MPRI 2–2 Models of programming languages: domains, categories, games

Problem 2: The monoidal structure of complete lattices

Thomas Ehrhard

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We continue the study of the category of complete lattices and linear maps. In this problem, we provide the main ingredients to exhibit a $*$-autonomous symmetric monoidal closed structure on **Csl**. In other words **Csl** is a model of MALL (Multiplicative Additive Linear Logic).

2. (a) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S\ f(x) \leq g(x)$), is a sup-CSL. We denote it as $S \multimap T$.

   (b) Given $x \in S$ define a function $x^* : S \to \bot$ by

   $$x^*(y) = \begin{cases} 1 & \text{if } y \not\leq x \\ 0 & \text{if } y \leq x \end{cases}$$

   Prove that $x^* \in S \multimap \bot$.

   (c) Given a sup-CSL S, we use S^{op} for the same set S equipped with the reverse order: $x \leq_{S^{\mathsf{op}}} y$ if $y \leq_S x$. Prove that the map $x \mapsto x^*$ is an order isomorphism from the poset S^{op} to $S \multimap \bot$. Warning: one must prove that it is increasing in both directions because an increasing bijection is not necessarily an order isomorphism! Call $\mathsf{k} : (S \multimap \bot) \to S^{\mathsf{op}}$ the inverse isomorphism.

   (d) **(*)** Given $f \in (S \multimap T)$ define $f^* : (T \multimap \bot) \to (S \multimap \bot)$ by $f^*(y') = y' f$. Prove that $f^* \in \mathbf{Csl}(T \multimap \bot, S \multimap \bot)$. Let $f^\perp \in \mathbf{Csl}(T^{\mathsf{op}}, S^{\mathsf{op}})$ be the associated morphism (through the iso k defined above, that is $f^\perp(y) = \mathsf{k}(f^*(y^*))$). Prove that

   $$\forall x \in S\, \forall y \in T \quad f(x) \leq y \Leftrightarrow x \leq f^\perp(y)\,.$$

   One says that f and $f^\perp$ define a Galois connection between S and T. Last prove that $f^{\perp\perp} = f$.

   (e) Given sup-CSLs S and T we define $S \otimes T$ as the set of all $I \subseteq S \times T$ such that

   - I is down-closed (that is: is $(x, y) \in I$ and $(x_0, y_0) \in S \times T$ are such that $x_0 \leq x$ and $y_0 \leq y$ then $(x_0, y_0) \in I$).
   - and, for all $A \subseteq S$ and $B \subseteq T$, if A and B satisfy $A \times B \subseteq I$ then $(\bigvee A, \bigvee B) \in I$.

   Prove that $(S \otimes T, \subseteq)$ is an inf-CSL (more precisely, it is closed under arbitrary intersections). As a consequence, it is also a sup-CSL: if $\mathcal{I} \subseteq S \otimes T$ then $\bigvee \mathcal{I} = \bigcap \{I \in S \otimes T \mid \bigcup \mathcal{I} \subseteq I\}$. But notice that in this sup-CSL, the sups are not defined as unions in general.

   (f) Prove that the least element of $S \otimes T$ is $0_{S \otimes T} = S \times \{0\} \cup \{0\} \times T$. [*Hint:* Remember that $\bigvee \emptyset = 0$ and that $\emptyset \times B = \emptyset$ for any B.]

   (g) We say that a map $f : S \times T \to U$ (where S, T, U are sup-CSLs) is *bilinear* if for all $A \subseteq S$ and $B \subseteq T$ we have $\bigvee f(A \times B) = f(\bigvee (A \times B)) = f(\bigvee A, \bigvee B)$. Prove that this condition is equivalent to the following:

   - for all $x \in S$ and $B \subseteq T$, one has $f(x, \bigvee B) = \bigvee_{y \in B} f(x, y)$

- and for all $y \in T$ and $A \subseteq S$, one has $f(\bigvee A, y) = \bigvee_{x \in A} f(x, y)$

that is, f is separately linear in both arguments.

(h) **(*)** Given $x \in S$ and $y \in T$ let $x \otimes y = {\downarrow}(x, y) \cup 0_{S \otimes T} \subseteq S \times T$. Prove that $x \otimes y \in S \otimes T$ and that the function $\tau : (x, y) \mapsto x \otimes y$ is a bilinear map $S \times T \to S \otimes T$.

Prove that, if $I \in S \otimes T$ then $I = \bigvee \{x \otimes y \mid x \in S,\ y \in T \text{ and } x \otimes y \subseteq I\}$.

(i) Let $(S, T) \multimap U$ be the set of all bilinear maps $S \times T \to U$ ordered pointwise (that is $f \leq g$ if $\forall (x, y) \in S \times T\ f(x, y) \leq g(x, y)$). Prove that $(S, T) \multimap U$ and $(S \multimap (T \multimap U))$ are isomorphic in **Csl**. Deduce from this fact that $(S, T) \multimap U$ is a sup-CSL.

(j) **(**)** Let S, T and U be sup-CSLs. If $f \in (S \otimes T \multimap U)$, then $f\,\tau \in ((S, T) \multimap U)$ by linearity of f and the function

$$\begin{aligned} \Phi : (S \otimes T \multimap U) &\to ((S, T) \multimap U) \\ f &\mapsto f\,\tau \end{aligned}$$

is a sup-CSL morphism (these facts are obvious).

Prove that Φ is injective and surjective. Deduce from this fact that Φ is an iso in **Csl**. [*Hint:* To prove surjectivity, given $h \in (S, T) \multimap U$, define $g : U \to \mathcal{P}(S \times T)$ by $g(z) = \{(x, y) \in S \times T \mid h(x, y) \leq z\}$, prove that $g \in \mathbf{Csl}(U^{\mathsf{op}}, (S \otimes T)^{\mathsf{op}})$ and then show that $\Phi(g^{\perp}) = h$. Use Question (d).]

Using the fact that $\multimap$ is a functor $\mathbf{Csl}^{\mathsf{op}} \times \mathbf{Csl} \to \mathbf{Csl}$ (which acts on morphisms by pre- and post-composition, we saw a special case in Question (d)) and using the isomorphism $\Phi \in \mathbf{Csl}(S \otimes T \multimap U, S \multimap (T \multimap U))$, it is not difficult to show that **Csl** is an SMC which is $*$-autonomous with dualizing object $\perp$. The main missing ingredient is the functorial action of the $\_ \otimes \_$ operation we have defined on objects. The interested reader is encouraged to work this out!