MPRI 2–2 Models of programming languages: domains, categories, games

Problem 5: Introduction to probabilistic coherence spaces (PCSs)

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The goal of this problem is to introduce some intuitions and methods which are useful to better understand PCSs.

Warning: here, we call ω-cpo any partially ordered set D where any increasing sequence $(x_n)_{n\in\mathbb{N}}$ of elements of D has a least upper bound (lub) $\sup_{n\in\mathbb{N}} x_n$. When dealing with probabilities, this notion of completeness is more relevant than the standard completeness used in domain theory which is based on arbitrary directed sets. Indeed at some point (not in the context of PCSs actually), one has to use the monotone convergence theorems where countability is absolutely crucial because measurable sets are closed under countable unions and not arbitrary unions.

Accordingly, if D and E are ω-cpos, an increasing function $f : D \to E$ is ω-Scott continuous if f commutes with the lubs of increasing sequences.

Notice that $[0,1] \subseteq \mathbb{R}$, ordered with the standard order relation on the real numbers, is an ω-cpo.

5. (a) Let S be the set of all families $\overrightarrow{a} = (a_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ a_n \in \mathbb{R}_{\geq 0}$ and

$$\sum_{n\in\mathbb{N}} a_n \leq 1\,.$$

We equip S with the product order: $\overrightarrow{a} \leq \overrightarrow{b}$ if $\forall n \in \mathbb{N}\ a_n \leq b_n$. Prove that S is an ω-cpo.

(b) Prove that the function $\sigma : S \to [0,1]$ defined by $\sigma(\overrightarrow{a}) = \sum_{n\in\mathbb{N}} a_n$ is increasing and ω-Scott continuous (in the sense that it commutes with the sups of increasing sequences).

(c) **(*)** Given $\overrightarrow{a} \in S$ we define the function

$$\begin{aligned}\mathsf{Fun}(\overrightarrow{a}) : [0,1] &\to [0,1]\\ t &\mapsto \sum_{n\in\mathbb{N}} a_n t^n\end{aligned}$$

Prove that $\mathsf{Fun}\,\overrightarrow{a}$ is an increasing and ω-Scott continuous function $[0,1] \to [0,1]$. [*Hint:* For ω-Scott continuity, let $(t_k)_{k\in\mathbb{N}}$ be an increasing sequence in $[0,1]$ and let t be its lub, prove that $f(t) \leq \sup_{k\in\mathbb{N}} f(t_k)$. To this end distinguish the two cases $t < 1$ and $t = 1$.]

(d) Prove that if $\overrightarrow{a}, \overrightarrow{b} \in S$ and $\mathsf{Fun}(\overrightarrow{a}) = \mathsf{Fun}(\overrightarrow{b})$ then $\overrightarrow{a} = \overrightarrow{b}$. [*Hint:* For $\overrightarrow{a} \in S$ and $f = \mathsf{Fun}(\overrightarrow{a})$, consider the right-hand derivatives $f^{(n)}(0^+)$.]

(e) We equip S with the pointwise order: $\overrightarrow{a} \leq \overrightarrow{b}$ if $\forall n \in \mathbb{N}\ a_n \leq b_n$. It is clear that $\overrightarrow{a} \leq \overrightarrow{b} \Rightarrow \forall t \in [0,1]\ \ \mathsf{Fun}(\overrightarrow{a})(t) \leq \mathsf{Fun}(\overrightarrow{b})(t)$. Prove that the converse implication does not hold.

Let $\mathcal{S}$ be the set of all functions $f : [0,1] \to [0,1]$ such that there is $\overrightarrow{a} \in S$ such that $f = \mathsf{Fun}(\overrightarrow{a})$. If $f \in \mathcal{S}$, we use $\mathsf{Tr}f$ for the unique $\overrightarrow{a} \in S$ such that $\mathsf{Fun}(\overrightarrow{a}) = f$. We equip $\mathcal{F}$ with the order isomorphism defined by $f \leq g$ if $\mathsf{Tr}(f) \leq \mathsf{Tr}(g)$, so that, equipped with this order relation, $\mathcal{F}$ a cpo.

(f) Let $(f_k)_{k\in\mathbb{N}}$ be an increasing sequence of elements of $\mathcal{S}$ and let $f = \sup_{k\in\mathbb{N}} f_k$. Prove that for all $t \in [0,1]$ the sequence $(f_k(t))_{k\in\mathbb{N}}$ is increasing and that $\forall t \in [0,1]\ f(t) = \sup_{k\in\mathbb{N}} f_k(t)$.

(g) Let $p \in [0,1]$. Given a function $f : [0,1] \to [0,1]$, we define another function $F_p(f) : [0,1] \to [0,1]$ by

$$F_p(f)(t) = (1-p)t + pf(t)^2\,.$$

Prove that if $f \in \mathcal{S}$ then $F_p(f) \in \mathcal{S}$.

(h) Prove that the function $F_p : \mathcal{S} \to \mathcal{S}$ is ω-Scott continuous. Let f_p be the least fixpoint of F_p. Prove that $f_p(0) = 0$.

(i) **(*)** Prove that, for $t \in [0,1]$,

$$\begin{aligned} f_p(t) &= \frac{1-\sqrt{1-4p(1-p)t}}{2p} \quad \text{if } p > 0 \\ f_0(t) &= t \end{aligned}$$

NB: For $p > 0$, $f_p(t)$ is a solution of a quadric equation which has two solutions in general, given by two continuous functions $g(t)$ and $h(t)$. The slight difficulty is to prove that one has to always choose the same solution (say $g(t)$) for $f_p(t)$, for all values of t.

(j) Express as simply as possible the function $g : [0,1] \to [0,1]$ given by $g(p) = f_p(1)$, prove that g is not derivable at $p = \frac{1}{2}$.