in\mathcal{B}\underline{\mathbf{Hpol}}_n(B,C))_{k=1}^{\infty}$ be a monotone

sequence and let $f : \underline{B} \to \underline{C}$ be the pointwise lub of this sequence which is well defined since for each $x \in \underline{B}$ we have

$$\|f_k(x)\| \leq \|\mathsf{L}_n f_k\| \, \|x\|^n \leq \frac{n^n}{n!} \|f_k\| \, \|x\|^n \leq \frac{n^n}{n!} \|x\|^n \, .$$

The sequence $(\mathsf{L}_n f_k)_{k=1}^\infty$ is monotone in $\frac{n^n}{n!}\mathcal{B}\underline{\mathbf{Sym}_n(B,C)}$ and has therefore a lub $h \in \mathcal{B}\underline{\mathbf{Sym}_n(B,C)}$ and remember that this lub is defined pointwise on $\underline{B}^n$. It follows that $f = \mathsf{M}_n(h) \in \underline{\mathbf{Hpol}}_n(B,C)$ and that we have

$$\forall x \in \underline{B} \quad f(x) = \sup_k f_k(x)\, .$$

Since $\forall k\ f_k \leq f$ by monotonicity of M_n it follows that $f = \sup_k f_k$. Last observe that $\|f\| \leq 1$ which ends the proof of Scott completeness of $\mathcal{B}\underline{\mathbf{Hpol}}_n(B,C)$.

So we have shown that $\underline{\mathbf{Hpol}}_n(B,C)$ is a cone, and also that the linear maps L_n and M_n are Scott-continuous.

Remark 9.5. It is important to notice that, contrarily to $\mathbf{Sym}_n(B,C)$, the canonical order relation of $\underline{\mathbf{Hpol}}_n(B,C)$ is not the pointwise order. As an example take $n=2$, $B = 1 \,\&\, 1$, $C = 1$, and consider $f, g \in \underline{\mathbf{Hpol}}_n(B,C)$ given by $f(x,y) = 2xy$ and $g(x,y) = x^2 + y^2$. Then

$$\begin{aligned}\mathsf{L}_2 f((x_1,y_1),(x_2,y_2)) &= \frac{1}{2}(f(x_1+x_2, y_1+y_2) - f(x_1,y_1) - f(x_2,y_2)))\\ &= x_1 y_2 + x_2 y_1\end{aligned}$$

and similarly $\mathsf{L}_2 g((x_1,y_1),(x_2,y_2)) = x_1x_2 + y_1y_2$ and therefore we do not have $\mathsf{L}_2 f \leq \mathsf{L}_2 g$ (take $(x_1,y_1) = (x_2,y_2) = (1,0)$) whereas $\forall (x,y) \in \underline{1 \,\&\, 1} = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\ f(x,y) \leq g(x,y)$.

Given $d \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(d,B)}$ and $m \in \mathcal{M}_d^C$ we define $\beta \triangleright m : \overline{d} \times \underline{\mathbf{Hpol}}_n(B,C) \to \mathbb{R}_{\geq 0}$ as usual by $(\beta \triangleright m)(r,f) = m(r, f(\beta(r)))$ for all $f \in \underline{\mathbf{Hpol}}_n(B,C)$. Notice that

$$\boldsymbol{\lambda} r \in \overline{d} \cdot m(r, f(\beta(r))) = \boldsymbol{\lambda} r \in \overline{d} \cdot m(r, \mathsf{L}_n f(\overline{\beta(r)}^n))$$

and this function is measurable because $\mathsf{L}_n f \in \underline{\mathbf{Sym}_n(B,C)}$, which implies that $\mathsf{L}_n f \circ \langle \beta, \ldots, \beta \rangle \in \underline{\mathsf{Path}(d,C)}$. Then it is easily checked that setting $\mathcal{M}_d = \{\beta \triangleright m \mid \beta \in \underline{\mathsf{Path}(d,B)}$ and $m \in \mathcal{M}_d^C\}$ we define a measurability structure on $\underline{\mathbf{Hpol}}_n(B,C)$ so that $E = (\underline{\mathbf{Hpol}}_n(B,C), (\mathcal{M}_d)_{d \in \mathbf{ar}})$ is a measurable cone that we denote as $\mathbf{Hpol}_n(B,C)$.

Lemma 9.6. $\mathsf{L}_n \in \mathbf{MCones}(\underline{\mathbf{Hpol}}_n(B,C), \frac{n^n}{n!}\mathbf{Sym}_n(B,C))$.

Proof. In view of what we know about L_n, it suffices to prove that $\mathsf{L}_n : \underline{\mathbf{Hpol}_n(B,C)} \to \underline{\mathbf{Sym}_n(B,C)}$ is measurable so let $d \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(d, \mathbf{Hpol}_n(B,C))}$, we must check that $\mathsf{L}_n \circ \eta \in \underline{\mathsf{Path}(d, \mathbf{Sym}_n(B,C))}$. Let $e \in \mathbf{ar}$ and $p \in \mathcal{M}_e^{\mathbf{Sym}_n(B,C)}$. We have $p = \vec{\beta} \triangleright m$ where $\vec{\beta} \in \underline{\mathsf{Path}(e,B)^n}$ and $m \in \mathcal{M}_e^B$ so that

$$\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \mathsf{L}_n(\eta(r))) = \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, \mathsf{L}_n(\eta(r))(\vec{\beta}(s)))$$

which, coming back to the definition of L_n, is measurable by measurability of addition and subtraction in $\mathbb{R}$ and linearity of $m(s, \_)$. ∎

Theorem 9.7. $\mathbf{Hpol}_n(B,C)$ *is an integrable cone and*

$$\mathsf{L}_n \in \mathbf{ICones}(\underline{\mathbf{Hpol}}_n(B,C), \frac{n^n}{n!}\mathbf{Sym}_n(B,C))\, .$$

Proof. We prove integrability of $E = \mathbf{Hpol}_n(B,C)$ so let $d \in \mathbf{ar}$, $\eta \in \underline{\mathsf{Path}(d,E)}$ and $\mu \in \underline{\mathsf{Meas}(d)}$, we define $f : \underline{B} \to \underline{C}$ by $f(x) = \int \eta(r)(x)\mu(dr)$ using the fact that C is an integrable cone. Since L_n is measurable we can define $h : \underline{B}^n \to \underline{C}$ by

$$h(\vec{x}) = \int \mathsf{L}_n(\eta(r))(\vec{x})\mu(dr)\,.$$

which is clearly symmetric. It is n-linear, Scott-continuous, measurable by Lemma 5.6, and integrable by the first statement of Theorem 7.1. So we have $h \in \underline{\mathbf{Sym}_n(B,C)}$ and $h(x,\dots,x) = \int \eta(r)(x)\mu(dx) = f(x)$ for all $x \in \underline{B}$ which proves that $f \in \underline{\mathbf{Hpol}_n(B,C)}$. Last let $p \in \mathcal{M}_0^{\mathbf{Hpol}_n(B,C)}$ so that $p = x \triangleright m$ for some $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$. We have $p(f) = m(f(x)) = m(\int \eta(r)(x)\mu(dx)) = \int m(\eta(r)(x))\mu(dr)$ by definition of an integral in C. So $p(f) = \int p(\eta(r))\mu(dr)$ by definition of p, which shows that f is the integral of η in $\mathbf{Hpol}_n(B,C)$ and hence that this measurable cone is also integrable.

The integrability of L_n results from its definition and from the fact that integrals commute with finite sums and differences. □

9.3. **The cone of analytic functions.**

Definition 9.8. A function $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic if it is bounded, and there is a sequence $(f_n \in \underline{\mathbf{Hpol}}_n(B,C))_{n\in\mathbb{N}}$ such that

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n=0}^{\infty} f_n(x)\,. \tag{9.2}$$

Such a sequence $(f_n)_{n\in\mathbb{N}}$ is called a *homogeneous polynomial decomposition* of f.

Notice that the precise meaning of (9.2) is that the monotone sequence $(\sum_{n=0}^{N} f_n(x))_{N\in\mathbb{N}}$ is bounded in $\underline{C}$ (in the sense of the norm) and has $f(x)$ as lub.

Lemma 9.9. *If $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic, then f has exactly one homogeneous polynomial decomposition.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a homogeneous polynomial decomposition of an analytic f that we can assume without loss of generality to range in $\mathcal{B}\underline{C}$ since f is bounded. Let $x' \in \underline{B'}$ and $x \in \mathcal{B}\underline{B}$. Let

$$\begin{aligned}\varphi : [0,1] &\to \mathbb{R}_{\geq 0}\\ t &\mapsto x'(f(tx))\,.\end{aligned}$$

We have $\varphi(t) = \sum_{n=0}^{\infty} x'(f_n(x))t^n$ by linearity and continuity of x' and hence

$$\forall n \in \mathbb{N} \quad x'(f_n(x)) = \frac{1}{n!}\varphi^{(n)}(0) = \frac{1}{n!}\frac{d^n}{dt^n}x'(f(tx)) \mid_{t=0}$$

so that if $(g_n)_{n\in\mathbb{N}}$ is another homogeneous polynomial decomposition of f we have $x'(f_n(x)) = x'(g_n(x))$ for all x, n and x'. Since this holds in particular for all $x' = m \in \mathcal{M}_0^B$ our claim is proven by (**Mssep**). □

If $f : \mathcal{B}\underline{B} \to C$ is analytic, we use $\mathsf{P}_n(f)$ for the nth component of its unique homogeneous polynomial decomposition and we set $\mathsf{D}_0^{(n)} = n!(\mathsf{L}_n \circ \mathsf{P}_n)$ so that $\mathsf{D}_0^{(n)} f \in \underline{\mathbf{Sym}_n(B,C)}$ and we have

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{x}^n)$$

which can be understood as the Taylor expansion of f, motivating the notation: $\mathsf{D}_0^{(n)} f$ can be understood as the nth derivative of f at 0, which is an n-linear symmetric function. As usual we say that f is measurable if, for all $d \in \mathbf{ar}$ and $\beta \in \underline{\mathsf{Path}(d,B)}$ one has $f \circ \beta \in \underline{\mathsf{Path}(d,C)}$.

We define now a cone of analytic and measurable functions $\mathcal{B}\underline{B} \to \underline{C}$ so let P be the set of these functions. We define the algebraic operations on P pointwise: if $f, g \in P$ then $f + g \in P$ since $(f+g)(x) = f(x) + g(x) = \sum_{n=0}^{\infty}(\mathsf{P}_n f(x) + \mathsf{P}_n g(x))$ by continuity of addition. Notice that if $f, g \in P$ then

$$f \leq g \Leftrightarrow \forall n \in \mathbb{N} \quad \mathsf{D}_0^{(n)} f \leq \mathsf{D}_0^{(n)} g\,.$$

since $f \leq g$ means that $\forall x \in \mathcal{B}B\ f(x) \leq g(x)$ and $\boldsymbol{\lambda} x \in \mathcal{B}B \cdot (g(x) - f(x)) \in P$. Each map $\mathsf{D}_0^{(n)} : P \to \underline{\mathbf{Sym}_n(B,C)}$ is linear by Lemmas 9.4 and 9.9.

We set as usual

$$\|f\| = \sup\{\|f(x)\| \mid x \in \mathcal{B}\underline{B}\}$$

and define in that way a cone. Let indeed $(f^k)_{k\in\mathbb{N}}$ be a monotone sequence in $\mathcal{B}P$. For each $k, n \in \mathbb{N}$ we have $\left\|\mathsf{D}_0^{(n)} f^k\right\| \leq n^n$ by Lemma 9.6 and the sequence $(\mathsf{D}_0^{(n)} f^k)_{k\in\mathbb{N}}$ is monotone and hence has a lub $h_n \in \underline{\mathbf{Sym}_n(B,C)}$ and we have

$$\forall x_1, \ldots, x_n \in \underline{B} \quad h_n(x_1, \ldots, x_n) = \sup_{k\in\mathbb{N}} \mathsf{D}_0^{(n)} f^k(x_1, \ldots, x_n)\,.$$

In particular we can define the homogeneous polynomial map $f_n = \mathsf{M}_n(\frac{1}{n!} h_n)$, which means

$$f_n(x) = \frac{1}{n!} h_n(\overline{x}^n) = \sup_{k\in\mathbb{N}} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n)$$

for all $x \in \mathcal{B}\underline{B}$, and hence f is analytic with

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} f_n(x) \\ &= \sum_{n=0} \sup_{k\in\mathbb{N}} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n) \\ &= \sup_{k\in\mathbb{N}} \sum_{n=0} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n) \\ &= \sup_{k\in\mathbb{N}} f^k(x) \end{aligned}$$

which shows that $(f_k)_{k\in\mathbb{N}}$ has f as lub in $\mathcal{B}P$ since f is clearly measurable (as usual by the monotone convergence theorem).

Then we define a family $\mathcal{M} = (\mathcal{M}_d)_{d\in\mathbf{ar}}$ of sets of measurability tests by stipulating that $p \in \mathcal{M}_d$ if $p = \beta \triangleright m$ where $\beta \in \mathcal{B}\underline{\mathsf{Path}(d,B)}$ and $m \in \mathcal{M}_d^C$, and, if $f \in P$ and $r \in \overline{d}$

then $p(r, f) = m(r, f(\beta(r)))$. It is easily checked that $(P, \mathcal{M})$ is a measurable cone, that we denote as $B \Rightarrow_{\mathsf{a}} C$.

We check that $B \Rightarrow_{\mathsf{a}} C$ is integrable so let $\eta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$ for some $d \in \mathbf{ar}$ and let $\mu \in \underline{\mathsf{Meas}(d)}$. We define a function $f : \mathcal{B}\underline{B} \to \underline{C}$ by

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \int_d^C \eta(r)(x)\mu(dr)\,.$$

This function is well defined because for each given $x \in \mathcal{B}\underline{B}$ one has $\boldsymbol{\lambda} r \in \overline{d} \cdot \eta(r)(x) \in \underline{\mathsf{Path}(d, C)}$. For each $r \in \overline{d}$ we can write

$$\eta(r)(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)$$

and hence

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_d^C \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_d^{\mathbf{Sym}_n(B,C)} \mathsf{D}_0^{(n)}(\eta(r))\mu(dr) \right) (\overline{x}^n)$$

by definition of integrals in the integrable cone $\mathbf{Sym}_n(B, C)$, and hence $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$. Let $p = (x \triangleright m) \in \mathcal{M}_0^{B \Rightarrow_{\mathsf{a}} C}$ where $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$, we have

$$\begin{aligned}
p(f) &= m(f(x)) \\
&= m\left(\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} m\left(\int \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) \right) \quad \text{by lin. and cont. of } m \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int m(\mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n))\mu(dr) \quad \text{by def. of integrals in } C \\
&= \int \left(\sum_{n=0}^{\infty} \frac{1}{n!} m(\mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)) \right) \mu(dr) \quad \text{by the monotone convergence th.} \\
&= \int m\left(\sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n) \right) \mu(dr) \\
&= \int m(\eta(r)(x))\mu(dr) = \int p(\eta(r))\mu(dr)
\end{aligned}$$

which shows that $f = \int \eta(r)\mu(dr)$, and hence the measurable cone $B \Rightarrow_{\mathsf{a}} C$ is integrable.

Lemma 9.10. *For each $n \in \mathbb{N}$, the function $\mathsf{P}_n : \underline{B \Rightarrow_{\mathsf{a}} C} \to \underline{\mathbf{Hpol}_n(B, C)}$ is linear, continuous, measurable, integrable and has norm ≤ 1.*

Proof. Linearity and continuity result straightforwardly from the fact that the homogeneous polynomial decomposition $(f_n = \mathsf{P}_n(f))_{n \in \mathbb{N}}$ of f is uniquely determined by its defining property:

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n \in \mathbb{N}} f_n(x)\,.$$

Let $\eta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$, we must check last that $\mathsf{P}_n \circ \eta \in \underline{\mathsf{Path}(d, \mathbf{Hpol}_n(B, C))}$ so let $e \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(e, B)}$ and $m \in \mathcal{M}_e^C$, we must prove that

$$\begin{aligned}\theta &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot (\beta \triangleright m)(s, \mathsf{P}_n(\eta(r))) \\ &= \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \mathsf{P}_n(\eta(r))(\beta(s)))\end{aligned}$$

is measurable $\overline{e + d} \to \mathbb{R}_{\geq 0}$. This results from the fact that

$$\theta(s, r) = \frac{1}{n!} \frac{d^n}{dt^n} m(s, \eta(r, t\beta(s))) \mid_{t=0}$$

and from the measurability and smoothness wrt. t of the map $(s, r, t) \mapsto m(s, \eta(r, t\varphi(s)))$. Indeed the following is standard: if $\mathcal{X}$ is a measurable space then if a function $\mathcal{X} \times [0, 1) \to \mathbb{R}_{\geq 0}$ is measurable, and smooth in its second argument, then so is its derivative wrt. its second argument.

Last we check integrability of P_n so let moreover $\mu \in \mathsf{Meas}(d)$, and let $p \in \mathcal{M}_0^{\mathbf{Hpol}_n(B,C)}$ so that $p = x \triangleright m$ for some $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$, we have

$$\begin{aligned}p(\mathsf{P}_n(\int_e^{\mathbf{Hpol}_n(B,C)} \eta(s)\mu(ds))) &= \frac{1}{n!} \frac{d^n}{dt^n} m(\int_e^C \eta(s, tx)\mu(ds))) \mid_{t=0} \\ &= \frac{1}{n!} \left(\frac{d^n}{dt^n} \int_e m(\eta(s, tx))\mu(ds) \right) \mid_{t=0} \\ &= \frac{1}{n!} \int_e \frac{d^n}{dt^n} m(\eta(s, tx)) \mid_{t=0} \mu(ds) \\ &= \int_e p(\mathsf{P}_n(\eta(s)))\mu(ds)\end{aligned}$$

by standard properties of integration.

The fact that $\|\mathsf{P}_n\| \leq 1$ results from the obvious fact that $\mathsf{P}_n f(x) \leq f(x)$ for all $x \in \mathcal{B}\underline{B}$. ☐

Theorem 9.11. *For all $n \in \mathbb{N}$ we have $\mathsf{D}_0^{(n)} \in \mathbf{ICones}(B \Rightarrow_{\mathsf{a}} C, n^n \mathbf{Sym}_n(B, C))$.*

Proof. Remember that $\mathsf{D}_0^{(n)} = n!(\mathsf{L}_n \circ \mathsf{P}_n)$ and apply Theorem 9.7 and Lemma 9.10. ☐

Theorem 9.12. *Any analytic function is stable.*

Proof. Immediate consequence of the definition of analytic functions and of Lemma 9.3. ☐

Remark 9.13. The converse is not true, as shown by Remark 8.29. Indeed since the stable function f introduced in that remark is actually linear, if f were analytic we would have $\mathsf{D}_0^{(1)} f = f$ and $\mathsf{D}_0^{(n)} f = 0$ if $n \neq 1$ which is not possible since f does not preserve integrals.

9.4. **The category of integrable cones and analytic functions.**

9.4.1. *Composing analytic functions.* We start with a special case of composition that we consider as the restriction of an analytic function to a *local cone* in the sense of Section 8.1.

Theorem 9.14. *Let $x \in \mathcal{B}\underline{B}$ and let $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$. Then the function $g : \mathcal{B}\underline{B_x} \to \underline{C}$ defined by $g(u) = f(x+u)$ is analytic, that is $g \in \underline{B_x \Rightarrow_{\mathsf{a}} C}$.*

Proof. Given $u \in \mathcal{B}\underline{B_x}$ we have

$$\begin{aligned} g(u) &= f(x+u) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{x+u}^n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k) \end{aligned}$$

so it suffices to show that for each $l \in \mathbb{N}$ the function $g_l : \mathcal{B}\underline{B_x} \to \underline{C}$ defined by

$$g_l(u) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k)$$

satisfies $g_l(u) = \varphi_l(\overline{u}^l)$ for some $\varphi_l \in \mathbf{Sym}_l(B_x, C)$. We show that we can set

$$\varphi_l(\vec{u}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\vec{u}, \overline{x}^k)$$

for all $\vec{u} = (u_1, \dots, u_l) \in \underline{B_x}^l$. So let $\vec{u} = (u_1, \dots, u_l) \in \underline{B_x}^l$ and let $\lambda \geq \max_{i=1}^{l} \|u_i\|_{\underline{B_x}}$ be such that $\lambda > 0$ so that for each i we have $\frac{1}{\lambda} u_i \in \mathcal{B}\underline{B_x}$.

For $N \in \mathbb{N}$ let $\varphi_l^N(\vec{u}) = \sum_{k=0}^{N} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\vec{u}, \overline{x}^k)$ so that $\varphi_l^N \in \underline{\mathbf{Sym}_l(B_x, C)}$; actually we even have $\varphi_l^N \in \underline{\mathbf{Sym}_l(B, C)}$. Observe that, setting $u = \frac{1}{l\lambda} \sum_{i=1}^{l} u_i \in \mathcal{B}\underline{B_x}$ we have $u_i \leq l\lambda u$ for each $i = 1, \dots, l$ so that

$$\begin{aligned} \varphi_l^N(\vec{u}) &\leq \varphi_l^N(\overline{l\lambda u}^l) = (l\lambda)^l \varphi_l^N(\overline{u}^l) \\ &\leq (l\lambda)^l g(u) = (l\lambda)^l f(x+u) \end{aligned}$$

so that $\left\|\varphi_l^N(\vec{u})\right\|_C \leq (l\lambda)^l \|f\|$ and since neither l nor λ depend in N the sequence $(\varphi_l^N(\vec{u}))_{N \in \mathbb{N}}$ is monotone in $(l\lambda)^l \mathcal{B}C$, it has a lub which is $\varphi_l(\vec{u})$ which is therefore well-defined and belongs to $(l\lambda)^l \mathcal{B}\underline{C}$. The fact that the map $\varphi_l : \underline{B_x} \to \underline{C}$ defined in that way is l-linear symmetric and Scott continuous results from the Scott-continuity of addition, scalar multiplication and

from the basic properties of lubs. The measurability and integrability of φ_l result as usual from the monotone convergence theorem. $\square$

Let $f \in \underline{\mathcal{B}B \Rightarrow_{\mathsf{a}} C}$ and $g \in \underline{C \Rightarrow_{\mathsf{a}} D}$, since $g(\mathcal{B}\underline{B}) \subseteq \mathcal{B}\underline{C}$, the function $g \circ f : \mathcal{B}\underline{B} \to \underline{D}$ is well defined and bounded. We assume first that $f(0) = 0$ so that the first term of the Taylor expansion of f vanishes and we have

$$\begin{aligned} g(f(x)) &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} g \left(\overline{\sum_{k=1}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(k)} f(\overline{x}^k)}^{\,n} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma : \overline{n} \to \mathbb{N}^+} \frac{n!}{\sigma!} \mathsf{D}_0^{(n)} g(\mathsf{D}_0^{(\sigma(1))} f(\overline{x}^{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} f(\overline{x}^{\sigma(n)}))\,. \end{aligned}$$

by multilinearity and continuity of the $\mathsf{D}_0^{(n)} f$'s, with the notation $\sigma! = \prod_{i=1}^{n} \sigma(i)!$. If $n, l \in \mathbb{N}$ we define $\mathsf{L}(n, l)$ as the set of all $\sigma : \overline{n} = \{1, \dots, n\} \to \mathbb{N}^+$ such that $\sum_{i=1}^{n} \sigma(i) = l$. This set is finite and empty as soon as $n > l$ (it is for obtaining this effect that we have assumed that $f(0) = 0$). We have

$$g(f(x)) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{n=0}^{l} \sum_{\sigma \in \mathsf{L}(n,l)} \frac{l!}{\sigma!} \mathsf{D}_0^{(n)} f(\mathsf{D}_0^{(\sigma(1))} g(\overline{x}^{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} g(\overline{x}^{\sigma(n)}))\,.$$

For each $l \in \mathbb{N}$, the function

$$\begin{aligned} h_l : \underline{B}^l &\to \underline{D} \\ (x_1, \dots, x_l) &\mapsto \sum_{n=0}^{l} \sum_{\sigma \in \mathsf{L}(n,l)} \frac{l!}{\sigma!} \mathsf{D}_0^{(n)} f(\mathsf{D}_0^{(\sigma(1))} g(x_1, \dots, x_{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} g(x_{l-\sigma(n)+1}, \dots, x_l)) \end{aligned}$$

is l-linear, measurable and integrable as a finite sum of such functions, however it is not necessarily symmetric (for instance, for $l = 4$, this sum contains the expression $\frac{4!}{(2!)^2} \mathsf{D}_0^{(2)} f(\mathsf{D}_0^{(2)} g(x_1, x_2), \mathsf{D}_0^{(2)} g(x_3, x_4)))$, but not $\frac{4!}{(2!)^2} \mathsf{D}_0^{(2)} f(\mathsf{D}_0^{(2)} g(x_1, x_3), \mathsf{D}_0^{(2)} g(x_2, x_4)))$, so we set

$$k_l(\vec{x}) = \frac{1}{l!} \sum_{\theta \in \mathfrak{S}_l} h_l(x_{\theta(1)}, \dots, x_{\theta(l)})$$

and k_l is again a finite sum of l-linear, measurable and integrable functions and hence obviously belongs to $\underline{\mathbf{Sym}_l(B, D)}$, and we have

$$g(f(x)) = \sum_{l=0}^{\infty} \frac{1}{l!} k_l(\overline{x}^l)$$

for each $x \in \mathcal{B}\underline{B}$ which proves that $g \circ f$ is analytic since this function is obviously bounded.

Now we don't assume anymore that $f(0) = 0$, and we define an obviously analytic function $f_0 \in \underline{\mathcal{B}B \to \mathcal{B}C_{f(0)}}$ by $f_0(x) = f(x) - f(0)$. By Lemma 9.14 the function $g_0 : \underline{\mathcal{B}C_{f(0)} \to D}$ given by $g_0(v) = g(f(0) + v)$ is analytic and hence $g \circ f = g_0 \circ f_0$ is analytic since $f_0(0) = 0$. The measurability of $g \circ f$ is obvious so $g \circ f \in \underline{B \Rightarrow_{\mathsf{a}} D}$.

This shows that we have defined a category **ACones** whose objects are the integrable cones and where a morphism from B to C is a $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$ such that $\|f\| \leq 1$. We aim now at proving that this category is cartesian closed.

Lemma 9.15. *For any measurable cones B, C we have* $\mathbf{ICones}(B, C) \subseteq \mathbf{ACones}(B, C)$.

This is obvious and shows that there is a forgetful faithful functor $\mathsf{Der_a} : \mathbf{ICones} \to \mathbf{ACones}$ which acts as the identity on objects and morphisms.

Proposition 9.16. *The category* **ACones** *has all (small) products.*

Proof. We already know that any family $(B_i)_{i\in I}$ of integrable cones has a product $B = \&_{i\in I} B_i$ in **ICones** with projections $(\mathsf{pr}_i \in \mathbf{ICones}(B, B_i))_{i\in I}$. We show that B is also the product of the family $(B_i)_{i\in I}$ with projections $(\mathsf{Der_a}(\mathsf{pr}_i))_{i\in I}$ in **ACones**. Remember that an element of $\underline{B}$ is a family $(x_i \in \underline{B_i})_{i\in I}$ such that the family $(\|x_i\|_{B_i})_{i\in I}$ is bounded in $\mathbb{R}_{\geq 0}$.

So let $(f_i \in \mathbf{ACones}(C, B_i))_{i\in I}$, it suffices to prove that the function $f : \mathcal{B}\underline{C} \to \underline{B}$ given by $f(y) = (f_i(y))_{i\in I}$ belongs to $\mathbf{ACones}(C, B)$. The fact that $\forall y \in \mathcal{B}\underline{C}\ f(y) \in \mathcal{B}\underline{B}$ results from the definition of the norm of B and from the fact that $\forall i \in I\ \ \|f_i\| \leq 1$. The measurability of f results trivially from its definition and from the definition of $\mathcal{M}^B$. We know that $f_i(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathsf{D}_0^{(n)} f_i(\overline{y}^n)$. For each $n \in \mathbb{N}$ the map $\varphi_n : \underline{C}^n \to \underline{B}$ defined by $\varphi_n(\vec{y}) = (\mathsf{D}_0^{(n)} f_i(\vec{y}))_{i\in I}$ belongs to $\underline{\mathbf{Sym}_n(C, B)}$ since we know that $\left\|\mathsf{D}_0^{(n)} f\right\| \leq n^n$ by Theorem 9.11. It follows that f is analytic since $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\varphi_n(\overline{y}^n)$. □

Notice that contrarily to **ICones** the category **ACones** has not all equalizers and therefore is not complete.

Theorem 9.17. *The category* **ACones** *is cartesian closed.*

Proof. We already know that $B \Rightarrow_{\mathsf{a}} C$ is an integrable cone and we have an obvious function

$$\begin{aligned}\mathsf{Ev} : \underline{(B \Rightarrow_{\mathsf{a}} C) \,\&\, B} = \underline{(B \Rightarrow_{\mathsf{a}} C)} \times \underline{B} &\to \underline{C} \\ (f, x) &\mapsto f(x)\end{aligned}$$

which satisfies $\|\mathsf{Ev}\| \leq 1$, we show that it is measurable. Let $\theta \in \underline{\mathsf{Path}(d, (B \Rightarrow_{\mathsf{a}} C) \,\&\, B)}$ for some $d \in \mathbf{ar}$, so that $\theta = \langle \eta, \beta \rangle$ where $\eta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$ and $\beta \in \underline{\mathsf{Path}(d, B)}$, we must prove that $\mathsf{Ev} \circ \langle \eta, \beta \rangle \in \underline{\mathsf{Path}(d, C)}$ so let $m \in \mathcal{M}_e^C$ for some $e \in \mathbf{ar}$, we must prove that the function

$$\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot m(s, \eta(r)(\beta(r))) : \overline{e + d} \to \mathbb{R}_{\geq 0}$$

is measurable. We build $p = (\beta \circ \mathsf{pr}_1) \triangleright (m \circ \mathsf{pr}_2) \in \mathcal{M}_{d+e}^{B \Rightarrow_{\mathsf{a}} C}$ and since $\eta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$ the map

$$\begin{aligned}\psi &= \boldsymbol{\lambda}(r_1, s, r_2) \in \overline{d + e + d} \cdot p(r_1, s, \eta(r_2)) \\ &= \boldsymbol{\lambda}(r_1, s, r_2) \in \overline{d + e + d} \cdot m(s, \eta(r_2)(\beta(r_1)))\end{aligned}$$

is measurable which shows that $\varphi = \boldsymbol{\lambda}(s, r) \in \overline{e + d} \cdot \psi(r, s, r)$ is measurable.

We prove that Ev is analytic. We have

$$\begin{aligned}\mathsf{Ev}(f,x) &= f(x)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathsf{D}_0^{(n)}f(\overline{x}^n)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\varphi_n(\overline{(f,x)}^{n+1})\\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(n+1)\varphi_n(\overline{(f,x)}^{n+1})\end{aligned}$$

where $\varphi_n : (\underline{(B \Rightarrow_{\mathsf{a}} C) \mathbin{\&} B})^{n+1} \to \underline{C}$ is given by

$$\varphi_n((f_1,x_1),\dots,(f_{n+1},x_{n+1})) = \frac{1}{n+1}\sum_{i=1}^{n+1}\mathsf{D}_0^{(n)}f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})$$

and therefore belongs to $\underline{\mathbf{Sym}_{n+1}(B,C)}$; the measurability of φ_n follows from that of $\mathsf{D}_0^{(n)}f$. If follows that Ev is analytic, with

$$\mathsf{D}_0^{(n)}\,\mathsf{Ev}((f_1,x_1),\dots,(f_n,x_n)) = \begin{cases}0 & \text{if } n=0\\ \sum_{i=1}^{n}\mathsf{D}_0^{(n)}f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) & \text{otherwise.}\end{cases}$$

Let D be an integrable cone and let $f \in \mathbf{ACones}(D \mathbin{\&} B, C)$. Given $z \in \mathcal{B}\underline{D}$ let $f_z : \mathcal{B}\underline{B} \to \underline{C}$ be given by $f_z(x) = f(z,x)$. We know that $f_z \in \underline{B \Rightarrow_{\mathsf{a}} C}$ by Theorem 9.14 applied at $(z,0) \in \mathcal{B}\underline{D \mathbin{\&} B}$ and by precomposing the obtained "local" analytic function $g : (D \mathbin{\&} B)_{(z,0)} \to C$ defined by $g(w,y) = f(z+w,y)$ with the obviously analytic function $x \mapsto (0,x)$: this composition of functions coincides with f_z.

We are left with proving that the function $g : \mathcal{B}\underline{D} \to \underline{B \Rightarrow_{\mathsf{a}} C}$ defined by $g(z) = f_z$ belongs to $\mathbf{ACones}(D, B \Rightarrow_{\mathsf{a}} C)$. It is obvious that $\|g\| \leq 1$ so let us check that g is measurable. Let $\delta \in \underline{\mathsf{Path}(d,D)}$, we must prove that $g \circ \delta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$ so let $e \in \mathbf{ar}$ and $p \in \mathcal{M}_e^{B \Rightarrow_{\mathsf{a}} C}$, we must prove that the function

$$\varphi = \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, g(\delta(r))) : \overline{e+d} \to \mathbb{R}_{\geq 0}$$

is measurable. We have $p = \beta \triangleright m$ where $\beta \in \underline{\mathsf{Path}(e,B)}$ and $m \in \mathcal{M}_e^D$ so that

$$\begin{aligned}\varphi &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, g(\delta(r))(\beta(s)))\\ &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot m(s, f(\delta(r),\beta(s)))\end{aligned}$$

is measurable because f is measurable and $\boldsymbol{\lambda}(r,s) \in \overline{d+e} \cdot (\delta(r), \beta(s)) \in \underline{\mathsf{Path}(d+e, D \mathbin{\&} B)}$. We are left with proving that g is analytic. For $z \in \mathcal{B}\underline{D}$ we have

$$\begin{aligned}
g(z) &= \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(z,x) \\
&= \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{(z,x)}^n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(n)} f(\overline{(z,0)+(0,x)}^n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \sum_{k=0}^{n} \binom{n}{k} \mathsf{D}_0^{(n)} f(\overline{(z,0)}^k, \overline{(0,x)}^{n-k}) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(n)} f(\overline{(z,0)}^k, \overline{(0,x)}^{n-k}) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{\infty} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f(\overline{(z,0)}^k, \overline{(0,x)}^l) = \sum_{k=0}^{\infty} \frac{1}{k!} h_k(\overline{z}^k)
\end{aligned}$$

where

$$h_k(z_1, \dots, z_k) = \sum_{l=0}^{\infty} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f((z_1,0), \dots, (z_k,0), \overline{(0,x)}^l)$$

is well defined for all $z_1, \dots, z_k \in \underline{D}$. Indeed, as usual it suffices to take some $\lambda > 0$ such that $\lambda \geq \|z_i\|_D$ for $i = 1, \dots, k$ and observe that for all $N \in \mathbb{N}$ one has, setting $z = \sum_{i=1}^{k} z_i$ so that $\frac{1}{k\lambda} z \in \mathcal{B}D$,

$$\begin{aligned}
\sum_{l=0}^{N} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f((z_1,0), \dots, (z_k,0), \overline{(0,x)}^l) &\leq h_k(\overline{z}^k) \\
&= (k\lambda)^k h_k(\frac{1}{k\lambda} z) \\
&\leq k!(k\lambda)^k g(\frac{1}{k\lambda} z)\,.
\end{aligned}$$

The map h_k is multilinear by Scott-continuity of the algebraic operations in any cone, it is obviously symmetric by the symmetry of the $\mathsf{D}_0^{(n)} f$. Scott continuity follows from that of the $\mathsf{D}_0^{(n)} f$ and from commutations of lubs. Last, measurability and integrability follow as usual from the monotone convergence theorem. So we have $h_k \in \mathbf{Sym}_n(D, B \Rightarrow_{\mathsf{a}} C)$ and this shows that g is analytic.

To prove that **ACones** is cartesian closed it suffices to prove that g is the unique morphism in $\mathbf{ACones}(D, B \Rightarrow_{\mathsf{a}} C)$ such that

$$\mathsf{Ev} \circ (g \mathbin{\&} \mathsf{Id}_B) = f$$

which results straightforwardly from the fact that Ev is defined exactly as in **Set**. $\square$

9.5. **The linear/non-linear adjunction, in the analytic case.**

Theorem 9.18. *The functor* $\mathsf{Der_a} : \mathbf{ICones} \to \mathbf{ACones}$ *has a left adjoint.*

Proof. By Theorem 5.17, it suffices to check that $\mathsf{Der_a}$ preserves all limits, that is all cartesian products and equalizers. Preservation of cartesian products being obvious, let us deal with equalizers. So let $f, g \in \mathbf{ICones}(B, C)$ and let (E, e) be their equalizer in $\mathbf{ICones}$: remember that $\underline{E} = \{x \in \underline{B} \mid f(x) = g(x)\}$ and that $e \in \mathbf{ICones}(E, B)$ is the obvious injection. Let $h \in \mathbf{ACones}(D, B)$ be such that $f \circ h = g \circ h$. This means that $h(\mathcal{B}\underline{D}) \subseteq \mathcal{B}\underline{E}$. We know that

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} h_n(z)$$

where $h_n \in \mathbf{Hpol}_n(D, B)$ is fully characterized by

$$\forall x' \in \underline{B'} \quad x'(h_n(z)) = \frac{d^n}{dt^n} x'(h(tz)) \mid_{t=0}, \tag{9.3}$$

see the proof of Lemma 9.9. We contend that $f \circ h_n = g \circ h_n$ so let $z \in \mathcal{B}\underline{D}$ and let $p \in \mathcal{M}_0^C$, we have

$$p(f(h_n(z))) = \frac{d^n}{dt^n} p(f(h(tz)))) \mid_{t=0},$$

by Equation (9.3) applied with $x' = p\, f \in \underline{B'}$ and hence $p(f(h_n(z))) = p(g(h_n(z)))$ which proves our contention by (**Mssep**). This shows that $h_n(\mathcal{B}\underline{D}) \subseteq \mathcal{B}\underline{E}$. Therefore since the operator L_n is defined in terms of addition, subtraction and multiplication by non-negative real numbers we have $\mathsf{L}_n h_n \in \underline{\mathbf{Sym}_n(B, E)}$ – measurability and integrability follow from the fact that measurability tests and integrals in E are defined as in B. Finally this shows that $\mathsf{D}_0^{(n)} h \in \underline{\mathbf{Sym}_n(B, E)}$ and hence $h \in \mathbf{ACones}(D, E)$ which shows that (E, e) is the equalizer of f, g in $\mathbf{ACones}$. $\square$

Let $\mathsf{E_a} : \mathbf{ACones} \to \mathbf{ICones}$ be the left adjoint of $\mathsf{Der_a}$, and let us introduce the notation $\Theta^{\mathsf{a}}_{B,C} : \mathbf{ICones}(\mathsf{E_a}\, B, C) \to \mathbf{ACones}(B, \mathsf{Der_a}\, C) = \mathbf{ACones}(B, C)$ for the associated natural bijection (remember that $\mathsf{Der_a}\, C = C$).

We use $(!^{\mathsf{a}}, \mathsf{der}^{\mathsf{a}}, \mathsf{dig}^{\mathsf{a}})$ for the induced comonad on $\mathbf{ICones}$ whose Kleisli category is (equivalent to) $\mathbf{ACones}$ since

$$\begin{aligned}
\mathbf{ICones}_{!^{\mathsf{a}}}(B, C) &= \mathbf{ICones}(!^{\mathsf{a}} B, C) \\
&= \mathbf{ICones}(\mathsf{E_a}\ \mathsf{Der_a}\, B, C) \\
&\simeq \mathbf{ACones}(\mathsf{Der_a}\, B, \mathsf{Der_a}\, C) \\
&= \mathbf{ACones}(B, C)\,.
\end{aligned}$$

Notice that actually $!^{\mathsf{a}} B = \mathsf{E_a}\, B$. Let $\mathsf{an}_B = \Theta^{\mathsf{a}}_{B,\Theta^{\mathsf{a}} B}(\mathsf{Id}_{\mathsf{E_a}\, B}) \in \mathbf{ACones}(B, !^{\mathsf{a}} B)$ be the unit of the adjunction, which is the "universal analytic map" on B in the sense that for any integrable cone C and any $f \in \mathbf{ACones}(B, C)$ one has $f = (\Theta^{\mathsf{a}}_{B,C})^{-1}(f) \circ \mathsf{an}_B$ (dropping the $\mathsf{Der_a}$ symbol since this functor acts as the identity on objects and morphisms considered as functions). In particular, any analytic function can be written as the composition of a linear, continuous and integrable map with an instance of an and this factorization is actually unique) since for $l \in \mathbf{ICones}(!^{\mathsf{a}} B, C)$, one has $\Theta^{\mathsf{a}}_{B,C}(l) = l \circ \mathsf{an}_B$. For any $x \in \mathcal{B}\underline{B}$ we set $x^{!_{\mathsf{a}}} = \mathsf{an}_B(x) \in \mathcal{B}\underline{\mathsf{E_a}\, B}$ so that, for $f \in \mathbf{ACones}(B, C)$ we have $f(x) = (\Theta^{\mathsf{a}}_{B,C})^{-1}(f)(x^{!_{\mathsf{a}}})$.

The counit $\mathsf{der}^{\mathsf{a}}_B \in \mathbf{ICones}(!^{\mathsf{a}} B, B)$ of the comonad $!^{\mathsf{a}}\_$ is also the counit of the adjunction. It satisfies therefore

$$\forall x \in \mathcal{B}\underline{B} \quad \mathsf{der}^{\mathsf{a}}_B(x^{!_{\mathsf{a}}}) = x\,.$$

The comultiplication $\mathsf{dig}^{\mathsf{a}}_B \in \mathbf{ICones}(!^{\mathsf{a}} B, !^{\mathsf{a}}!^{\mathsf{a}} B) = \mathbf{ICones}(\mathsf{E_a}\,\mathsf{Der_a}\,B, \mathsf{E_a}\,\mathsf{Der_a}\,\mathsf{E_a}\,\mathsf{Der_a}\,B)$ is defined by $\mathsf{dig}^{\mathsf{a}}_B = \mathsf{E_a}\,\mathsf{an}_B$ so that we have

$$\forall x \in \mathcal{B}B \quad \mathsf{dig}^{\mathsf{a}}_B(x^{!_{\mathsf{a}}}) = x^{!_{\mathsf{a}}!_{\mathsf{a}}}\,.$$

since by naturality of an_B we have $\mathsf{Der_a}\,\mathsf{E_a}\,\mathsf{an}_B \circ \mathsf{an}_B = \mathsf{an}_{\mathsf{Der_a}\,\mathsf{E_a}\,B} \circ \mathsf{an}_B$ in $\mathbf{ACones}$.

Lemma 9.19. *Let $f \in \mathbf{ICones}(B, C)$ and $x \in \mathcal{B}\underline{B}$. We have $(!^{\mathsf{a}} f)(x^{!_{\mathsf{a}}}) = f(x)^{!_{\mathsf{a}}}$.*

Proof. Identical to that of Lemma 8.24. □

Consider the two functors $L, R : \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ defined on objects by $L(B, C, D) = (B \Rightarrow_{\mathsf{a}} (C \multimap D))$ and $R(B, C, D) = (C \multimap (B \Rightarrow_{\mathsf{a}} D))$.

Lemma 9.20. *Let B, C, D be integrable cones. There is an isomorphism in $\mathbf{ICones}$ from $L(B, C, D) = (B \Rightarrow_{\mathsf{a}} (C \multimap D))$ to $R(B, C, D) = (C \multimap (B \Rightarrow_{\mathsf{a}} D))$ which is natural in B, C and D.*

Proof. Identical to that of Lemma 8.25. □

Then we have

$$\begin{aligned}
\mathbf{ICones}(!^{\mathsf{a}}(B_1 \mathbin{\&} B_2), C) &\simeq \mathbf{ACones}(B_1 \mathbin{\&} B_2, C)\\
&\simeq \mathbf{ACones}(B_1, B_2 \Rightarrow_{\mathsf{a}} C)\\
&\simeq \mathbf{ICones}(!^{\mathsf{a}} B_1, B_2 \Rightarrow_{\mathsf{a}} C)\\
&\simeq \mathbf{ICones}(1, !^{\mathsf{a}} B_1 \multimap (B_2 \Rightarrow_{\mathsf{a}} C))\\
&\simeq \mathbf{ICones}(1, B_2 \Rightarrow_{\mathsf{a}} (!^{\mathsf{a}} B_1 \multimap C)) \text{ by Lemma 9.20}\\
&\simeq \mathbf{ACones}(\top, B_2 \Rightarrow_{\mathsf{a}} (!^{\mathsf{a}} B_1 \multimap C))\\
&\simeq \mathbf{ACones}(B_2, (!^{\mathsf{a}} B_1 \multimap C))\\
&\simeq \mathbf{ICones}(!^{\mathsf{a}} B_2, (!^{\mathsf{a}} B_1 \multimap C))\\
&\simeq \mathbf{ICones}(!^{\mathsf{a}} B_2 \otimes !^{\mathsf{a}} B_1, C)\\
&\simeq \mathbf{ICones}(!^{\mathsf{a}} B_1 \otimes !^{\mathsf{a}} B_2, C)
\end{aligned}$$

by a sequence of natural bijections and hence by Lemma 1.1 we have a natural isomorphism $\mathsf{m}^2_{B_1,B_2}$ in

$$\mathbf{ICones}(!^{\mathsf{a}} B_1 \otimes !^{\mathsf{a}} B_2, !^{\mathsf{a}}(B_1 \mathbin{\&} B_2))\,.$$

Similarly we define an iso $\mathsf{m}^0 \in \mathbf{ICones}(1, !^{\mathsf{a}}\top)$. Then one can prove using Lemma 1.1 again that $!^{\mathsf{a}}$ is a strong monoidal comonad.

Theorem 9.21. *Equipped with the strong monoidal monad $!^{\mathsf{s}}$, the category $\mathbf{ICones}$ is a Seely category in the sense of* [Mel09].

9.6. **The coalgebra structure of** $\mathsf{Meas}(d)$**.** Let $d \in \mathbf{ar}$. In Section 7 we defined the Dirac path $\boldsymbol{\delta}^d \in \mathcal{B}\underline{\mathsf{Path}(d, \mathsf{Meas}(d))}$ which maps $r \in \overline{d}$ to $\boldsymbol{\delta}^d(r)$, the Dirac measure at r. In Section 9.5 we introduced the universal stable map $\mathsf{an}_{\mathsf{Meas}(d)} \in \mathbf{ACones}(\mathsf{Meas}(d), !^{\mathsf{a}}\mathsf{Meas}(d))$. Since morphisms in **ACones** are measurable we have

$$\mathsf{an}_{\mathsf{Meas}(d)} \circ \boldsymbol{\delta}^d \in \mathcal{B}\underline{\mathsf{Path}(d, !^{\mathsf{a}}\mathsf{Meas}(d))}$$

and we define

$$\mathsf{h}^{\mathsf{a}}_d = \mathcal{I}_d^{\mathsf{Meas}(d)}(\mathsf{an}_{\mathsf{Meas}(d)} \circ \boldsymbol{\delta}^d) \in \mathbf{ICones}(\mathsf{Meas}(d), !^{\mathsf{a}}\mathsf{Meas}(d))$$

using Theorem 7.1. In other words $\mathsf{h}^{\mathsf{a}}_d$ is defined by

$$\mathsf{h}^{\mathsf{a}}_d(\mu) = \int \boldsymbol{\delta}^d(r)^{!_{\mathsf{a}}} \mu(dr)$$

and satisfies $\mathsf{h}^{\mathsf{a}}_d(\boldsymbol{\delta}^d(r)) = \boldsymbol{\delta}^d(r)^{!_{\mathsf{a}}}$.

Theorem 9.22. *Equipped with* h_d*, the object* $\mathsf{Meas}(d)$ *of* **ICones** *is a coalgebra of the comonad* $!^{\mathsf{a}}\_$*.*

Proof. Same as that of Theorem 8.28. ☐

9.7. **The category of measurable functions as a full subcategory of the Eilenberg Moore category.** A Polish space is a complete metric space which has a countable dense subset.

Lemma 9.23. *Let $\mathcal{X}$ be a Polish space, equipped with its standard Borel σ-algebra $\sigma_{\mathcal{X}}$. Let μ be a probability measure on $\mathcal{X}$ and assume that $\forall U \in \sigma_{\mathcal{X}}\ \mu(U) \in \{0,1\}$. Then μ is a Dirac measure.*

Proof. Given $r \in \mathcal{X}$ and $\varepsilon \geq 0$ we use $B(r,\varepsilon) \subseteq \mathcal{X}$ for the closed ball of radius ε centered at r. Let D be a countable dense subset of $\mathcal{X}$. Let $F \subseteq \mathcal{X}$ be closed and such that $\mu(F) = 1$ and let $\varepsilon > 0$, we have $F \subseteq \bigcup_{r \in D \cap F} B(r,\varepsilon)$ and hence $1 = \mu(F) \leq \sum_{r \in D \cap F} \mu(B(r,\varepsilon))$ and hence $\exists r \in D \cap F\ \mu(B(r,\varepsilon)) = 1$. We define a sequence $(r_n)_{n\in\mathbb{N}}$ of elements of D such that $\forall n \in \mathbb{N}\ \mu(B(r_n, 2^{-n})) = 1$ as follows. We obtain r_0 by applying the property above with $F = \mathcal{X}$ and $\varepsilon = 1$. We get r_{n+1} by applying the property above with $F = B(r_n, 2^{-n})$ and $\varepsilon = 2^{-(n+1)}$. Then the sequence $(r_n)_{n\in\mathbb{N}}$ is Cauchy and has therefore a limit r and we have $\{r\} = \bigcap_{n\in\mathbb{N}} B(r_n, 2^{-n})$ so that $\mu(\{r\}) = \inf_{n\in\mathbb{N}} \mu(B(r_n, 2^{-n})) = 1$ since μ is a measure. It follows that $\mu(U) = 0$ for any measurable U such that $r \notin U$ since we must have $\mu(\{r\} \cup U) = 1$ and hence $\mu = \boldsymbol{\delta}^{\mathcal{X}}(r)$. ☐

Lemma 9.24. *If $\mathcal{X}$ and $\mathcal{Y}$ are measurable spaces such that the σ-algebra of $\mathcal{Y}$ contains all singletons (this is true in particular if $\mathcal{Y}$ is a Polish space), and if κ is a kernel from $\mathcal{X}$ to $\mathcal{Y}$ such that for all $r \in \mathcal{X}$ the measure $\kappa(r)$ is a Dirac measure on $\mathcal{Y}$, then there is a uniquely defined measurable function $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\kappa = \boldsymbol{\delta}^{\mathcal{Y}} \circ \varphi$.*

This is obvious.

Theorem 9.25. *Let $d, e \in$ **ar** be such that $\overline{e}$ is a Polish space and let f be a morphism from* $\mathsf{Meas}(d)$ *to* $\mathsf{Meas}(e)$ *in* **ICones**. *Then f is a coalgebra morphism from* $(\mathsf{Meas}(d), \mathsf{h}^{\mathsf{a}}_d)$ *to* $(\mathsf{Meas}(e), \mathsf{h}^{\mathsf{a}}_e)$ *iff there is a $\varphi \in \mathbf{ar}(d,e)$ such that $f = \mathsf{Meas}(\varphi) = \varphi_*$.*

As a consequence, if we assume that $\overline{d}$ is a Polish space for all $d \in \mathbf{ar}$, then $\mathbf{ar}$ is a full subcategory of the Eilenberg Moore category of the comonad $!^{\mathsf{a}}\_$ through the Meas *functor.*

Most measurable spaces which appear in probability theory are Polish spaces: discrete spaces, all products of the real line, the Cantor Space and the Baire Space, the Hilbert Cube, all the closed subspaces of these spaces *etc.* So the restriction to Polish spaces is not a serious one.

Proof. Saying that f is a coalgebra morphism means that the following diagram commutes in **ICones**:

$$\begin{tikzcd} \mathsf{Meas}(d) \arrow[r, "f"] \arrow[d, "\mathsf{h}^{\mathsf{a}}_d"'] & \mathsf{Meas}(e) \arrow[d, "\mathsf{h}^{\mathsf{a}}_e"] \\ !^{\mathsf{a}}\mathsf{Meas}(d) \arrow[r, "!^{\mathsf{a}} f"] & !^{\mathsf{a}}\mathsf{Meas}(e) \end{tikzcd}$$

which, by Theorem 7.2, is equivalent to

$$\forall r \in \overline{d} \quad (f(\boldsymbol{\delta}^d(r)))^{!_{\mathsf{a}}} = \int_s^{!^{\mathsf{a}}\mathsf{Meas}(e)} \boldsymbol{\delta}^e(s)^{!_{\mathsf{a}}} f(\boldsymbol{\delta}^d(r))(ds) \tag{9.4}$$

and this equation trivially holds if $f = \varphi_*$. Assume conversely that f satisfies (9.4). Let V be a measurable subset of $\overline{e}$ and let $g \in \mathbf{ACones}(\mathsf{Meas}(e), 1)$ be defined by $g(\nu) = \nu(V)^2$ and let $g_0 = (\Theta^{\mathsf{a}}_{\mathsf{Meas}(d),1})^{-1}(g) \in \mathbf{ICones}(\mathsf{Meas}(d), 1)$ which is characterized by $\forall \mu \in \underline{\mathsf{Meas}(d)}\ g(\mu) = g_0(\mu^{!_{\mathsf{a}}})$. We have $g_0(f(\boldsymbol{\delta}^d(r))^{!_{\mathsf{a}}}) = g(f(\boldsymbol{\delta}^d(r))) = f(\boldsymbol{\delta}^d(r))(V)^2$ and, since g_0 preserves integrals,

$$\begin{aligned} f(\boldsymbol{\delta}^d(r))(V)^2 &= g_0\big(\int_s^{!^{\mathsf{a}}\mathsf{Meas}(e)} \boldsymbol{\delta}^e(s)^{!_{\mathsf{a}}} f(\boldsymbol{\delta}^d(r))(ds)\big) \quad \text{by Equation (9.4)} \\ &= \int_e g_0(\boldsymbol{\delta}^e(s)^{!_{\mathsf{a}}}) f(\boldsymbol{\delta}^d(r))(ds) \\ &= \int_e \boldsymbol{\delta}^e(s)(V)^2 f(\boldsymbol{\delta}^d(r))(ds) \\ &= \int_e \boldsymbol{\delta}^e(s)(V) f(\boldsymbol{\delta}^d(r))(ds) \\ &= f(\boldsymbol{\delta}^d(r))(V) \end{aligned}$$

so we have $f(\boldsymbol{\delta}^d(r))(V) \in \{0, 1\}$ for all measurable $V \subseteq \mathcal{Y}$. Let $g \in \mathbf{ACones}(\mathsf{Meas}(e), 1)$ be defined now by $g(\nu) = 1$ and let $g_0 = (\Theta^{\mathsf{a}}_{\mathsf{Meas}(d),1})^{-1}(g) \in \mathbf{ICones}(\mathsf{Meas}(d), 1)$, we have $g_0(f(\boldsymbol{\delta}^d(r))^{!_{\mathsf{a}}}) = g(f(\boldsymbol{\delta}^d(r))) = 1$ and, since g_0 preserves integrals,

$$\begin{aligned} 1 = g_0\big(\int_s^{!^{\mathsf{a}}\mathsf{Meas}(e)} \boldsymbol{\delta}^e(s)^{!_{\mathsf{a}}} f(\boldsymbol{\delta}^d(r))(ds)\big) &= \int_e g_0(\boldsymbol{\delta}^e(s)^{!_{\mathsf{a}}}) f(\boldsymbol{\delta}^d(r))(ds) \\ &= \int_e f(\boldsymbol{\delta}^d(r))(ds) \\ &= f(\boldsymbol{\delta}^d(r))(\overline{e}) \end{aligned}$$

and hence the measure $f(\boldsymbol{\delta}^d(r))$ is a Dirac measure by Lemma 9.23 and it follows that $f = \mathsf{Meas}(\varphi) = \varphi_*$ for a uniquely determined $\varphi \in \mathbf{ar}(d, e)$ by Lemma 9.24. $\square$

Remark 9.26. The same theorem can be proved, exactly in the same way, for the stable exponential in the setting of Section 8.6 since the analytic functions g used in the proof are stable functions.

9.8. **Fixpoints operators in SCones and ACones.** In this section $\mathcal{C}$ denotes **SCones** or **ACones**. Remember that $\mathcal{C}$ is a CCC having the following property: its objects are integrable cones and any $f \in \mathcal{C}(B, B)$ has a least fixpoint $\in \mathcal{B}\underline{B}$ which is the lub of the monotone sequence $(f^n(0) \in \mathcal{B}\underline{B})_{n \in \mathbb{N}}$ since f is Scott continuous.

It is completely standard to apply this property to the map $\mathcal{Z} \in \mathcal{C}((B \Rightarrow B) \Rightarrow B, (B \Rightarrow B) \Rightarrow B)$ given by

$$\mathcal{Z}(F)(f) = f(F(f))$$

which is well-defined and belongs to $\mathcal{C}((B \Rightarrow B) \Rightarrow B, (B \Rightarrow B) \Rightarrow B)$ by cartesian closedness of $\mathcal{C}$. The least fixpoint $\mathcal{Y}$ of $\mathcal{Z}$ is an element of $\mathcal{C}(B \Rightarrow B, B)$ which is easily seen to satisfy

$$\mathcal{Y}(f) = \sup_{n=0}^{\infty} f^n(0)$$

and is therefore a least fixpoint operator that we have proven here to be a morphism in $\mathcal{C}$, that is, a stable or an analytic map depending on the considered category $\mathcal{C}$. This morphism $\mathcal{Y}$ is the key ingredient to interpret recursively defined functional programs in the CCC $\mathcal{C}$.

10. Probabilistic coherence spaces as integrable cones

So far we have seen several ways of building integrable cones: as spaces of measures, or of paths, as products and tensor products, as spaces of analytic maps *etc.* We describe here another source of integrable cones: the probabilistic coherence spaces.

We use $\overline{\mathbb{R}_{\geq 0}}$ for the completed real half-line, that is $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R} \cup \{\infty\}$, considered as a semi-ring with multiplication satisfying $0\infty = 0$, which is the only possible choice since we want multiplication to be Scott continuous.

Let I be a set. If $i \in I$ we use $\mathsf{e}(i)$ for the element of $(\mathbb{R}_{\geq 0})^I$ such that $\mathsf{e}(i)_j = \delta_{i,j}$.

If $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ we define $\mathcal{P}^\perp \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ by

$$\mathcal{P}^\perp = \{x' \in (\mathbb{R}_{\geq 0})^I \mid \forall x \in \mathcal{P}\ \sum_{i \in I} x_i x'_i \leq 1\}$$

and we use the notation $\langle x, x' \rangle = \sum_{i \in I} x_i x'_i$. As usual we have $\mathcal{P} \subseteq \mathcal{Q} \Rightarrow \mathcal{Q}^\perp \Rightarrow \mathcal{P}^\perp$ and $\mathcal{P} \subseteq \mathcal{P}^{\perp\perp}$, and as a consequence $\mathcal{P}^\perp = \mathcal{P}^{\perp\perp\perp}$. In other words, it is equivalent to say that $\mathcal{P} = \mathcal{P}^{\perp\perp}$ or to say that $\mathcal{P} = \mathcal{Q}^\perp$ for some $\mathcal{Q}$.

Theorem 10.1. *Let $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$, one has $\mathcal{P} = \mathcal{P}^{\perp\perp}$ iff the following conditions hold*

- *$\mathcal{P}$ is convex (that is, if $x, y \in \mathcal{P}$ and $\lambda \in [0, 1]$ then $\lambda x + (1 - \lambda)y \in \mathcal{P}$)*
- *$\mathcal{P}$ is down-closed for the product order*
- *and, for any sequence $(x(n))_{n \in \mathbb{N}}$ of element of $\mathcal{P}$ which is monotone for the pointwise order, the pointwise lub $\in \overline{\mathbb{R}_{\geq 0}}^I$ of this sequence belongs to $\mathcal{P}$.*

A proof is outlined in [Gir04] and a complete proof can be found in [Ehr22].

Definition 10.2. A probabilistic coherence space (PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is a set which is at most countable[15] and $\mathsf{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ satisfies

- $\mathsf{P}X = \mathsf{P}X^{\perp\perp}$
- for all $a \in |X|$ there is $x \in \mathsf{P}X$ such that $x_a > 0$
- and for all $a \in |X|$ the set $\{x_a \mid x \in \mathsf{P}X\} \subseteq \mathbb{R}_{\geq 0}$ is bounded.

The 2nd and 3rd conditions are required to keep the coefficients finite and are dual of each other.

Given two sets I, J, a vector $u \in \overline{\mathbb{R}_{\geq 0}}^I$ and a matrix $w \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$, we define $w \cdot u \in \overline{\mathbb{R}_{\geq 0}}^J$ by

$$w \cdot u = \left(\sum_{i \in I} w_{i,j} u_i \right)_{j \in J}$$

and then, given $v \in \overline{\mathbb{R}_{\geq 0}}^J$, observe that

$$\langle w \cdot u, v \rangle = \langle w, u \mathbin{\overline{\otimes}} v \rangle = \sum_{i \in I, j \in J} w_{i,j} u_i v_j \in \overline{\mathbb{R}_{\geq 0}}$$

where $u \mathbin{\overline{\otimes}} v \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$ is defined by $(u \mathbin{\overline{\otimes}} v)_{i,j} = u_i v_j$ (we use this notation here to avoid confusions with the tensor operations we have introduced for integrable cones).

Given $w_1 \in \overline{\mathbb{R}_{\geq 0}}^{I_1 \times I_2}$ and $w_2 \in \overline{\mathbb{R}_{\geq 0}}^{I_2 \times I_3}$, one defines $w_2\, w_1 \in \overline{\mathbb{R}_{\geq 0}}^{I_1 \times I_3}$ (product of matrices written in reversed order) by

$$(w_2\, w_1)_{i_1, i_3} = \sum_{i_2 \in I_2} (w_1)_{i_1, i_2} (w_2)_{i_2, i_3} \,.$$

It is easily checked that, given PCSs X and Y, one defines a PCS $X \multimap Y$ by $|X \multimap Y| = |X| \times |Y|$ and

$$\mathsf{P}(X \multimap Y) = \{t \in (\mathbb{R}_{\geq 0})^{|X \multimap Y|} \mid \forall x \in \mathsf{P}X \ t \cdot x \in \mathsf{P}Y\} \,.$$

Indeed one can check that

$$\mathsf{P}(X \multimap Y) = \{x \mathbin{\overline{\otimes}} y' \mid x \in \mathsf{P}X \text{ and } y' \in \mathsf{P}X^{\perp}\}^{\perp} \,.$$

Then given $s \in \mathsf{P}(X \multimap Y)$ and $t \in \mathsf{P}(Y \multimap Z)$ one has

$$t\, s \in \mathsf{P}(X \multimap Z)$$

and the diagonal matrix $\mathsf{Id} = (\delta_{a,a'})_{(a,a') \in |X \multimap X|}$ belongs to $\mathsf{P}(X \multimap X)$. This defines the category **Pcoh** of probabilistic coherence spaces.

Let $t \in \mathbf{Pcoh}(X, Y)$. We use $\mathsf{fun}(t) : \mathsf{P}X \to \mathsf{P}Y$ for the function defined by $\mathsf{fun}(t)(x) = t \cdot x$.

The orthogonal (or linear negation) $X^{\perp}$ of a PCS X is defined by $|X^{\perp}| = |X|$ and $\mathsf{P}(X^{\perp}) = (\mathsf{P}X)^{\perp}$ so that $X^{\perp\perp} = X$. We use $\perp$ for the PCS such that $|\perp| = \{*\}$ and $\mathsf{P}\perp = [0, 1]$. Setting $1 = \perp^{\perp}$ we have obviously $1 = \perp$ and $X^{\perp}$ is trivially isomorphic to $X \multimap \perp$. Under this iso, the function $\mathsf{fun}(x') : \mathsf{P}X \to [0, 1]$ associated with $x' \in \mathsf{P}X^{\perp}$ is given by $\mathsf{fun}(x')(x) = \langle x, x' \rangle$.

Any PCS X induces a measurable cone $\mathsf{ic}(X)$ defined by $\underline{\mathsf{ic}(X)} = \{\lambda x \mid x \in \mathsf{P}X \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}$ with algebraic operations defined in the obvious pointwise way. Notice that if $t \in$

[15] This countability assumption is crucial in the present setting, again because our use of the monotone convergence theorem.

$\mathbf{Pcoh}(X,Y)$ we can extend $\mathsf{fun}(t)$ to a function $\underline{\mathsf{ic}(X)} \to \underline{\mathsf{ic}(Y)}$ by setting $\mathsf{fun}(t)(x) = \lambda^{-1}\,\mathsf{fun}(t)(\lambda x)$ for any $\lambda > 1$ such that $\lambda x \in \mathsf{P}X$ (the function does not depend on the choice of λ).

The norm of this cone is defined by

$$\|x\|_{\mathsf{ic}(X)} = \sup_{x' \in \mathcal{B}B'} \langle x, x' \rangle = \inf\{\lambda > 0 \mid x \in \lambda \mathsf{P}X\}$$

so that $\mathcal{B}\underline{\mathsf{ic}(X)} = \mathsf{P}X$. The measurability structure of $\mathsf{ic}(X)$ is given by $\mathcal{M}_d^{\mathsf{ic}(X)} = \mathcal{M}_0^{\mathsf{ic}(X)}$ for all $d \in \mathbf{ar}$ and $\mathcal{M}_0^{\mathsf{ic}(X)} = \{\mathsf{fun}(x') \mid x' \in \mathsf{P}X^\perp\}$.

Lemma 10.3. *Let X be a PCS and let $\mathcal{P} \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ be such that $\mathsf{P}X = \mathcal{P}^\perp$. Then*

$$\|x\|_{\mathsf{ic}(X)} = \sup_{x' \in \mathcal{P}} \langle x, x' \rangle .$$

The proof is easy. Notice that for any $a \in |X|$ one has $\mathsf{e}(a) \in \underline{\mathsf{ic}(X)}$ by the second condition in the definition of a PCS, and that $\|\mathsf{e}(a)\|$ is not necessarily equal to 1.

Lemma 10.4. *Let X be a PCS and $d \in \mathbf{ar}$. A function $\beta : \overline{d} \to \underline{\mathsf{ic}(X)}$ is a measurable path of the measurable cone $\mathsf{ic}(X)$ iff $\beta(\overline{d})$ is bounded and, for all $a \in |X|$, the function $\boldsymbol{\lambda} r \in \overline{d} \cdot \beta(r)_a : \overline{d} \to \mathbb{R}_{\geq 0}$ is measurable.*

Proof. The $\Rightarrow$ direction results from the observation that, for any $a \in |X|$ there is a $\lambda > 0$ such that $\lambda \mathsf{e}(a) \in \mathsf{P}X^\perp$. For the $\Leftarrow$ direction let $\beta : \overline{d} \to \mathsf{P}X$ be such that the function $\boldsymbol{\lambda} r \in \overline{d} \cdot \beta(r)_a$ is measurable for all $a \in |X|$. Let $x' \in \mathsf{P}X^\perp$, we must prove that $\varphi = \boldsymbol{\lambda} r \in \overline{d} \cdot \langle \beta(r), x' \rangle$ is measurable. Since $|X|$ is countable, this results from the monotone convergence theorem and from the fact that

$$\varphi(r) = \sum_{a \in |X|} \beta(r)_a x'_a .$$

$\square$

Theorem 10.5. *For any PCS X the measurable cone $\mathsf{ic}(X)$ is integrable.*

Proof. Let $\beta : \overline{d} \to \mathsf{P}X$ be a measurable path (by Lemma 10.4 this is equivalent to saying that $\beta_a = \boldsymbol{\lambda} r \in \overline{d} \cdot \beta(r)_a$ is measurable $\overline{d} \to \mathbb{R}_{\geq 0}$ for all $a \in |X|$ since $\forall r \in \overline{d}\ \ \|\beta(r)\| \leq 1$). Let $\mu \in \underline{\mathsf{Meas}(d)}$. We define $x \in (\mathbb{R}_{\geq 0})^{|X|}$ by

$$x_a = \int \beta_a(r)\mu(dr)$$

which is a well defined element of $\mathbb{R}_{\geq 0}$ since the function β_a is bounded by definition of a PCS. Let $x' \in \mathsf{P}X^\perp$, we have, applying the monotone convergence theorem,

$$\begin{aligned}
\langle x, x' \rangle &= \sum_{a \in |X|} \left(\int \beta_a(r)\mu(dr) \right) x'_a \\
&= \sum_{a \in |X|} \int \left(\beta_a(r)) x'_a \right) \mu(dr) \\
&= \int \langle \beta(r), x' \rangle \mu(dr) \leq \|\mu\|
\end{aligned}$$

so if $\lambda > 0$ is such that $\lambda \|\mu\| \leq 1$ we get $\langle \lambda x, x' \rangle \leq 1$ for all $x' \in \mathsf{P}X^\perp$ so that $x \in \underline{\mathsf{ic}(X)}$. The equation $\langle x, x' \rangle = \int \langle \beta(r), x' \rangle \mu(dr)$ which holds for all $x' \in \mathsf{P}X^\perp$ shows that x is the integral of β over μ by definition of $\mathcal{M}^{\mathsf{ic}(X)}$. $\square$

Theorem 10.6. *If* $t \in \mathsf{P}(X \multimap Y)$ *then* $\mathsf{fun}(t) \in \mathbf{ICones}(\mathsf{ic}(X), \mathsf{ic}(Y))$ *and extended to morphisms in that way, the operation* ic *is a full and faithful functor* $\mathbf{Pcoh} \to \mathbf{ICones}$.

Proof. The fact that $\mathsf{fun}(t)$ is linear and continuous is easy (the proof can be found in [DE11] for instance). Measurability and integral preservation of $\mathsf{fun}(t)$ boil down again to the monotone convergence theorem. Faithfulness results from the fact that t is completely determined by the action of $\mathsf{fun}(t)$ on the elements $\mathsf{e}(a)$ of $\underline{\mathsf{ic}(X)}$ (for all $a \in |X|$; remember that indeed $\forall a \in |X|\ \mathsf{e}(a) \in \underline{\mathsf{ic}(X)}$). Last let $f \in \mathbf{ICones}(\mathsf{ic}(X), \mathsf{ic}(Y))$. We define $t \in (\mathbb{R}_{\geq 0})^{|X \multimap Y|}$ by $t_{a,b} = f(\mathsf{e}(a))_b$. Given $x \in \mathsf{P}X$ and $y' \in \mathsf{P}Y^\perp$ we have

$$\begin{aligned}
\langle t \cdot x, y' \rangle &= \sum_{a \in |X|, b \in |Y|} t_{a,b} x_a y'_b \\
&= \sum_{a \in |X|, b \in |Y|} f(\mathsf{e}(a))_b x_a y'_b \\
&= \sum_{b \in |Y|} f(x)_b y'_b \quad \text{by linearity and continuity of } f \\
&= \langle f(x), y' \rangle \leq 1
\end{aligned}$$

since $\|f\| \leq 1$, which shows that $t\,x \in \mathsf{P}Y$ and hence $t \in \mathbf{Pcoh}(X, Y)$. The equation $\langle t \cdot x, y' \rangle = \langle f(x), y' \rangle$ for all $y' \in \mathsf{P}Y^\perp$ also shows that $t \cdot x = f(x)$ and hence the functor ic is full. $\square$

We use $X \mathbin{\overline{\otimes}} Y$ for the tensor product operation in $\mathbf{Pcoh}$, that is $|X \mathbin{\overline{\otimes}} Y| = |X| \times |Y|$ and $\mathsf{P}(X \mathbin{\overline{\otimes}} Y) = \{x \mathbin{\overline{\otimes}} y \mid x \in \mathsf{P}X \text{ and } y \in \mathsf{P}Y\}^{\perp\perp} = \left(X \multimap Y^\perp\right)^\perp$.

Theorem 10.7. *If* X, Y *are PCSs then* fun *is a natural iso from the integrable cones* $\mathsf{ic}(X \multimap Y)$ *to the integrable cones* $\mathsf{ic}(X) \multimap \mathsf{ic}(Y)$ *in* $\mathbf{ICones}$.

Proof sketch. We know by Theorem 10.6 that fun is an iso of cones. We need to prove that fun and fun^{-1} are measurable and that fun preserve integrals (then fun^{-1} also preserves integrals by injectivity of fun).

Let $d \in \mathbf{ar}$ and $\eta \in \underline{\mathsf{Path}(d, \mathsf{ic}(X \multimap Y))}$, we show that

$$\mathsf{fun} \circ \eta \in \underline{\mathsf{Path}(d, \mathsf{ic}(X) \multimap \mathsf{ic}(Y))}$$

so let $e \in \mathbf{ar}$, $\beta \in \underline{\mathsf{Path}(e, \mathsf{ic}(X))}$ and $m \in \mathcal{M}_e^{\mathsf{ic}(Y)}$ meaning that $m = \mathsf{fun}(y')$ for some $y' \in \mathsf{P}Y^\perp$, we have

$$\begin{aligned}
\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot (\beta \triangleright m)(s, \mathsf{fun}(\eta(r))) &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \langle \mathsf{fun}(\eta(r))(\beta(s)), y' \rangle \\
&= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \langle \eta(r) \cdot \beta(s), y' \rangle \\
&= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \sum_{(a,b) \in |X| \times |Y|} \eta(r)_{a,b} \beta(s)_a y'_b
\end{aligned}$$

and this function is measurable as a countable sum of measurable functions. Conversely let now $\eta \in \underline{\mathsf{Path}(d, \mathsf{ic}(X) \multimap \mathsf{ic}(Y))}$, we must prove that $\mathsf{fun}^{-1} \circ \eta \in \underline{\mathsf{Path}(d, \mathsf{ic}(X \multimap Y))}$ so let

$e \in \mathbf{ar}$ and $p \in \mathcal{M}_e^{\mathsf{ic}(X \multimap Y)}$, that is $p = \mathsf{fun}(z)$ for some $z \in \mathsf{P}(X \multimap Y)^\perp = \mathsf{P}(X \mathbin{\overline{\otimes}} Y^\perp)$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot p(s, \mathsf{fun}^{-1}(\eta(r))) &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \langle z, \mathsf{fun}^{-1}(\eta(r))\rangle \\ &= \boldsymbol{\lambda}(s,r) \in \overline{e+d} \cdot \sum_{(a,b) \in |X| \times |Y|} z_{a,b} \eta(r)(\mathsf{e}(a))_b\end{aligned}$$

which is measurable as a countable sum of measurable functions since we know that for all a, b the function $\boldsymbol{\lambda} r \in \overline{d} \cdot \eta(\mathsf{e}(a))_b$ is measurable by our assumption that η is a measurable path.

The fact that fun preserves integrals results from the pointwise definition of integration in $\mathsf{ic}(X) \multimap \mathsf{ic}(Y)$. □

Theorem 10.8. *There is a natural isomorphism* $\varphi_{X,Y} \in \mathbf{ICones}(\mathsf{ic}(X) \otimes \mathsf{ic}(Y), \mathsf{ic}(X \mathbin{\overline{\otimes}} Y))$.

Proof sketch. The map $\boldsymbol{\lambda}(x,y) \in \underline{\mathsf{ic}(X)} \times \underline{\mathsf{ic}(Y)} \cdot x \mathbin{\overline{\otimes}} y$ is easily seen to be bilinear, Scott continuous, measurable and separately integrable so that we have an associated $\varphi_{X,Y} \in \mathbf{ICones}(\mathsf{ic}(X) \otimes \mathsf{ic}(Y), \mathsf{ic}(X \mathbin{\overline{\otimes}} Y))$ characterized by $\varphi_{X,Y}(x \otimes y) = x \mathbin{\overline{\otimes}} y$. We define now $\psi_{X,Y} : \mathsf{ic}(X \mathbin{\overline{\otimes}} Y) \to \mathsf{ic}(X) \otimes \mathsf{ic}(Y)$. First given $(a,b) \in |X \mathbin{\overline{\otimes}} Y|$ we set $\psi_{X,Y}(\mathsf{e}(a,b)) = \mathsf{e}(a) \otimes \mathsf{e}(b)$. Next given $z \in \underline{\mathsf{ic}(X \mathbin{\overline{\otimes}} Y)}$ such that $\mathsf{supp}(z) = \{(a,b) \in |X \mathbin{\overline{\otimes}} Y| \mid z_{(a,b)} \neq 0\}$ is finite we set $\psi_{X,Y}(z) = \sum_{(a,b) \in |X \overline{\otimes} Y|} z_{(a,b)} \mathsf{e}(a) \otimes \mathsf{e}(b)$ which is a well defined finite sum in the cone $\underline{\mathsf{ic}(X) \otimes \mathsf{ic}(Y)}$. We contend that

$$\|\psi_{X,Y}(z)\|_{\mathsf{ic}(X) \otimes \mathsf{ic}(Y)} \leq \|z\|_{\mathsf{ic}(X \overline{\otimes} Y)}$$

so let $\varepsilon > 0$ and assume without loss of generality that $\|z\| \leq 1$. By Theorem 3.8 there is $g \in \mathcal{B}\underline{\mathsf{ic}(X) \otimes \mathsf{ic}(Y) \multimap \perp}$ such that $\|\psi_{X,Y}(z)\|_{\mathsf{ic}(X) \otimes \mathsf{ic}(Y)} \leq g(\psi_{X,Y}(z)) + \varepsilon$. Let $h \in \mathcal{B}\,(\mathsf{ic}(X) \multimap (\mathsf{ic}(Y) \multimap \perp))$ be the the bilinear morphism associated to g by the iso of Theorem 6.6. We have

$$\begin{aligned}g(\psi_{X,Y}(z)) &= \sum_{(a,b) \in |X \otimes Y|} z_{(a,b)} g(\mathsf{e}(a) \otimes \mathsf{e}(b)) \\ &= \sum_{(a,b) \in |X \otimes Y|} z_{(a,b)} h(\mathsf{e}(a), \mathsf{e}(b)) \leq 1\end{aligned}$$

because $(h(\mathsf{e}(a), \mathsf{e}(b)))_{(a,b) \in |X \otimes Y|} \in \mathsf{P}(X \otimes Y)^\perp$ by Theorem 10.7 and by our assumption that $\|z\| \leq 1$. So we have $\|\psi_{X,Y}(z)\|_{\mathsf{ic}(X) \otimes \mathsf{ic}(Y)} \leq 1 + \varepsilon$ and since this holds for all $\varepsilon > 0$ our contention is proven. Now let z be any element of $\underline{\mathsf{ic}(X \mathbin{\overline{\otimes}} Y)}$ and assume again that $\|z\| \leq 1$. Let $(I_n)_{n \in \mathbb{N}}$ be a monotone sequence of finite sets such that $\bigcup I_n = |X| \times |Y|$ and let $z(n) \in \underline{\mathsf{ic}(X \mathbin{\overline{\otimes}} Y)}$ be defined by

$$z(n)_{(a,b)} = \begin{cases} z_{(a,b)} & \text{if } (a,b) \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

so that the sequence $(z(n))_{n \in \mathbb{N}}$ is monotone and has z as lub in $\underline{\mathsf{ic}(X \mathbin{\overline{\otimes}} Y)}$. The sequence $(\psi_{X,Y}(z(n)))_{n \in \mathbb{N}}$ is monotone and all its elements have norm ≤ 1 since each $z(n)$ has finite support and norm ≤ 1 and hence it has a lub in $\underline{\mathsf{ic}(X) \otimes \mathsf{ic}(Y)}$. It is easy to check that this lub does not depend on the choice of the I_n's, so we can set $\psi_{X,Y}(z) = \sup_{n \in \mathbb{N}} \psi_{X,Y}(z(n))$

so that actually

$$\psi_{X,Y}(z) = \sum_{(a,b)\in|X|\times|Y|} z_{a,b}\mathsf{e}(a)\otimes\mathsf{e}(b)\,.$$

The proof that $\psi_{X,Y} \in \mathbf{ICones}(\mathsf{ic}(X\otimes Y), \mathsf{ic}(X)\otimes\mathsf{ic}(Y))$ follows the standard pattern and it is obvious that it is the inverse of $\varphi_{X,Y}$. □

Remark 10.9. There is a similar natural morphism $\varphi_X \in \mathbf{ICones}(!^{\mathsf{a}}\mathsf{ic}(X), \mathsf{ic}(!X))$ for all PCS X such that $\varphi_X(x^{!_{\mathsf{a}}}) = x^!$ for all $x \in \mathsf{P}X$. We conjecture that this morphism is an iso.

10.1. **The Cantor Space as an equalizer of Pcoh morphisms.** Since **ICones** is a complete category, any equalizer of two parallel **Pcoh** morphisms is an integrable cone. This cone needs not be a PCS (contrarily to the larger category **ICones**, the category **Pcoh** is not complete). We will illustrate this fact on a concrete example showing that interesting cones arise very simply as such equalizers.

Consider the PCS S whose web is the set $\{0,1\}^{<\omega}$ of finite sequences of 0's and 1's and where $x \in (\mathbb{R}_{\geq 0})^{|S|}$ belongs to $\mathsf{P}S$ if, for any $u \subseteq |X|$ which is an antichain (meaning that if $s, s' \in u$ then $s \leq s' \Rightarrow s = s'$ where $\leq$ is the prefix order), one has $\sum_{s\in u} x_s \leq 1$. Since $\mathsf{P}S = \mathcal{A}^\perp$ where $\mathcal{A}$ is the set of all characteristic functions of antichains, S is a PCS (the second and third conditions of Definition 10.2 result from the observation that any singleton is an antichain). Notice that

$$\|x\|_S = \sup_{x'\in\mathcal{A}} \langle x, x'\rangle \tag{10.1}$$

by Lemma 10.3.

The PCS S is the "least solution" (in the sense explained in [DE11, ET19]) of the equation $S = 1 \,\&\, (S \oplus S)$.

There is a morphism $\theta \in \mathbf{Pcoh}(S,S)$ which is given by

$$\theta_{s,t} = \begin{cases} 1 & \text{if } t = ta \text{ for some } a \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

where we use simple juxtaposition for concatenation. Indeed given an antichain u and $x \in \mathsf{P}S$ we have

$$\sum_{t\in u} (\theta\cdot x)_t = \sum_{s\in v} x_s \leq 1$$

where $v = \{sa \mid s \in u \text{ and } a \in \{0,1\}\}$ is an antichain since u is an antichain. Let C be the integrable cone which is the equalizer of θ and Id_S, considered as morphisms of **ICones** through the full and faithful functor ic.

Theorem 10.10. *The integrable cone C is isomorphic to $\mathsf{Meas}(\mathcal{C})$ where $\mathcal{C}$ is the Cantor Space equipped with the Borel sets of its usual topology (the product topology of $\{0,1\}^\omega$ where $\{0,1\}$ has the discrete topology).*

Proof. We have

$$\underline{C} = \{x \in \mathsf{ic}(S) \mid \theta\cdot x = x\}\,,$$

that is, an element of $\underline{C}$ is an $x \in (\mathbb{R}_{\geq 0})^{|S|}$ such that $x \in \mathsf{P}S$ and

$$\forall s \in |S| \quad x_s = x_{s0} + x_{s1}\,.$$

Given $s \in |S|$ we set $\uparrow s = \{\alpha \in \mathcal{C} \mid s < \alpha\} \subseteq \mathcal{C}$, which is a clopen of $\mathcal{C}$. Let U be an open subset of $\mathcal{C}$, the set $\downarrow U$ of all $s \in |S|$ which are minimal such that $\uparrow s \subseteq U$ is an antichain, and we have

$$U = \bigcup \{\uparrow s \mid s \in \downarrow U\} \tag{10.2}$$

by definition of the topology of $\mathcal{C}$. Given $x \in \underline{\mathsf{ic}(S)}$ we define a function $\mathsf{meas}(x) : \mathcal{O}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ on the open sets of $\mathcal{C}$ by

$$\mathsf{meas}(x)(U) = \sum_{s \in \downarrow U} x_s$$

and we have $\mathsf{meas}(x)(U) \leq \|x\|$ by our assumption that $x \in \underline{\mathsf{ic}(X)}$. This function $\mathsf{meas}(x)$ is additive (that is $\mathsf{meas}(x)(\bigcup_{i \in I} U_i) = \sum_{i \in I} \mathsf{meas}(x)(U_i)$ for any countable family $(U_i)_{i \in I}$ of pairwise disjoint open subsets of $\mathcal{C}$). And so $\mathsf{meas}(x)$ extends to a uniquely defined finite measure on the Borel sets of the Cantor Space, that is to an element of $\underline{\mathsf{Meas}(\mathcal{C})}$.

The fact that the map $\mathsf{meas} : \underline{\mathsf{ic}(S)} \to \underline{\mathsf{Meas}(\mathcal{C})}$ is linear and continuous is routine, and its measurability and integrability results as usual from the monotone convergence theorem. So we have $\mathsf{meas} \in \mathbf{ICones}(\mathsf{ic}(S), \mathsf{Meas}(\mathcal{C}))$ and hence by restriction $\mathsf{meas} \in \mathbf{ICones}(\underline{C}, \mathsf{Meas}(\mathcal{C}))$ since $\|\mathsf{meas}(x)\| = \mathsf{meas}(x)(\mathcal{C}) \leq \|x\|$ for all $x \in \underline{C}$.

Let $\mu \in \underline{\mathsf{Meas}(\mathcal{C})}$, we define $\mathsf{rep}(\mu) \in (\mathbb{R}_{\geq 0})^{|S|}$ by $\mathsf{rep}(\mu)_s = \mu(\uparrow s)$. Given an antichain $u \subseteq |S|$ notice that the clopens $(\uparrow s)_{s \in u}$ are pairwise disjoint and that $U = \bigcup_{s \in u} \uparrow s$ is open and hence measurable, so, since μ is a measure, we have

$$\sum_{s \in u} \mathsf{rep}(\mu)_s = \sum_{s \in u} \mu(\uparrow s) = \mu(\bigcup_{s \in u} \uparrow s) = \mu(U) \leq \mu(\mathcal{C}) \,.$$

Since this holds for any antichain u we have shown that $\mathsf{rep}(\mu) \in \mathsf{P}S$. Notice that for any $s \in |S|$ we have $\uparrow s = \uparrow s0 \cup \uparrow s1$ and that this union is disjoint, so that $\mu(\uparrow s) = \mu(\uparrow s0) + \mu(\uparrow s1)$ since μ is a measure, that is $\mathsf{rep}(\mu) \in \underline{C}$. Again, checking that rep is linear, continuous, measurable and integrable is routine; as an example let us prove the last property so let $d \in \mathbf{ar}$ and let $\kappa \in \underline{\mathsf{Path}(d, \mathsf{Meas}(\mathcal{C}))}$. Let $m \in \mathcal{M}_0^C$, that is $m = \mathsf{fun}(x')$ for some $x' \in \mathsf{P}S^{\perp}$. We have

$$\begin{aligned}
m(\int^C \mathsf{rep}(\kappa(r))\mu(dr)) &= \sum_{s \in |S|} x'_s(\int^{\mathsf{ic}(S)} \mathsf{rep}(\kappa(r))\mu(dr))_s \\
&= \sum_{s \in |S|} x'_s \int \mathsf{rep}(\kappa(r))_s \mu(dr) \\
&= \int (\sum_{s \in |S|} x'_s \kappa(r)(\uparrow s))\mu(dr) \\
&= \int m(\mathsf{rep}(\kappa(r)))\mu(dr) \,.
\end{aligned}$$

By Formula (10.1) we have $\|\mathsf{rep}\| \leq 1$ and hence $\mathsf{rep} \in \mathbf{ICones}(\mathsf{Meas}(\mathcal{C}), C)$.

We conclude the proof by observing that meas and rep are inverse of each other. Let first $x \in \underline{C}$, we have, for all $s \in |S|$,

$$\mathsf{rep}(\mathsf{meas}(x))_s = \mathsf{meas}(x)(\uparrow s) = x_s$$

since $\downarrow\uparrow s = \{s\}$. Let now $\mu \in \mathsf{Meas}(\mathcal{C})$ and let $U \in \mathcal{O}(\mathcal{C})$ we have

$$\mathsf{meas}(\mathsf{rep}(\mu))(U) = \sum_{s \in \downarrow U} \mathsf{rep}(\mu)_s = \sum_{s \in \downarrow U} \mu(\uparrow s) = \mu(U)$$

by Formula (10.2). It follows that $\mathsf{meas}(\mathsf{rep}(\mu)) = \mu$. □

Conclusion

Elaborating on earlier work by the first author (together with Michele Pagani and Christine Tasson) on a denotational semantics based on measurable cones and by the second author on a notion of convex QBSs where integration is the fundamental algebraic operation, we have developed a theory of integration for measurable cones, introducing the category of *integrable cones* and of linear morphisms preserving integrals. We have shown that this category is a model of intuitionistic Linear Logic featuring exponential resource comonads; for defining the tensor product and the exponential we have used the special adjoint functor theorem which avoids providing explicit combinatorial constructions of these objects.

The model obtained in that way has many pleasant properties, for instance it contains the category of measurable spaces and finite kernels as well as the category of probabilistic coherence spaces as full subcategories and, for any measurable space, the cone of finite measures on this space is a coalgebra of the exponential comonad, meaning that the measurable spaces feature a canonical sampling primitive based on the general operation of integration that we have introduced.

In future work we will explain how this model can be used for interpreting call-by-value or even call-by-push-value probabilistic functional programming languages with continuous data-types (interpreted as the aforementioned coalgebras) as well as recursive types.

Acknowledgments

This work has been partly funded by the ANR PRC project Probabilistic Programming Semantics (PPS), ANR-19-CE48-0014.

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