MPRI 2–2 Models of programming languages: domains, categories, games

Problem 4: The Eilenberg-Moore of the relational model of LL

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The goal of this problem is to study the Eilenberg Moore category $\mathbf{Rel}^!$ of $\mathbf{Rel}$, the relational model of LL.

Let P be an object of $\mathbf{Rel}^!$ (the category of coalgebras of $!\_$). Remember that $P = (\underline{P}, \mathsf{h}_P)$ where $\underline{P}$ is an object of $\mathbf{Rel}$ (a set) and $\mathsf{h}_P \in \mathbf{Rel}(\underline{P}, !\underline{P})$ satisfies the following commutations:

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\ & \searrow^{\underline{P}} & \downarrow{\scriptstyle \mathsf{der}_{\underline{P}}} \\ & & \underline{P} \end{array} \qquad \begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\ {\scriptstyle \mathsf{h}_P}\downarrow & & \downarrow{\scriptstyle \mathsf{dig}_{\underline{P}}} \\ !\underline{P} & \xrightarrow{!\mathsf{h}_P} & !!\underline{P} \end{array}$$

4. (a) Check that these commutations mean:
   - for all $a, a' \in \underline{P}$, one has $(a, [a']) \in \mathsf{h}_P$ iff $a = a'$
   - and for all $a \in \underline{P}$ and $m_1, \dots, m_k \in !\underline{P}$, one has $(a, m_1 + \cdots + m_k) \in \mathsf{h}_P$ iff there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, m_i) \in \mathsf{h}_P$ for $i = 1, \dots, k$.

   Intuitively, $(a, [a_1, \dots, a_k])$ means that a can be decomposed into "$a_1 + \cdots + a_k$" where the "+" is the decomposition operation associated with P.

   (b) Prove that if P is an object of $\mathbf{Rel}^!$ such that $\underline{P} \neq \emptyset$ then there is at least one element e of $\underline{P}$ such that $(e, [\,]) \in \mathsf{h}_P$. Explain why such an e could be called a "coneutral element of P".

If P and Q are objects of $\mathbf{Rel}^!$, remember that an $f \in \mathbf{Rel}^!(P, Q)$ (morphism of coalgebras) is an $f \in \mathbf{Rel}(\underline{P}, \underline{Q})$ such that the following diagram commutes

$$\begin{array}{ccc} \underline{P} & \xrightarrow{f} & \underline{Q} \\ {\scriptstyle \mathsf{h}_P}\downarrow & & \downarrow{\scriptstyle \mathsf{h}_Q} \\ !\underline{P} & \xrightarrow{!f} & !\underline{Q} \end{array}$$

(c) Check that this commutation means that for all $a \in \underline{P}$ and $b_1, \dots, b_k \in \underline{Q}$, the two following properties are equivalent
   - there is $b \in \underline{Q}$ such that $(a, b) \in f$ and $(b, [b_1, \dots, b_k]) \in \mathsf{h}_Q$
   - there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, b_i) \in f$ for $i = 1, \dots, k$.

The object 1 of $\mathbf{Rel}$ (the set $\{*\}$) can be equipped with a structure of coalgebra (still denoted 1) with $\mathsf{h}_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$.

(d) Prove that the elements of $\mathbf{Rel}^!(1, P)$ can be identified with the subsets x of $\underline{P}$ such that, for all $a_1, \dots, a_k \in \underline{P}$, one has $a_1, \dots, a_k \in x$ iff there exists $a \in x$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$. We call *values* of P these subsets of $\underline{P}$ and denote as $\mathsf{val}(P)$ the set of these values.

Prove that an element of $\mathsf{val}(P)$ is never empty and that $\mathsf{val}(P)$, equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to $\subseteq$) is still a value.

Remember that if E is an object of $\mathbf{Rel}$ then $(!E, \mathsf{dig}_E)$ is an object of $\mathbf{Rel}^!$ (the free coalgebra generated by E, that we can identify with an object of the Kleisli category $\mathbf{Rel}_!$).

(e) Exhibit an order isomorphism between the cpos $\mathsf{val}(!E, \mathsf{dig}_E)$ and $\mathcal{P}(E)$ (both ordered by set inclusion).

For any categorical model $\mathcal{L}$ of LL, the category $\mathcal{L}^!$ is cartesian; we specialize the corresponding general definitions to the case $\mathcal{L} = \mathbf{Rel}$. The product of P_1 and P_2 is $P_1 \otimes P_2$, the coalgebra defined by $\underline{P_1 \otimes P_2} = \underline{P_1} \otimes \underline{P_2}$ and $\mathsf{h}_{P_1 \otimes P_2}$ is the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2}} !\underline{P_1} \otimes !\underline{P_2} \xrightarrow{\mu^2_{\underline{P_1}, \underline{P_2}}} !(\underline{P_1} \otimes \underline{P_2})$$

where $\mu^2_{E_1,E_2} \in \mathbf{Rel}(!E_1 \otimes !E_2, !(E_1 \otimes E_2))$ is the lax monoidality natural transformation of $!\_$, remember that in $\mathbf{Rel}$ we have

$$\mu^2_{E_1,E_2} = \{(([a_1, \dots, a_k], [b_1, \dots, b_k]), [(a_1, b_1), \dots, (a_k, b_k)]) \mid \\ k \in \mathbb{N} \text{ and } (a_1, b_1), \dots, (a_k, b_k) \in E_1 \times E_2\}\,.$$

Concretely, we have simply that $((a_1, a_2), [(a_1^1, a_2^1), \dots, (a_1^k, a_2^k)]) \in \mathsf{h}_{P_1 \otimes P_2}$ iff $(a_i, [a_i^1, \dots, a_i^k]) \in \mathsf{h}_{P_i}$ for $i = 1, 2$.

(f) Prove that $P_1 \otimes P_2$, equipped with projections $(\mathsf{pr}_l^{\otimes})_{l=1,2}$ defined by:

$$\mathsf{pr}_1^{\otimes} = \{((a, e), a) \mid a \in \underline{P_1} \text{ and } (e, []) \in \mathsf{h}_{P_2}\}\,,$$

and similarly for $\mathsf{pr}_2^{\otimes}$, is the cartesian product of P_1 and P_2 in $\mathbf{Rel}^!$. Warning: $\mathcal{L}^!$ is always cartesian when $\mathcal{L}$ is a model of LL; you are not expected to give a general proof of this fact, just a verification that this is true in $\mathbf{Rel}^!$.

(g) Let $(f_l \in \mathbf{Rel}^!(P_l, Q_l))_{l=1,2}$. Prove that the morphism $\langle f_1\, \mathsf{pr}_1^{\otimes}, f_2\, \mathsf{pr}_2^{\otimes} \rangle^{\otimes} \in \mathbf{Rel}^!(P_1 \otimes P_2, Q_1 \otimes Q_2)$ coincides with $f_1 \otimes f_2$ (which, by definition, belongs to $\mathbf{Rel}(\underline{P_1} \otimes \underline{P_2}, \underline{Q_1} \otimes \underline{Q_2})$).

(h) Prove that 1 is the terminal object of $\mathbf{Rel}^!$.

(i) Let $(P_l)_{l \in I}$ be a collection of objects of $\mathbf{Rel}^!$ (I is an arbitrary indexing set). Remember that for each $l \in I$ we have an injection morphism

$$\mathsf{in}_l = \{(a, (l, a)) \mid a \in \underline{P_l}\} \in \mathbf{Rel}(\underline{P_l}, \oplus_{k \in I} \underline{P_k})$$

and that for any collection $(t_l \in \mathbf{Rel}(\underline{P_l}, \underline{Q}))$, there is exactly one morphism $t \in \mathbf{Rel}(\oplus_{l \in I} \underline{P_l}, h_Q)$ such that $\forall l \in I\ t\, \mathsf{in}_l = t_l$, namely

$$t = [t_l]_{l \in I} = \{((l, a), b) \mid l \in I \text{ and } (a, b) \in t_l\}\,.$$

In other words, $(\oplus_{l \in I} \underline{P_l}, (\mathsf{in}_l)_{l \in I})$ is the coproduct of the $\underline{P_l}$'s in $\mathbf{Rel}$. Given a multiset $m = [a_1, \dots, a_k]$ and $l \in I$, we set $l \cdot m = [(l, a_1), \dots, (l, a_k)]$. Let

$$h = \{((l, a), l \cdot m) \mid l \in I \text{ and } (a, m) \in \mathsf{h}_{P_l}\} \in \mathbf{Rel}(X, !X)$$

where $X = \oplus_{l \in I} \underline{P_l}$. Prove that (X, h) is an object of $\mathbf{Rel}^!$, for which we will use the notation $\oplus_{l \in I} P_l$.

(j) Exhibit an order isomorphism between the cpo $\mathsf{val}(\oplus_{l \in I} P_l)$ and $\bigcup_{l \in I} \{l\} \times \mathsf{val}(P_l)$ ordered as follows: $(l, x) \leq (l', x')$ iff $l = l'$ and $x \subseteq x'$.

(k) Is it always true that if $x_1, x_2 \in \mathsf{val}(P)$ then $x_1 \cup x_2 \in \mathsf{val}(P)$?

(l) We admit that $(\mathsf{in}_l \in \mathbf{Rel}^!(P_l, \oplus_{k\in I} P_k))_{l\in I}$ and that, given $(f_l \in \mathbf{Rel}^!(P_l, Q))_{l\in I}$, one has $([f_l]_{l\in I}) \in \mathbf{Rel}^!(\oplus_{l\in I} P_l, Q)$. The patient reader can check these facts, which show that $(\oplus_{l\in I} P_l, (\mathsf{in}_l)_{l\in I})$ is the coproduct of the P_l's in the category $\mathbf{Rel}^!$.

Let P be an object of $\mathbf{Rel}^!$. Using the universal property of the coproduct, exhibit a morphism $d \in \mathbf{Rel}^!(\oplus_{l\in I}(P \otimes P_l), P \otimes (\oplus_{l\in I})P_l)$, describe d explicitely and prove that is an iso.

(m) So cartesian products distribute over coproducts in $\mathbf{Rel}^!$, prove that the similar property does not hold in $\mathbf{Rel}$.