MPRI 2–2 Models of programming languages: domains, categories, games

Problem 3: Linear and stable functions on coherence spaces

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A coherence spaces is a structure $E = (|E|, \coh_E)$ where $\coh_E$ is a binary symmetric and reflexive relation on the set $|E|$. The strict coherence relation is $\scoh_E$ is defined by: $a \scoh_E b$ if $a \coh_E b$ and $a \neq b$, which is a symmetric and antireflexive relation.

We use $\mathsf{Cl}(E)$ for the set of cliques of E, that is $\mathsf{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \coh_E a'\}$. Ordered by inclusion, this is a cpo.

We shall say that a family $(x_i \in \mathsf{Cl}(E))_{i \in I}$ is *summable* if $\forall i, j \in I\ i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$. When this is the case, we set $\sum_{i \in I} x_i = \bigcup_{i \in I} x_i$. Accordingly, if $x_1, x_2 \in \mathsf{Cl}(E)$ are summable, we write $x_1 + x_2 = x_1 \cup x_2$.

Let E and F be coherence spaces and let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$. We use the following terminology (warning: Paul-André Melliès may have used another one!).

- f is Scott-continuous if it is increasing, and, for any directed set $D \subseteq \mathsf{Cl}(E)$ (remember that this means that $D \neq \emptyset$ and $\forall x_1, x_2 \in D\, \exists x \in D\ x_1, x_2 \subseteq x$), we have $f(\bigcup D) = \bigcup f(D) = \bigcup\{f(x) \mid x \in D\}$. Since f is increasing, this condition is equivalent to $f(\bigcup D) \subseteq \bigcup f(D)$.

- f is stable if it is Scott-continuous and

$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cap x_2) = f(x_1) \cap f(x_2)$$

  that is, since f is increasing:

$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cap x_2) \supseteq f(x_1) \cap f(x_2)\,.$$

- f is linear if it is stable and

$$f(\emptyset) = \emptyset$$
$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cup x_2) = f(x_1) \cup f(x_2)\,.$$

  that is, since f is increasing:

$$f(\emptyset) = \emptyset$$
$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cup x_2) \subseteq f(x_1) \cup f(x_2)\,.$$

Remember also that if E and F are coherence spaces, the coherence space $E \multimap F$ is defined by $|E \multimap F| = |E| \times |F|$ and, for $(a, b), (a', b') \in |E \multimap F|$, one has $(a, b) \coh_{E \multimap F} (a', b')$ if:

$$a \coh_E a' \Rightarrow (b \coh_F b' \text{ and } (b = b' \Rightarrow a = a'))\,.$$

We define a coherence space $\mathbb{D}$ by $|\mathbb{D}| = \{0, 1\}$ with $0 \scoh_{\mathbb{D}} 1$, that is $\mathbb{D} = 1 \,\&\, 1$.

We use **Cohs** for the category of coherence spaces and stable functions.

3. (a) Let E and F be coherence spaces and let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be increasing and Scott-continuous. Prove that f is linear if and only if it is additive, that is:
   - $f(\varnothing) = \varnothing$
   - and for any $x_1, x_2 \in \mathsf{Cl}(E)$, if x_1 and x_2 are summable in $\mathsf{Cl}(E)$, then $f(x_1)$ and $f(x_2)$ are summable in $\mathsf{Cl}(F)$ and one has $f(x_1 + x_2) = f(x_1) + f(x_2)$.

   (b) Let $\mathsf{S}E = (\mathbb{D} \multimap E)$, prove that there is an order isomorphism

$$\mathsf{Cl}(\mathsf{S}E) \simeq \{(x,u) \in \mathsf{Cl}(E)^2 \mid x \text{ and } u \text{ are summable}\} \text{ (with the product order).}$$

   If $x, u \in \mathsf{Cl}(E)$ are summable, we use $\langle\!\langle x, u \rangle\!\rangle$ for the corresponding element of $\mathsf{Cl}(\mathsf{S}E)$.

   (c) Let $\langle\!\langle x_1, u_1 \rangle\!\rangle, \langle\!\langle x_2, u_2 \rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$, prove that $\langle\!\langle x_1, u_1 \rangle\!\rangle \cap \langle\!\langle x_2, u_2 \rangle\!\rangle = \langle\!\langle x_1 \cap x_2, u_1 \cap u_2 \rangle\!\rangle$. Assuming moreover that $\langle\!\langle x_1, u_1 \rangle\!\rangle \cup \langle\!\langle x_2, u_2 \rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$, prove that $(x_1 + u_1) \cup (x_2 + u_2) \in \mathsf{Cl}(E)$ and that $(x_1 + u_1) \cap (x_2 + u_2) = (x_1 \cap x_2) + (u_1 \cap u_2)$.

   (d) Let $u \in \mathsf{Cl}(E)$. We define a coherence space E_u by $|E_u| = \{a \in |E| \mid u \text{ and } \{a\} \text{ are summable}\}$ and $a \coh_{E_u} a'$ if $a \coh_E a'$, so that $\mathsf{Cl}(E_u) = \{x \in \mathsf{Cl}(E) \mid x \text{ and } u \text{ are summable}\}$ as easily checked. Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be Scott-continuous and stable.
   We define a function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathcal{P}(|F|)$ by

$$\Delta_u f(x) = f(x+u) \setminus f(x)\,.$$

   Check that $\Delta_u f(x) \in \mathsf{Cl}(F)$. Prove that $\Delta_u f$ is increasing and Scott-continuous. Prove that $\Delta_u f$ is stable.

   (e) **(*)** Conversely let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be Scott-continuous and assume that, for all $u \in \mathsf{Cl}(E)$, the function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathsf{Cl}(F)$ is increasing. Prove that f is stable.

   (f) **(*)** Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be stable. We define a function

$$\begin{aligned} \Delta f : \mathsf{S}E &\to \mathsf{S}F \\ \langle\!\langle x, u \rangle\!\rangle &\mapsto \langle\!\langle f(x), \Delta_u f(x) \rangle\!\rangle\,. \end{aligned}$$

   Prove that Δf is a stable function.

   (g) Prove that the operation defined in Question **(f)** is a functor $\Delta : \mathbf{Cohs} \to \mathbf{Cohs}$ which acts on objects by $\Delta E = \mathsf{S}E$. In other words prove that $\Delta\,\mathsf{Id} = \mathsf{Id}$ and, for $f \in \mathbf{Cohs}(E,F)$ and $g \in \mathbf{Cohs}(F,G)$, one has $\Delta(g \circ f) = \Delta g \circ \Delta f$.

We use $\mathsf{Cl}_{\mathsf{fin}}(E)$ for the set of finite cliques of E.

   (h) Remember that if $f \in \mathbf{Cohs}(E,F)$, one defines $\mathsf{Tr} f$ as the set of all pairs $(x_0, b) \in \mathsf{Cl}_{\mathsf{fin}}(E) \times |F|$ such that $b \in f(x_0)$ and x_0 is minimal with this property, that is: $\forall x \subseteq x_0\ b \in f(x) \Rightarrow x = x_0$. Prove that

$$\begin{aligned} \mathsf{Tr}(\Delta f) = \{(\langle\!\langle x_0, \varnothing \rangle\!\rangle, (0,b)) \in \mathsf{Cl}_{\mathsf{fin}}(\mathsf{S}E) \times |\mathsf{S}F| \mid (x_0, b) \in \mathsf{Tr} f\} \\ \{(\langle\!\langle x_0, u_0 \rangle\!\rangle, (1,b)) \in \mathsf{Cl}_{\mathsf{fin}}(\mathsf{S}E) \times |\mathsf{S}F| \mid (x_0 + u_0, b) \in \mathsf{Tr} f \text{ and } u_0 \neq \varnothing\}\,. \end{aligned}$$