MPRI 2–2 Models of programming languages: domains, categories, games

Problem 2: The monoidal structure of complete lattices

Thomas Ehrhard

December 16, 2024

We continue the study of the category of complete lattices and linear maps. In this problem, we provide the main ingredients to exhibit a $*$-autonomous symmetric monoidal closed structure on **Csl**. In other words **Csl** is a model of MALL (Multiplicative Additive Linear Logic).

2. (a) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S\ f(x) \leq g(x)$), is a sup-CSL. We denote it as $S \multimap T$.

Solution: Let F be a set of linear functions $S \to T$. Let $g : S \to T$ be defined by

$$g(x) = \bigvee_{f \in F} f(x)\,.$$

Let $A \subseteq S$, we have

$$\begin{aligned} g(\bigvee A) &= \bigvee_{f\in F} f(\bigvee A) \\ &= \bigvee_{f\in F} \bigvee_{x \in A} f(x) \quad \text{by linearity of the elements of } A \\ &= \bigvee_{x\in A} \bigvee_{f \in F} f(x) \quad \text{easy property of } \bigvee \\ &= \bigvee g(A) \end{aligned}$$

which shows that $g \in S \multimap T$. By definition we have $f \leq g$ for each $f \in F$. Let now $h \in S \multimap T$ such that $f \leq h$ for each $f \in F$. This means that, for each $x \in S$, we have $f(x) \leq h(x)$ for all $f \in F$, and hence $g(x) \leq h(x)$. Therefore $g \leq h$ and we have shown that g is the sup of the set F.

(b) Given $x \in S$ define a function $x^* : S \to \bot$ by

$$x^*(y) = \begin{cases} 1 & \text{if } y \not\leq x \\ 0 & \text{if } y \leq x \end{cases}$$

Prove that $x^* \in S \multimap \bot$.

Solution: Let $A \subseteq S$. We have

$$\begin{aligned} x^*(\bigvee A) = 1 &\Leftrightarrow \bigvee A \not\leq x \\ &\Leftrightarrow \exists y \in A \; y \not\leq x \\ &\Leftrightarrow \exists y \in A \; x^*(y) = 1 \\ &\Leftrightarrow \bigvee x^*(y) = 1 \end{aligned}$$

(c) Given a sup-CSL S, we use S^{op} for the same set S equipped with the reverse order: $x \leq_{S^{\mathsf{op}}} y$ if $y \leq_S x$. Prove that the map $x \mapsto x^*$ is an order isomorphism from the poset S^{op} to $S \multimap \bot$. Warning: one must prove that it is increasing in both directions because an increasing bijection is not necessarily an order isomorphism! Call $\mathsf{k} : (S \multimap \bot) \to S^{\mathsf{op}}$ the inverse isomorphism.

Solution: Let $x_1, x_2 \in S$ with $x_1 \leq x_2$. Given $y \in S$, one has ${x_1}^*(y) = 0 \Leftrightarrow y \leq x_1 \Rightarrow y \leq x_2 \Leftrightarrow {x_2}^*(y) = 0$ and hence ${x_2}^* \leq {x_1}^*$. Hence the map $x \mapsto x^*$ is increasing $S^{\mathsf{op}} \to (S \multimap \bot)$. Given $x' \in S \multimap \bot$ let $\mathsf{k}(x') = \bigvee\{x \in S \mid x'(x) = 0\} \in S$. If $x'_1 \leq x'_2$ then $x'_2(x) = 0 \Rightarrow x'_1(x) = 0$ and hence $\mathsf{k}(x'_2) \leq_S \mathsf{k}(x'_1)$ so k is increasing $(S \multimap \bot) \to S^{\mathsf{op}}$. Now let $x \in S$, we have

$$\mathsf{k}(x^*) = \bigvee\{y \in S \mid x^*(y) = 0\} = \bigvee\{y \in S \mid y \leq x\} = x$$

and let $x' \in S \multimap \bot$, for each $y \in S$ we have

$$\mathsf{k}(x')^*(y) = 0 \Leftrightarrow y \leq \mathsf{k}(x') \Leftrightarrow x'(y) = 0$$

because $x'(\mathsf{k}(x')) = 0$ by linearity of x'.

(d) **(*)** Given $f \in (S \multimap T)$ define $f^* : (T \multimap \bot) \to (S \multimap \bot)$ by $f^*(y') = y' f$. Prove that $f^* \in \mathbf{Csl}(T \multimap \bot, S \multimap \bot)$. Let $f^\perp \in \mathbf{Csl}(T^{\mathsf{op}}, S^{\mathsf{op}})$ be the associated morphism (through the iso k defined above, that is $f^\perp(y) = \mathsf{k}(f^*(y^*))$). Prove that

$$\forall x \in S \, \forall y \in T \quad f(x) \leq y \Leftrightarrow x \leq f^\perp(y) \, .$$

One says that f and $f^\perp$ define a Galois connection between S and T. Last prove that $f^{\perp\perp} = f$.

Solution: Let $B' \subseteq T \multimap \bot$. The function $f^*(\bigvee B') \in S \multimap \bot$ is given by $f^*(\bigvee B')(x) = (\bigvee B')(f(x)) = \bigvee_{y' \in B'} y'(f(x)) = \bigvee_{y' \in B'} f^*(y')(x) = (\bigvee_{y' \in B'} f^*(y'))(x)$ which proves that f^* is linear.
Let $x \in S$ and $y \in T$, one has

$$\begin{aligned} x \leq f^\perp(y) &\Leftrightarrow x \leq \mathsf{k}(f^*(y^*)) \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid f^*(y^*)(x_1) = 0\} \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid y^*(f(x_1)) = 0\} \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\} \\ &\Leftrightarrow f(x) \leq y \end{aligned}$$

In the last equivalence we have $f(x) \leq y \Rightarrow x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\}$ because if $f(x) \leq y$ then $x \in \{x_1 \in S \mid f(x_1) \leq y\}$ and conversely if $x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\}$ then $f(x) \leq f(\bigvee\{x_1 \in S \mid f(x_1) \leq y\}) = \bigvee\{f(x_1) \in S \mid f(x_1) \leq y\} \leq y$.
Let $f \in \mathbf{Csl}(S, T)$ so that $f^\perp \in \mathbf{Csl}(T^{\mathsf{op}}, S^{\mathsf{op}})$ and $f^{\perp\perp} \in \mathbf{Csl}(S, T)$. We have $x \leq_S f^\perp(y) \Leftrightarrow f(x) \leq_T y$ and $y \leq_{T^{\mathsf{op}}} f^{\perp\perp}(x) \Leftrightarrow f^\perp(y) \leq_{S^{\mathsf{op}}} x$ that is $y \geq_T f^{\perp\perp}(x) \Leftrightarrow f^\perp(y) \geq_S$

x. So we get $\forall x \in S, y \in T$ $f(x) \leq_T y \Leftrightarrow f^{\perp\perp}(x) \leq_T y$. Taking $y = f(x)$ and then $y = f^{\perp\perp}(x)$ we get $f^{\perp\perp}(x) = f(x)$.

(e) Given sup-CSLs S and T we define $S \otimes T$ as the set of all $I \subseteq S \times T$ such that

- I is down-closed (that is: if $(x, y) \in I$ and $(x_0, y_0) \in S \times T$ are such that $x_0 \leq x$ and $y_0 \leq y$ then $(x_0, y_0) \in I$).
- and, for all $A \subseteq S$ and $B \subseteq T$, if A and B satisfy $A \times B \subseteq I$ then $(\bigvee A, \bigvee B) \in I$.

Prove that $(S \otimes T, \subseteq)$ is an inf-CSL (more precisely, it is closed under arbitrary intersections). As a consequence, it is also a sup-CSL: if $\mathcal{I} \subseteq S \otimes T$ then $\bigvee \mathcal{I} = \bigcap \{I \in S \otimes T \mid \bigcup \mathcal{I} \subseteq I\}$. But notice that in this sup-CSL, the sups are not defined as unions in general.

Solution: Let $\mathcal{I} \subseteq S \otimes T$ and let $I = \bigcap \mathcal{I}$. Let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq I$. For each $J \in S \otimes T$ we have $A \times B \subseteq J$ and hence $(\bigvee A, \bigvee B) \in J$. It follows that $(\bigvee A, \bigvee B) \in I$.

(f) Prove that the least element of $S \otimes T$ is $0_{S \otimes T} = S \times \{0\} \cup \{0\} \times T$. [*Hint:* Remember that $\bigvee \emptyset = 0$ and that $\emptyset \times B = \emptyset$ for any B.]

Solution: Notice first that $0_{S \otimes T}$ is down-closed. Let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq 0_{S \otimes T}$. Notice that we must have $A \subseteq \{0\}$ or $B \subseteq \{0\}$. Indeed otherwise we can find $x \in A \setminus \{0\}$ and $y \in B \setminus \{0\}$, but then we have $(x, y) \in A \times B \subseteq 0_{S \otimes T}$ which is not possible. It follows that $(\bigvee A, \bigvee B) \in 0_{S \otimes T}$. Let now $I \in S \otimes T$, then we have $\emptyset = \emptyset \times T \subseteq I$ and hence $(0, 1) = (\bigvee \emptyset, \bigvee T) \in I$. Similarly $(1, 0) \in I$, which shows that $0_{S \otimes T} \subseteq I$ since I is down-closed.

(g) We say that a map $f : S \times T \to U$ (where S, T, U are sup-CSLs) is *bilinear* if for all $A \subseteq S$ and $B \subseteq T$ we have $\bigvee f(A \times B) = f(\bigvee (A \times B)) = f(\bigvee A, \bigvee B)$. Prove that this condition is equivalent to the following:

- for all $x \in S$ and $B \subseteq T$, one has $f(x, \bigvee B) = \bigvee_{y \in B} f(x, y)$
- and for all $y \in T$ and $A \subseteq S$, one has $f(\bigvee A, y) = \bigvee_{x \in A} f(x, y)$

that is, f is separately linear in both arguments.

Solution: Assume first that f is bilinear, then with these notations we have $f(x, \bigvee B) = f(\bigvee \{x\} \times B) = \bigvee f(\{x\} \times B) = \bigvee_{y \in B} f(x, y)$. Conversely assume that f is separately linear, given $A \subseteq S$ and $B \subseteq T$, we have $f(\bigvee A, \bigvee B) = \bigvee_{b \in B} f(\bigvee A, y) = \bigvee_{y \in B} \bigvee_{x \in A} f(x, y) = \bigvee f(A \times B)$.

(h) **(*)** Given $x \in S$ and $y \in T$ let $x \otimes y = \downarrow(x, y) \cup 0_{S \otimes T} \subseteq S \times T$. Prove that $x \otimes y \in S \otimes T$ and that the function $\tau : (x, y) \mapsto x \otimes y$ is a bilinear map $S \times T \to S \otimes T$.
Prove that, if $I \in S \otimes T$ then $I = \bigvee \{x \otimes y \mid x \in S,\ y \in T \text{ and } x \otimes y \subseteq I\}$.

Solution: First $x \otimes y$ is down-closed as a union of down-closed sets. Next let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq x \otimes y$. For any $x_1 \in A \setminus \{0\}$ and $y_1 \in B \setminus \{0\}$ we must have $x_1 \leq x$ and $y_1 \leq y$ since $(x_1, y_1) \in (A \times B) \setminus 0_{S \otimes T}$, it follows that $(\bigvee A, \bigvee B) \leq (x, y)$. This shows that $x \otimes y \in S \otimes T$.
To prove the bilinearity of τ we must show that $\bigvee \tau(A \times B) = \bigvee A \otimes \bigvee B$. We have $\bigvee \tau(A \times B) \subseteq \bigvee A \otimes \bigvee B$ because the map τ is clearly increasing so it suffices to prove the converse inclusion $\bigvee A \otimes \bigvee B \subseteq \bigvee \tau(A \times B)$. This amounts to proving that for any $I \in S \otimes T$, if $\bigcup \tau(A \times B) \subseteq I$ then $\bigvee A \otimes \bigvee B \subseteq I$. Since we already know that $0_{S \otimes T} \subseteq I$, it suffices to see that $(\bigvee A, \bigvee B) \in I$. We know that $\bigcup \tau(A \times B) \subseteq I$, that is $\bigcup_{(x,y) \in A \times B} (\downarrow x \times \downarrow y) \cup 0_{S \otimes T} \subseteq I$ and hence $A \times B \subseteq I$ so that $(\bigvee A, \bigvee B) \in I$ since $I \in S \otimes T$.

To prove the last equation, it suffices to show that $I \subseteq \bigcup\{x\otimes y \mid x \in S,\ y \in T \text{ and } x\otimes y \subseteq I\}$, so let $(x, y) \in I$, it will be enough to prove that $x \otimes y \subseteq I$. This results from $0_{S\otimes T} \subseteq I$ and $\downarrow(x, y) \subseteq I$ because I is down-closed.

(i) Let $(S, T) \multimap U$ be the set of all bilinear maps $S \times T \to U$ ordered pointwise (that is $f \leq g$ if $\forall (x, y) \in S \times T\ f(x, y) \leq g(x, y)$). Prove that $(S, T) \multimap U$ and $(S \multimap (T \multimap U))$ are isomorphic in **Csl**. Deduce from this fact that $(S, T) \multimap U$ is a sup-CSL.

Solution: Given $f \in (S, T) \multimap U$ let $\lambda(f) : S \to U^T$ be defined by $\lambda(f)(x)(y) = f(x, y)$. By bilinearity of f, for each x the function $\lambda(f)(x) : T \to U$ is linear, and the map $\lambda(f)$ itself is linear because $T \multimap U$ is ordered pointwise. The fact that λ is an order isomorphism is an easy verification.

(j) **(**)** Let S, T and U be sup-CSLs. If $f \in (S \otimes T \multimap U)$, then $f\,\tau \in ((S, T) \multimap U)$ by linearity of f and the function

$$\begin{aligned} \Phi : (S \otimes T \multimap U) &\to ((S, T) \multimap U) \\ f &\mapsto f\,\tau \end{aligned}$$

is a sup-CSL morphism (these facts are obvious).

Prove that Φ is injective and surjective. Deduce from this fact that Φ is an iso in **Csl**. [*Hint:* To prove surjectivity, given $h \in (S, T) \multimap U$, define $g : U \to \mathcal{P}(S \times T)$ by $g(z) = \{(x, y) \in S \times T \mid h(x, y) \leq z\}$, prove that $g \in \mathbf{Csl}(U^{\mathsf{op}}, (S \otimes T)^{\mathsf{op}})$ and then show that $\Phi(g^{\perp}) = h$. Use Question (d).]

Solution: Let $f \in (S \otimes T \multimap U)$. Let $I \in S \otimes T$, we have

$$\begin{aligned} f(I) &= f(\bigvee\{x \otimes y \mid x \in S,\ y \in T \text{ and } x \otimes y \subseteq I\}) && \text{by Question (h)} \\ &= \bigvee\{f(x \otimes y) \mid x \in S,\ y \in T \text{ and } x \otimes y \subseteq I\} && \text{by linearity of } f \\ &= \bigvee\{\Phi(f)(x, y) \mid x \in S,\ y \in T \text{ and } x \otimes y \subseteq I\} \end{aligned}$$

and hence Φ is injective.

Let $h \in (S, T) \multimap U$. We define $g : U \to \mathcal{P}(S \times T)$ by

$$g(z) = \{(x, y) \in S \times T \mid h(x, y) \leq z\}\,.$$

Then by bilinearity of h we have $g(z) \in S \otimes T$. Moreover, if $C \subseteq U$, we have

$$\begin{aligned} g(\bigwedge C) &= \{(x, y) \in S \times T \mid h(x, y) \leq \bigwedge C\} \\ &= \{(x, y) \in S \times T \mid \forall z \in C\ h(x, y) \leq z\} \\ &= \bigcap_{z \in C} g(z) \end{aligned}$$

which shows that $g \in \mathbf{Csl}(U^{\mathsf{op}}, (S \otimes T)^{\mathsf{op}})$ and hence $g^{\perp} \in \mathbf{Csl}(S \otimes T, U)$.

We prove that $\Phi(g^{\perp}) = h$, which will prove that Φ is surjective as announced.

By definition of $g^{\perp}$, we have

$$\forall z \in U,\, I \in S \otimes T \quad g(z) \leq_{(S\otimes T)^{\mathsf{op}}} I \Leftrightarrow z \leq_{U^{\mathsf{op}}} g^{\perp}(I)$$

but

$$\begin{aligned} g(z) \leq_{(S\otimes T)^{\mathsf{op}}} I &\Leftrightarrow I \subseteq \{(x, y) \in S \times T \mid h(x, y) \leq z\} \\ &\Leftrightarrow h(I) \subseteq \downarrow z \end{aligned}$$

hence

$$\forall z \in U,\, I \in S \otimes T \quad h(I) \subseteq {\downarrow} z \Leftrightarrow g^{\perp}(I) \leq z$$

that is

$$\forall z \in U,\, I \in S \otimes T \quad \bigvee h(I) \leq z \Leftrightarrow g^{\perp}(I) \leq z$$

so that $g^{\perp}(I) = \bigvee h(I)$. Therefore $\Phi(g^{\perp})(x, y) = \bigvee \{h(x_0, y_0) \mid (x_0, y_0) \in x \otimes y\} = f(x, y)$ since, if $(x_0, y_0) \in x \otimes y$ we have either $x_0 = 0$ or $y_0 = 0$ (and then $h(x_0, y_0) = 0$ because h is bilinear), or $(x_0, y_0) \leq (x, y)$.

Using the fact that $\multimap$ is a functor $\mathbf{Csl}^{\mathsf{op}} \times \mathbf{Csl} \to \mathbf{Csl}$ (which acts on morphisms by pre- and post-composition, we saw a special case in Question (d)) and using the isomorphism $\Phi \in \mathbf{Csl}(S \otimes T \multimap U, S \multimap (T \multimap U))$, it is not difficult to show that **Csl** is an SMCC which is $*$-autonomous with dualizing object $\perp$. The main missing ingredient is the functorial action of the $\_ \otimes \_$ operation we have defined on objects. The interested reader is encouraged to work this out!