MPRI 2–2 Models of programming languages: domains, categories, games

Problem 3: Linear and stable functions on coherence spaces

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A coherence spaces is a structure $E = (|E|, \coh_E)$ where $\coh_E$ is a binary symmetric and reflexive relation on the set $|E|$. The strict coherence relation is $\scoh_E$ is defined by: $a \scoh_E b$ if $a \coh_E b$ and $a \neq b$, which is a symmetric and antireflexive relation.

We use $\mathsf{Cl}(E)$ for the set of cliques of E, that is $\mathsf{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \coh_E a'\}$. Ordered by inclusion, this is a cpo.

We shall say that a family $(x_i \in \mathsf{Cl}(E))_{i \in I}$ is *summable* if $\forall i, j \in I\ i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$. When this is the case, we set $\sum_{i \in I} x_i = \bigcup_{i \in I} x_i$. Accordingly, if $x_1, x_2 \in \mathsf{Cl}(E)$ are summable, we write $x_1 + x_2 = x_1 \cup x_2$.

Let E and F be coherence spaces and let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$. We use the following terminology (warning: Paul-André Melliès may have used another one!).

- f is Scott-continuous if it is increasing, and, for any directed set $D \subseteq \mathsf{Cl}(E)$ (remember that this means that $D \neq \emptyset$ and $\forall x_1, x_2 \in D\ \exists x \in D\ x_1, x_2 \subseteq x$), we have $f(\bigcup D) = \bigcup f(D) = \bigcup \{f(x) \mid x \in D\}$. Since f is increasing, this condition is equivalent to $f(\bigcup D) \subseteq \bigcup f(D)$.

- f is stable if it is Scott-continuous and

$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cap x_2) = f(x_1) \cap f(x_2)$$

  that is, since f is increasing:

$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cap x_2) \supseteq f(x_1) \cap f(x_2)\,.$$

- f is linear if it is stable and

$$f(\emptyset) = \emptyset$$
$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cup x_2) = f(x_1) \cup f(x_2)\,.$$

  that is, since f is increasing:

$$f(\emptyset) = \emptyset$$
$$\forall x_1, x_2 \quad x_1 \cup x_2 \in \mathsf{Cl}(E) \Rightarrow f(x_1 \cup x_2) \subseteq f(x_1) \cup f(x_2)\,.$$

Remember also that if E and F are coherence spaces, the coherence space $E \multimap F$ is defined by $|E \multimap F| = |E| \times |F|$ and, for $(a, b), (a', b') \in |E \multimap F|$, one has $(a, b) \coh_{E \multimap F} (a', b')$ if:

$$a \coh_E a' \Rightarrow (b \coh_F b' \text{ and } (b = b' \Rightarrow a = a'))\,.$$

We define a coherence space $\mathbb{D}$ by $|\mathbb{D}| = \{0, 1\}$ with $0 \scoh_{\mathbb{D}} 1$, that is $\mathbb{D} = 1 \mathbin{\&} 1$.

We use **Cohs** for the category of coherence spaces and stable functions.

3. (a) Let E and F be coherence spaces and let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be increasing and Scott-continuous. Prove that f is linear if and only if it is additive, that is:

   - $f(\varnothing) = \varnothing$
   - and for any $x_1, x_2 \in \mathsf{Cl}(E)$, if x_1 and x_2 are summable in $\mathsf{Cl}(E)$, then $f(x_1)$ and $f(x_2)$ are summable in $\mathsf{Cl}(F)$ and one has $f(x_1 + x_2) = f(x_1) + f(x_2)$.

Solution: Assume first that f is linear. Let $x_1, x_2 \in \mathsf{Cl}(E)$ be summable, that is $x_1 \cup x_2 \in \mathsf{Cl}(E)$ and $x_1 \cap x_2 = \varnothing$. Since f is increasing we have $f(x_1) \cup f(x_2) \subseteq f(x_1 \cup x_2) \in \mathsf{Cl}(F)$, and since f is stable we have $f(x_1) \cap f(x_2) = f(x_1 \cap x_2) = f(\varnothing) = \varnothing$ by linearity of f, so $f(x_1)$ and $f(x_2)$ are summable. We have $f(x_1 + x_2) = f(x_1) + f(x_2)$ by linearity of f.

Assume conversely that f is additive. We prove first that f is stable, so let $x_1, x_2 \in \mathsf{Cl}(E)$ be such that $x_1 \cup x_2 \in \mathsf{Cl}(E)$, we must prove that $f(x_1) \cap f(x_2) \subseteq f(x_1 \cap x_2)$. Let $x_1' = x_1 \setminus (x_1 \cap x_2) \in \mathsf{Cl}(E)$ then x_1' and $x_1 \cap x_2$ are summable and $x_1' + (x_1 \cap x_2) = x_1$. Since f is additive we have $f(x_1') \cap f(x_1 \cap x_2) = \varnothing$ and $f(x_1) = f(x_1') + f(x_1 \cap x_2)$. Similarly, setting $x_2' = x_2 \setminus (x_1 \cap x_2) \in \mathsf{Cl}(E)$ we have $f(x_2') \cap f(x_1 \cap x_2) = \varnothing$ and $f(x_2) = f(x_2') + f(x_1 \cap x_2)$. Since $x_1 \cup x_2 \in \mathsf{Cl}(E)$ and $x_1' \cap x_2' = \varnothing$, x_1' and x_2' are summable, so that $f(x_1') \cap f(x_2') = \varnothing$. Using distributivity of $\cap$ over $\cup$ we get $f(x_1) \cap f(x_2) = f(x_1 \cap x_2)$.

With the same assumptions we must also prove that $f(x_1 \cup x_2) \subseteq f(x_1) \cup f(x_2)$. We have

$$\begin{aligned}
f(x_1 \cup x_2) &= f(x_1' + (x_1 \cap x_2) + x_2') \\
&= f(x_1') + f(x_1 \cap x_2) + f(x_2') \\
&= (f(x_1') \cup f(x_1 \cap x_2)) \cup (f(x_1 \cap x_2) \cup f(x_2')) \\
&= (f(x_1') + f(x_1 \cap x_2)) \cup (f(x_1 \cap x_2) + f(x_2')) \\
&= f(x_1) \cup f(x_2)
\end{aligned}$$

and hence f is linear.

(b) Let $\mathsf{S}E = (\mathbb{D} \multimap E)$, prove that there is an order isomorphism

$$\mathsf{Cl}(\mathsf{S}E) \simeq \{(x, u) \in \mathsf{Cl}(E)^2 \mid x \text{ and } u \text{ are summable}\} \text{ (with the product order).}$$

If $x, u \in \mathsf{Cl}(E)$ are summable, we use $\langle\!\langle x, u \rangle\!\rangle$ for the corresponding element of $\mathsf{Cl}(\mathsf{S}E)$.

Solution: We have $|\mathbb{D}| = \{0, 1\}$ with $0 \frown_{\mathbb{D}} 1$. We have $|\mathsf{S}E| = \{0, 1\} \times |E|$, and the coherence is given by $(i, a) \coh_{\mathsf{S}E} (j, b)$ if $i = j$ and $a \coh_E b$, or $i \neq j$ and $a \frown_E b$. In particular $(i, a) \coh_{\mathsf{S}E} (j, b) \Rightarrow a \coh_E b$.

Given $x \in \mathsf{Cl}(\mathsf{S}E)$ and $i \in \{0, 1\}$, we set $\pi_i(x) = \{a \in |E| \mid (i, a) \in x\}$. From the description above of $\mathsf{S}E$, $\pi_0(x)$ and $\pi_1(x)$ are summable cliques of E.

Conversely if $x, u \in \mathsf{Cl}(E)$ are summable, then $\langle\!\langle x, u \rangle\!\rangle = \{0\} \times x \cup \{1\} \times u \in \mathsf{Cl}(\mathsf{S}E)$.

(c) Let $\langle\!\langle x_1, u_1 \rangle\!\rangle, \langle\!\langle x_2, u_2 \rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$, prove that $\langle\!\langle x_1, u_1 \rangle\!\rangle \cap \langle\!\langle x_2, u_2 \rangle\!\rangle = \langle\!\langle x_1 \cap x_2, u_1 \cap u_2 \rangle\!\rangle$. Assuming moreover that $\langle\!\langle x_1, u_1 \rangle\!\rangle \cup \langle\!\langle x_2, u_2 \rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$, prove that $(x_1 + u_1) \cup (x_2 + u_2) \in \mathsf{Cl}(E)$ and that $(x_1 + u_1) \cap (x_2 + u_2) = (x_1 \cap x_2) + (u_1 \cap u_2)$.

Solution: For the first equation we have

$$\begin{aligned}
\langle\!\langle x_1, u_1 \rangle\!\rangle \cap \langle\!\langle x_2, u_2 \rangle\!\rangle &= (\{0\} \times x_1 \cup \{1\} \times u_1) \cap (\{0\} \times x_2 \cup \{1\} \times u_2) \\
&= (\{0\} \times x_1 \cap \{0\} \times x_2) \cup (\{0\} \times x_1 \cap \{1\} \times u_2) \\
&\quad \cup (\{1\} \times u_1 \cap \{0\} \times x_2) \cup (\{1\} \times u_1 \cap \{1\} \times u_2) \\
&= \{0\} \times (x_1 \cap x_2) \cup \varnothing \cup \varnothing \cup \{1\} \times (u_1 \cap u_2) \\
&= \langle\!\langle x_1 \cap x_2, u_1 \cap u_2 \rangle\!\rangle .
\end{aligned}$$

Assume moreover that $\langle\!\langle x_1, u_1\rangle\!\rangle \cup \langle\!\langle x_2, u_2\rangle\!\rangle = \{0\} \times (x_1 \cup x_2) \cup \{1\} \times (u_1 \cup u_2) \in \mathsf{Cl}(\mathsf{S}E)$, that is: $x_1 \cup x_2 \cup u_1 \cup u_2 \in \mathsf{Cl}(E)$ and $(x_1 \cup x_2) \cap (u_1 \cup u_2) = \varnothing$. The announced equation results from the fact that intersections distribute over unions and $x_1 \cap u_2, x_2 \cap u_1 \subseteq (x_1 \cup x_2) \cap (u_1 \cup u_2) = \varnothing$. Since $\{0\} \times (x_1 \cup x_2) \cup \{1\} \times (u_1 \cup u_2) \in \mathsf{Cl}(\mathsf{S}E)$, we have $x_1 \cup x_2 \cup u_1 \cup u_2 \in \mathsf{Cl}(E)$, that is $(x_1 + u_1) \cup (x_2 + u_2) \in \mathsf{Cl}(E)$.

(d) Let $u \in \mathsf{Cl}(E)$. We define a coherence space E_u by $|E_u| = \{a \in |E| \mid u \text{ and } \{a\} \text{ are summable}\}$ and $a \coh_{E_u} a'$ if $a \coh_E a'$, so that $\mathsf{Cl}(E_u) = \{x \in \mathsf{Cl}(E) \mid x \text{ and } u \text{ are summable}\}$ as easily checked. Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be Scott-continuous and stable.
We define a function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathcal{P}(|F|)$ by

$$\Delta_u f(x) = f(x + u) \setminus f(x)\,.$$

Check that $\Delta_u f(x) \in \mathsf{Cl}(F)$. Prove that $\Delta_u f$ is increasing and Scott-continuous. Prove that $\Delta_u f$ is stable.

Solution: We have $f(x+u) \setminus f(x) \subseteq f(x+u) \in \mathsf{Cl}(E_{f(u)})$ and hence $f(x+u) \setminus f(x) \in \mathsf{Cl}(E)$.
Let $x \subseteq x' \in \mathsf{Cl}(E_u)$, we prove that $\Delta_u f(x) \subseteq \Delta_u f(x')$ so let $b \in \Delta_u f(x) = f(x+u) \setminus f(x)$. If $b \notin \Delta_u f(x')$, this means that $b \in f(x')$ since we have $b \in f(x' + u)$ because f is increasing. Since $x+u, x' \subseteq x'+u$, we must have $b \in f((x+u) \cap x')$ because f is stable. But $(x+u) \cap x' = x$ since $u \cap x' = \varnothing$, which contradicts the fact that $b \notin \Delta_u f(x)$.
Let $D \subseteq \mathsf{Cl}(E_u)$ be directed. Since $\Delta_u f$ is increasing, we know that $\bigcup \Delta_u f(D) \subseteq \Delta_u f(\bigcup D)$, so let us prove the converse inclusion and let $b \in \Delta_u f(\bigcup D) = f((\bigcup D) + u) \setminus f(\bigcup D)$. The set $D + u = \{x + u \mid x \in D\}$ is directed in $\mathsf{Cl}(E)$ (remember that for all $x \in D$, x and u are summable). And we have $\bigcup(D + u) = (\bigcup D) + u$. So by continuity of f we have $f((\bigcup D) + u) = \bigcup f(D + u)$ and hence there is $x \in D$ such that $b \in f(x + u)$. Since $b \notin f(\bigcup D) \supseteq f(x)$, we have $b \in \Delta_u f(x)$.
Last let $x_1, x_2 \in \mathsf{Cl}(E_u)$ be such that $x_1 \cup x_2 \in \mathsf{Cl}(E_u)$ (which is equivalent to $x_1 \cup x_2 \in \mathsf{Cl}(E)$). We only have to prove that $\Delta_u f(x_1) \cap \Delta_u f(x_2) \subseteq \Delta_u f(x_1 \cap x_2)$, so let $b \in \Delta_u f(x_1) \cap \Delta_u f(x_2)$, that is $b \in f(x_1+u) \cap f(x_2+u)$ and $b \notin f(x_1) \cup f(x_2)$. Since $(x_1+u) \cup (x_2+u) = (x_1 \cup x_2) + u \in \mathsf{Cl}(E)$, we have $b \in f((x_1 + u) \cap (x_2 + u))$ by stability of f. Moreover $b \notin f(x_1 \cap x_2)$ because f is increasing, and hence $b \in \Delta_u f(x_1 \cap x_2)$, since $(x_1 + u) \cap (x_2 + u) = (x_1 \cap x_2) + u$. So $\Delta_u f$ is stable as contended.

(e) **(*)** Conversely let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be Scott-continuous and assume that, for all $u \in \mathsf{Cl}(E)$, the function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathsf{Cl}(F)$ is increasing. Prove that f is stable.

Solution: Let $x_1, x_2 \in \mathsf{Cl}(E_u)$ with $x_1 \cup x_2 \in \mathsf{Cl}(E_u)$, and let us prove that $f(x_1) \cap f(x_2) \subseteq f(x_1 \cap x_2)$. Let $b \in f(x_1) \cap f(x_2)$ and assume that $b \notin f(x_1 \cap x_2)$. Let $x = x_1 \cap x_2$ and $u = x_1 \setminus (x_1 \cap x_2)$. By our assumptions we have $b \in \Delta_u f(x)$ since $x + u = x_1$. Moreover we have $x \in U_u$, and also $x_2 \in U_u$ since $x_1 \cup x_2 \in \mathsf{Cl}(E)$ and $x_2 \cap (x_1 \setminus (x_1 \cap x_2)) = \varnothing$. Since $\Delta_u f$ is increasing we have $\Delta_u f(x) \subseteq \Delta_u f(x_2)$ and hence $b \in \Delta_u f(x_2) = f(x_2 + (x_1 \setminus (x_1 \cap x_2))) \setminus f(x_2) = f(x_1 \cup x_2) \setminus f(x_2)$ which contradicts our assumption that $b \in f(x_2)$.

(f) **(*)** Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be stable. We define a function

$$\begin{aligned} \Delta f : \mathsf{S}E &\to \mathsf{S}F \\ \langle\!\langle x, u\rangle\!\rangle &\mapsto \langle\!\langle f(x), \Delta_u f(x)\rangle\!\rangle\,. \end{aligned}$$

Prove that Δf is a stable function.

Solution: If $z = \langle\!\langle x, u\rangle\!\rangle, z' = \langle\!\langle x', u'\rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$, then $z \subseteq z'$ holds iff $x \subseteq x'$ and $u \subseteq u'$. Remember also that we set $\pi_0 z = x$ and $\pi_1 z = u$. A subset D of $\mathsf{Cl}(\mathsf{S}E)$ is directed iff $\pi_i D$ is directed for $i = 0, 1$, and $\bigcup D = \langle\!\langle \bigcup \pi_0 D, \bigcup \pi_1 D\rangle\!\rangle$.

With the notations above assume that $z \subseteq z'$, we have $f(x) \subseteq f(x')$ because f is increasing, and $\Delta_u f(x) \subseteq \Delta_u f(x')$ by Question **(d)**. Last we have $\Delta_u f(x') \subseteq \Delta_{u'} f(x')$ because f is increasing. This shows that Δf is increasing.
Next let $D \subseteq \mathsf{Cl}(\mathsf{S}E)$ be directed. Let $\langle\!\langle x, u \rangle\!\rangle = \bigcup D$. We have

$$
\begin{aligned}
\Delta f(\bigcup D) &= \langle\!\langle f(x), \Delta_u f(x) \rangle\!\rangle \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup \Delta_u f(\pi_0 D) \rangle\!\rangle \quad \text{since } f \text{ and } \Delta_u f \text{ are Scott-continuous} \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} (f(\pi_0 z + u) \setminus f(\pi_0 z)) \rangle\!\rangle \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} (f(\pi_0 z + \bigcup_{z' \in D} \pi_1 z') \setminus f(\pi_0 z)) \rangle\!\rangle \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} (\bigcup_{z' \in D} f(\pi_0 z + \pi_1 z') \setminus f(\pi_0 z)) \rangle\!\rangle \quad \text{by continuity of } f \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} \bigcup_{z' \in D} (f(\pi_0 z + \pi_1 z') \setminus f(\pi_0 z)) \rangle\!\rangle \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} (f(\pi_0 z + \pi_1 z) \setminus f(\pi_0 z)) \rangle\!\rangle \quad \text{because } D \text{ is directed} \\
&= \langle\!\langle \bigcup f(\pi_0 D), \bigcup_{z \in D} \Delta_{\pi_1 z} f(\pi_0 z) \rangle\!\rangle \\
&= \bigcup_{z \in D} \langle\!\langle f(z), \Delta_{\pi_1 z} f(\pi_0 z) \rangle\!\rangle = \bigcup_{z \in D} \Delta f(z)
\end{aligned}
$$

Let $z_i = \langle\!\langle x_i, u_i \rangle\!\rangle \in \mathsf{Cl}(\mathsf{S}E)$ for $i = 1, 2$ be such that $z_1 \cup z_2 \in \mathsf{S}E$, we prove that $\Delta f(z_1) \cap \Delta f(z_2) \subseteq \Delta f(z_1 \cap z_2)$. We have

$$
\begin{aligned}
\Delta f(z_1) \cap \Delta f(z_2) &= \langle\!\langle f(x_1), \Delta_{u_1} f(x_1) \rangle\!\rangle \cap \langle\!\langle f(x_2), \Delta_{u_2} f(x_2) \rangle\!\rangle \\
&= \langle\!\langle f(x_1) \cap f(x_2), \Delta_{u_1} f(x_1) \cap \Delta_{u_2} f(x_2) \rangle\!\rangle
\end{aligned}
$$

by Question **(d)** and

$$\Delta f(z_1 \cap z_2) = \langle\!\langle f(x_1 \cap x_2), \Delta_{u_1 \cap u_2} f(x_1 \cap x_2) \rangle\!\rangle$$

so since f is stable we only have to prove that $\Delta_{u_1} f(x_1) \cap \Delta_{u_2} f(x_2) \subseteq \Delta_{u_1 \cap u_2} f(x_1 \cap x_2)$. Let $b \in \Delta_{u_1} f(x_1) \cap \Delta_{u_2} f(x_2)$, we have $b \in f(x_1 + u_1) \cap f(x_2 + u_2)$, and since $(x_1 + u_1) \cup (x_2 + u_2) \in \mathsf{Cl}(E)$ by Question **(d)** we have $b \in f((x_1 + u_1) \cap (x_2 + u_2))$ since f is stable. Therefore $b \in f((x_1 \cap x_2) + (u_1 \cap u_2))$ by Question **(d)** again. Moreover, since $b \in \Delta_{u_1} f(x_1)$ we have $b \notin f(x_1)$ and hence $b \notin f(x_1 \cap x_2)$ since f is increasing. Therefore $b \in \Delta_{u_1 \cap u_2} f(x_1 \cap x_2)$.

(g) Prove that the operation defined in Question **(f)** is a functor $\Delta : \mathbf{Cohs} \to \mathbf{Cohs}$ which acts on objects by $\Delta E = \mathsf{S}E$. In other words prove that $\Delta\, \mathsf{Id} = \mathsf{Id}$ and, for $f \in \mathbf{Cohs}(E, F)$ and $g \in \mathbf{Cohs}(F, G)$, one has $\Delta(g \circ f) = \Delta g \circ \Delta f$.

Solution: We have

$$\Delta\, \mathsf{Id}_E(\langle\!\langle x, u \rangle\!\rangle) = \langle\!\langle x, (x + u) \setminus x \rangle\!\rangle = \langle\!\langle x, u \rangle\!\rangle$$

and hence $\Delta\, \mathsf{Id}_E = \mathsf{Id}_{\mathsf{S}E}$.
Next we have

$$
\begin{aligned}
\Delta g(\Delta f(\langle\!\langle x, u \rangle\!\rangle)) &= \Delta g(\langle\!\langle f(x), f(x + u) \setminus f(x) \rangle\!\rangle) \\
&= \langle\!\langle g(f(x)), g(f(x) + (f(x + u) \setminus f(x))) \setminus g(f(x)) \rangle\!\rangle \\
&= \langle\!\langle g(f(x)), g(f(x + u)) \setminus g(f(x)) \rangle\!\rangle \\
&= \Delta(g \circ f)(\langle\!\langle x, u \rangle\!\rangle)\,.
\end{aligned}
$$

We use $\mathsf{Cl}_{\mathsf{fin}}(E)$ for the set of finite cliques of E.

(h) Remember that if $f \in \mathbf{Cohs}(E, F)$, one defines $\mathsf{Tr} f$ as the set of all pairs $(x_0, b) \in \mathsf{Cl}_{\mathsf{fin}}(E) \times |F|$ such that $b \in f(x_0)$ and x_0 is minimal with this property, that is: $\forall x \subseteq x_0\ b \in f(x) \Rightarrow x = x_0$. Prove that

$$\begin{aligned}\mathsf{Tr}(\Delta f) = &\{(\langle\!\langle x_0, \varnothing \rangle\!\rangle, (0, b)) \in \mathsf{Cl}_{\mathsf{fin}}(\mathsf{S}E) \times |\mathsf{S}F| \mid (x_0, b) \in \mathsf{Tr} f\} \\ &\{(\langle\!\langle x_0, u_0 \rangle\!\rangle, (1, b)) \in \mathsf{Cl}_{\mathsf{fin}}(\mathsf{S}E) \times |\mathsf{S}F| \mid (x_0 + u_0, b) \in \mathsf{Tr} f \text{ and } u_0 \neq \varnothing\}\,.\end{aligned}$$

Solution: Easy!