MPRI 2–2 Models of programming languages: domains, categories, games

Problem 1: General properties of Complete Lattices

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The signs **(*)** and **(**)** point out more difficult and interesting questions. These are of course completely subjective indications!

In this problem we start the study of a denotational model of Linear Logic which is based on complete sup-semilattices and linear maps.

This model of LL has two related interesting properties, which are not so common among the known models of LL.

- It is not "based on webs", that is, contrarily to what happens in the model of coherence spaces and in the model **Rel** of sets and relations which will be a running example in my lectures, the elements of its objects cannot always be described as subsets of a set of atoms (remember that the cliques of a coherence space are subsets of the web of the coherence space, for instance).
- This model is a category which is *complete*, that is, it has all (projective) limits and also all colimits.

The existence of such models suggests that LL, as a logical system, might be extended with more limits and colimits than those which are already represented as LL logical connectives (namely, the additive connectives $\top$, &, 0 and $\oplus$).

When $\mathcal{E}$ is a category and X and Y are objects of $\mathcal{E}$, we use $\mathcal{E}(X, Y)$ for the set of morphisms from X to Y in $\mathcal{E}$. Given $f \in \mathcal{E}(X, Y)$ and $g \in \mathcal{E}(Y, Z)$ we simply use $g\, f$ for the composition of f with g, rather that $g \circ f$, to insist on the intiuition that the morphisms that we consider here are "linear" in some sense.

A *complete sup-semilattice* (most often we will simply say "complete semilattice", CSL or sup-CSL) is a partially ordered set S (the order relation will always be denoted as $\leq$ or $\leq_S$) such that each subset A of S has a least upper bound $\bigvee A \in S$ (also called "lub", "sup" or "supremum" of A). Remember that $\bigvee A$ is uniquely characterized by

- $\forall x \in A\ x \leq \bigvee A$
- $\forall x \in S\ (\forall y \in A\ y \leq x) \Rightarrow \bigvee A \leq x$.

In particular S must have two elements $0 = \bigvee \emptyset$ which is its least element and $1 = \bigvee S$ which is its greatest element. So a sup-semilattice is never empty.

A subset A of S is *down-closed* if for all $x \in A$ and all $y \in S$, if $y \leq x$ then $y \in A$. Given $x \in S$ we set $\downarrow x = \{y \in S \mid y \leq x\}$.

Each subset A of a complete semi-lattice S has a greatest lower bound (glb, inf or infimum) $\bigwedge A$, which is given by

$$\begin{aligned}\bigwedge A &= \bigvee \{y \in S \mid \forall x \in A\ y \leq x\} \\ &= \bigvee \bigcap_{x \in A} \downarrow x\,.\end{aligned}$$

Dually a complete *inf-semilattice* (or inf-CSL) is a partially ordered set S where each subset A has a glb $\bigwedge A$. Each inf-semilattice is also a sup-semilattice, with least upper bounds given by

$$\bigvee A = \bigwedge\{y \in S \mid \forall x \in A\ x \leq y\}$$

for all $A \subseteq S$.

A *linear morphism* of CSLs from S to T is a function $f : S \to T$ such that for all $A \subseteq S$ $f(\bigvee A) = \bigvee f(A)$ where we define as usual $f(A) = \{f(x) \mid x \in A\}$. Notice that this implies that f is monotone: given $x \leq y$ in S we have $f(y) = f(\bigvee\{x, y\}) = f(x) \vee f(y)$, that is $f(x) \leq f(y)$. Let **Csl** be the category whose objects are the sup-semilattices and morphisms are the linear maps of sup-semilattices. We use $\perp = \{0 < 1\}$ for the object of **Csl** which has exactly two elements.

It is easy to check that **Csl** is cartesian. The product of a family $(S_j)_{j\in J}$ of objects of **Csl** is the usual cartesian product $\prod_{j\in J} S_j$ equipped with the product order and projection defined in the usual way. We also use $S = \&_{j\in J} S_j$ for this product and $\pi_j \in \mathbf{Csl}(S, S_j)$ for the projections. The terminal object is $\top = \{0\}$.

The category **Rel** has all sets as objets, and $\mathbf{Rel}(E, F)$ is the set of all relations from E to F, that is, of all $s \subseteq E \times F$. The identity at E in **Rel** is $\mathsf{Id}_E = \{(a, a) \mid a \in E\}$ and composition is the usual composition of relations, that is, if $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, G)$, their composition is

$$t\,s = \{(a, c) \in E \times G \mid \exists b \in F\ (a, b) \in s \text{ and } (b, c) \in t\}\,.$$

1. In this problem we study some general properties of this category **Csl**, in particular we show that it is complete, that it contains **Rel** as a "full subcategory", and we develop an example of a limit of a diagram in **Rel** (an equalizer actually) which is a complete semilattice of which we will see that it cannot be considered as an object of **Rel**.

   (a) Given a function f from S to T, prove that the following properties are equivalent:

   1. f is an isomorphism in **Csl**;
   2. f is a bijection and is a morphism of **Csl**;
   3. f is an increasing bijection whose inverse is also increasing.

   Solution: We prove (1)⇒(2) ⇒(3)⇒(1).
   Assume (1). Let f^{-1} be the inverse of f in **Csl**. Since composition and identites are defined in **Csl** as in **Set**, the function f^{-1} is the inverse of the function f, so f is a bijection and hence (2) holds.
   Assume (2). Then, as we have seen above, f is increasing, and this is also true of f^{-1} since $f^{-1} \in \mathbf{Csl}(T, S)$, so (3) holds.
   Last assume (3) and let us prove (1). Let $A \subseteq S$, we must prove that $f(\bigvee A) = \bigvee_{x\in A} f(x)$. We have $f(\bigvee A) \geq \bigvee_{x\in A} f(x)$ because f is increasing, since $\forall x \in A\ x \leq \bigvee A$. Let $B = \{f(x) \mid x \in A\} \subseteq T$. For the same reason, using the fact that f^{-1} is increasing, we have $f^{-1}(\bigvee B) \geq \bigvee_{y\in B} f^{-1}(y)$, that is $f^{-1}(\bigvee_{x\in A} f(x)) \geq \bigvee A$ and hence, since f is increasing, we have $\bigvee_{x\in A} f(x) \geq f(\bigvee A)$. We have proven that $f \in \mathbf{Csl}(S, T)$, and for the same reason $f^{-1} \in \mathbf{Csl}(T, S)$, so that f is an iso in **Csl**.

   (b) Given a set E we denote as $\mathcal{P}(E)$ its powerset (that is, the set of all of its subsets) ordered by inclusion, so that $\mathcal{P}(E)$ is a sup-semilattice with $\bigvee A = \bigcup A$ for each $A \subseteq \mathcal{P}(E)$. Given $t \in \mathbf{Rel}(E, F)$ we define $\mathsf{fun}(t) : \mathcal{P}(E) \to \mathcal{P}(F)$ by $\mathsf{fun}(t)(x) = t \cdot x = \{b \in F \mid \exists a \in x\ (a, b) \in t\}$. Prove that $\mathsf{fun}(t) \in \mathbf{Csl}(\mathcal{P}(E), \mathcal{P}(F))$ and that, for each $f \in \mathbf{Csl}(\mathcal{P}(E), \mathcal{P}(F))$ there is exactly one $t = \mathsf{tr}(f) \in \mathbf{Rel}(E, F)$ such that $f = \mathsf{fun}(t)$. In other words, the functor $L : \mathbf{Rel} \to \mathbf{Csl}$ which maps E to $\mathcal{P}(E)$ and t to $\mathsf{fun}(t)$ is full and faithful. So one can consider **Rel** as a "full subcategory" of **Csl**.

Solution: The fact that $\mathsf{fun}(t)$ commutes with unions follows immediately from the definition, hence $\mathsf{fun}(t)$ belongs to $\mathbf{Csl}(\mathcal{P}(E), \mathcal{P}(F)))$. Let now $f \in \mathbf{Csl}(\mathcal{P}(E), \mathcal{P}(F))$ and let us set $t = \mathsf{tr}(f) = \{(a, b) \mid b \in f(\{a\})\} \in \mathbf{Rel}(E, F)$. Let $x \in \mathcal{P}(E)$, we have

$$\begin{aligned} \mathsf{fun}(t)(x) &= \{b \in F \mid \exists a \in x\ (a, b) \in t\} \\ &= \bigcup_{a \in x} f(\{a\}) \\ &= f(x) \end{aligned}$$

since f commutes with unions. This shows that $t \mapsto \mathsf{fun}(t)$ is surjective. On the other hand given $t \in \mathbf{Rel}(E, F)$ we have

$$\mathsf{tr}(\mathsf{fun}(t)) = \{(a, b) \in E \times F \mid b \in \mathsf{fun}(t)(\{a\})\} = t$$

by definition of $\mathsf{fun}(t)$ which shows that $t \mapsto \mathsf{fun}(t)$ is injective.

(c) Prove that the category **Csl** has all equalizers, in other words: given objects S and T of **Csl** and $f, g \in \mathbf{Csl}(S, T)$ there is an object U of **Csl** and a morphism $e \in \mathbf{Csl}(U, S)$ such that

- $f\,e = g\,e$
- and, for each object V of **Csl** and each morphism $h \in \mathbf{Csl}(V, S)$ such that $f\,h = g\,h$, there is exactly one morphism $h_0 \in \mathbf{Csl}(V, U)$ such that $h = e\,h_0$.

By a standard theorem of category theory, together with the fact that all products exist in **Csl**, this shows that this category is complete, that is, all (projective) limits exist in **Csl**.

Solution: We take $U = \{x \in S \mid f(x) = g(x)\}$, equipped with the induced order relation (that is $x \leq_U y$ if $x \leq_S y$). Given $A \subseteq U$ we have $A \subseteq S$ so let x_0 be the lub of A in S. Since f and g are linear we have

$$\begin{aligned} f(x_0) &= \bigvee f(A) \\ &= \bigvee g(A) \quad \text{since } A \subseteq U \\ &= g(x_0) \end{aligned}$$

and hence $x_0 \in U$.

Next one proves that x_0 is the lub of A *in* U. First given $x \in A$ one has $x \leq_S x_0$ and hence $x \leq_U x_0$ since $x, x_0 \in U$. Next let $y \in U$ be such that $\forall x \in A\ x \leq_U y$, we have $\forall x \in A\ x \leq_S y$ and hence $x_0 \leq_S y$, that is $x_0 \leq_U y$ since $x_0, y \in U$.

The inclusion map $e : U \to S$ (that is $e(x) = x$) is linear since we have seen that the lubs are computed in U exactly as in S.

Let now V be a sup-semilattice and $h \in \mathbf{Csl}(V, S)$ be such that $f\,h = g\,h$. This means that actually $\forall v \in V\ h(v) \in U$. So we can define $h_0 : V \to U$ by $h_0(v) = h(v)$. Again, the linearity of h_0 results from the fact that the sups in U are computed exactly as in S so that $h_0 \in \mathbf{Csl}(V, U)$. Last the uniqueness of h_0 results from the fact that e is injective.

The *Cantor space* is the set $\{0,1\}^\omega$ of all infinites sequences α of 0 and 1 equipped with the following topology (which is the product topology of the discrete space $\{0,1\}$): a subset U of $\{0,1\}^\omega$ is open iff for each $\alpha \in U$ there is a finite prefix w of α such that, for each $\beta \in \{0,1\}^\omega$, if w is a prefix of β then $\beta \in U$.

Hence a subset F of $\{0,1\}^\omega$ is closed (that is $\{0,1\}^\omega \setminus F$ is open) iff it has the following property: if $\alpha \in \{0,1\}^\omega$ is such that, for each finite prefix w of α there exists $\beta \in F$ such that w is a prefix of β, then $\alpha \in F$. As in all topological spaces, if $\mathcal{F}$ is a set of closed subsets then $\bigcap \mathcal{F}$ is closed (you are advised to check this directly using the characterization above of closed subsets).

So the set of closed subsets of $\{0,1\}^\omega$ ordered by inclusion is an inf-CSL: each subset $\mathcal{F}$ of this set has a greatest lower bound, namely $\bigcap \mathcal{F}$. Hence the set of closed subsets of $\{0,1\}^\omega$ ordered by inclusion is also a CSL: the lub of a set of closed sets is the *closure* of its union (= the intersection of all closed sets which contain this union).

We use $\mathcal{C}$ for the CSL of closed subsets of the Cantor space ordered by inclusion.

We use $\{0,1\}^*$ for the set of finite sequences of 0's and 1's, and if $w \in \{0,1\}^*$ and $\alpha \in \{0,1\}^\omega$, we write $w < \alpha$ when w is a prefix of α.

(d) Prove that for each $\alpha \in \{0,1\}^\omega$, one has $\{\alpha\} \in \mathcal{C}$.

> **Solution:** Let $\beta \in \{0,1\}^\omega$ be such that each finite prefix has an extension in $\{\alpha\}$: this means that each finite prefix of β is a prefix of α, so $\beta = \alpha$. Hence $\{\alpha\}$ is closed. All singletons are closed in all topological spaces which are T_1-separated (which is the case of the Cantor space, which is actually Hausdorff, that is, T_2).

(e) Prove that if $w \in \{0,1\}^*$ then the set $E(w) = \{\alpha \in \{0,1\}^\omega \mid w < \alpha\}$ is at the same time open and closed (one says that it is a clopen).

> **Solution:** If $\alpha \in \{0,1\}^\omega$ is such that $\alpha \in E(w)$ then $w < \alpha$, and we have $\forall \beta \in \{0,1\}^\omega\ w < \beta \Rightarrow \beta \in E(w)$, so $E(w)$ is open.
> Now let $\alpha \in \{0,1\}^\omega$ be such that $\forall v < \alpha\, \exists \beta \in E(w)\ v < \beta$. Let $v \in \{0,1\}^*$ be the prefix of α which has the same length as w, there must be $\beta \in E(w)$ such that $v < \beta$. So we have $v < \beta$ and $w < \beta$ and hence $v = w$ (the prefix order is a total order on the set of prefixes of each given element of $\{0,1\}$). This shows that $\alpha \in E(w)$ and hence $E(w)$ is closed.

(f) **(**)** Let $W = \{0,1\}^*$. If $w = \langle a_1, \dots, a_n \rangle \in W$ and $a \in \{0,1\}$ let $wa = \langle a_1, \dots, a_n, a \rangle$. Let $\theta = \{(wa, w) \mid w \in W \text{ and } a \in \{0,1\}\} \in \mathbf{Rel}(W,W)$. Let (C, c) be the equalizer of $\mathsf{Id}, \mathsf{fun}(\theta) \in \mathbf{Csl}(\mathcal{P}(W), \mathcal{P}(X))$ (so that C is a sup-semilattice and $c \in \mathbf{Csl}(C, \mathcal{P}(W))$ by Question (c)). Exhibit an order isomorphism between C and $\mathcal{C}$. Describe the lub operation in this complete lattice.

> **Solution:** By Question (c), we know that $C = \{x \subseteq W \mid \theta \cdot x = x\}$. So for $x \subseteq W$, the condition $x \in C$ means:
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> - $\theta \cdot x \subseteq x$, that is, if $wa \in x$ then $w \in x$, that is, x is prefix-closed
> - and $x \subseteq \theta \cdot x$ that is, if $w \in x$ then there is $a \in \{0,1\}$ such that $wa \in x$: each element of x has an extension in x.
>
> So we decide to see such an $x \in C$ as the set of all prefixes of the elements of a subset of $\{0,1\}^\omega$. More precisely let
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> $$\varphi(x) = \{\alpha \in \{0,1\}^\omega \mid \forall w \in W\ w < \alpha \Rightarrow w \in x\}.$$
>
> We prove that $\varphi(x) \in \mathcal{C}$, that is, $\varphi(x)$ is a closed subset of $\{0,1\}^\omega$. Let $\alpha \in \{0,1\}^\omega$ be such that for all $w < \alpha$ there is $\beta \in \{0,1\}^\omega$ such that $\beta \in \varphi(x)$ and $w < \beta$. This implies $\forall w \in W\ w < \alpha \Rightarrow w \in x$ and hence $\alpha \in \varphi(x)$, so $\varphi(x)$ is closed. Notice that the map φ is increasing (with respect to set inclusion).

Conversely given $F \in \mathcal{C}$ let

$$\psi(F) = \{w \in W \mid \exists \alpha \in F \ w < \alpha\} .$$

Then we clearly have $\psi(F) \in C$ and it is also clear that ψ is an increasing function.
Let us prove that $\varphi(\psi(F)) = F$ for all $F \in \mathcal{C}$. We first prove that $F \subseteq \varphi(\psi(F))$ so let $\alpha \in F$. For all $w < \alpha$ we have $w \in \psi(F)$ by definition of ψ, and hence $\alpha \in \varphi(\psi(F))$ by definition of φ. Conversely let $\alpha \in \varphi(\psi(F))$. This means $\forall w \in W \ w < \alpha \Rightarrow w \in \psi(F)$ that is $\forall w \in W \ w < \alpha \Rightarrow \exists \beta \in F \ w < \beta$ which implies $\alpha \in F$ because F is closed. So we have proven that $\varphi \circ \psi = \mathsf{Id}$, we prove now that $\psi \circ \varphi = \mathsf{Id}$.
Let $x \in C$, we prove first that $x \subseteq \psi(\varphi(x))$. Let $w \in x$. Using the assumption that $x \in C$ we can build by induction a sequence $a_1, a_2, \dots$ of elements of $\{0,1\}$ such that, for all $n \in \mathbb{N}$, one has $wa_1 \dots a_n \in x$. So let $\alpha = wa_1a_2 \dots \in \{0,1\}^\omega$. If $w' < \alpha$ we have either $w' = wa_1 \dots a_n$ for some n, or w' is a prefix of w. Hence $w' \in x$. This shows that $\alpha \in \varphi(x)$. Since $w < \alpha$ it follows that $w \in \psi(\varphi(x))$. Conversely let $w \in \psi(\varphi(x))$. Let $\alpha \in \varphi(x)$ be such that $w < \alpha$. By definition of $\varphi(x)$, we have $w \in x$.
By Question (a), φ is an isomorphism in **Csl**.
As seen in Question (c), the lub operation in C is defined as in $\mathcal{P}(W)$, and hence it is just union.

Given a CSL S, we say that $x \in S$ is *prime* if

$$\forall A \subseteq S \quad x \leq \bigvee A \Rightarrow \exists y \in A \ x \leq y .$$

(g) **(*)** Prove that, for a set E, the prime elements of $\mathcal{P}(E) \in$ **Csl** are exactly the singletons. Prove that $\mathcal{C}$ has no prime elements.
[*Hint:* For the second part, prove first that if $F \in C$ is prime, it must be a singleton $\{\alpha\}$ and then prove that no such singleton is prime. For this remember that, for a collection $\mathcal{F}$ of closed subsets of $\{0,1\}^\omega$, the closed set $\bigvee \mathcal{F}$ is the closure of $\bigcup \mathcal{F}$ (the intersection of all closed sets which contain $\bigcup \mathcal{F}$). So build a set $\mathcal{F}$ of shape $\mathcal{F} = \{\{\alpha(n)\} \mid n \in \mathbb{N}\}$ where $\alpha(n) \to_{n\to\infty} \alpha$ for the topology of the Cantor space and $\forall n \in \mathbb{N} \ \alpha(n) \neq \alpha$.]

Solution: For the first part observe that for each $x \subseteq E$ one has $x = \bigcup_{a \in x} \{a\}$. So if x is prime we must have $x \subseteq \{a\}$ for some $a \in X$. We cannot have $x = \emptyset$ since $\emptyset = \bigcup \emptyset$ and hence $\emptyset$ is not prime. Conversely it is obvious that if a is a singleton then $\{a\}$ is prime.
Concerning $\mathcal{C}$ remember first from Question (d) that each singleton $\{\alpha\}$ is closed, that is $\{\alpha\} \in \mathcal{C}$. Now let F be closed and assume that F is not a singleton. If $F = \emptyset$ then F is not prime because $\bigvee \emptyset = \emptyset$. So let $\alpha, \beta \in F$ with $\alpha \neq \beta$. Let $w < \alpha$ be such that $w \not< \beta$. Remember from Question (e) that the set $E(w) = \{\gamma \in \{0,1\}^\omega \mid w < \gamma\}$ is closed and open. Hence $F \cap E(w)$ and $F \setminus E(w)$ are both closed and satisfy $(F \cap E(w)) \vee (F \setminus E(w)) = F$ since

$$F = (F \cap E(w)) \cup (F \setminus E(w)) \subseteq (F \cap E(w)) \vee (F \setminus E(w)) \subseteq F,$$

so F is not prime: remember that $\alpha, \beta \in F$, but we don't have $F \subseteq F \cap E(w)$ since $\beta \notin F \cap E(w)$ and we don't have $F \subseteq F \setminus E(w)$ because $\alpha \notin F \setminus E(w)$.
Now we prove that $\{\alpha\}$ is never prime, whatever be $\alpha = \langle a_1, a_2, \dots \rangle$. For each $n \in \mathbb{N}$ let $\alpha(n) \in \{0,1\}^\omega$ be defined (for instance) by

$$\alpha(n) = \langle a_1, \dots, a_{n-1}, 1 - a_n, 0, 0, \dots \rangle$$

so that $\alpha(n) \to_{n\to\infty} \alpha$ (for the topology of the Cantor space). It follows that

$$\alpha \in \bigvee_{n=1}^{\infty} \{\alpha(n)\}$$

> but by construction $\alpha \neq \alpha(n)$ for all n. It follows that $\{\alpha\}$ is not prime and hence $\mathcal{C}$ has no prime elements.

This strongly suggests that the category **Rel** is not complete, and more precisely that it has no equalizer for the two maps $\theta, \mathsf{Id} \in \mathbf{Rel}(W, W)$ because the equalizer of $\mathsf{fun}(\theta)$ and Id in **Csl** is not an object of **Rel**. Indeed this equalizer C is an infinite CSL which is isomorphic to $\mathcal{C}$, a CSL which has no prime elements, whereas the only set E such that the CSL $(\mathcal{P}(E), \subseteq)$ has no prime element is $E = \emptyset$. In the last questions of this problem, we prove rigorously that θ and Id have no equalizer in **Rel**.

(h) Let $\mathcal{E}$ be a category and let $f_1, f_2 \in \mathcal{E}(X, Y)$ be two morphisms. Let Z be an object of $\mathcal{E}$ and $e \in \mathcal{E}(Z, X)$ be such that (Z, e) is an equalizer of f_1 and f_2. Remember that this means

- $f_1\, e = f_2\, e$
- and, for each object U of $\mathcal{E}$ and each $h \in \mathcal{E}(U, X)$ such that $f_1\, h = f_2\, h$, there is exactly one $h_0 \in \mathcal{E}(U, Z)$ such that $h = e\, h_0$.

Prove that e is a monomorphism. This means intuitively that e is "injective". The precise definition is: given two morphisms $k_1, k_2 \in \mathcal{E}(V, Z)$, if $e\, k_1 = e\, k_2$ then $k_1 = k_2$.

> **Solution:** With the notations above, assume that $e\, k_1 = e\, k_2 \in \mathcal{E}(V, X)$ and let us use g for this morphism. We have $f_1\, e = f_2\, e$ and hence $f_1\, g = f_2\, g$ and so there is a unique morphism $k \in \mathcal{E}(V, E)$ such that $e\, k = g$. Now both $k = k_1$ and $k = k_2$ satisfy this equation, hence $k_1 = k_2$.

(i) Prove that if a morphism t in **Rel** is a monomorphism, then the associated function $\mathsf{fun}(t)$ is injective.

> **Solution:** We use 1 for a chosen set which has exactly one element: $1 = \{*\}$.
> Let $t \in \mathbf{Rel}(E, F)$ be a monomorphism. Let $x_1, x_2 \in \mathcal{P}(E)$ and assume that $\mathsf{fun}(t)(x_1) = \mathsf{fun}(t)(x_2)$. Let $s_i = \{(*, a) \mid a \in x_i\} \in \mathbf{Rel}(1, E)$ for $i = 1, 2$. For $i = 1, 2$, we have $t\, s_i = \{(*, a) \mid a \in \mathsf{fun}(t)(x_i)\}$ and hence $t\, s_1 = t\, s_2$ so that $s_1 = s_2$ since t is a mono. Therefore $x_1 = \mathsf{fun}(s_1)(\{*\}) = \mathsf{fun}(s_2)(\{*\}) = x_2$.

(j) **(**)** Complete the proof that the equalizer of θ and Id does not exist in **Rel**.

> **Solution:** In what follows, to increase readability, we identify the sets $\mathbf{Rel}(G, H)$ and $\mathbf{Csl}(\mathcal{P}(G), \mathcal{P}(H))$ for all sets G and H, through the bijection $\mathsf{fun}(\_)$.
> Towards a contradiction, assume that this equalizer (E, e) exists in **Rel** so that E is a set and $e \in \mathbf{Csl}(\mathcal{P}(E), \mathcal{P}(W))$. Let $f \in \mathbf{Csl}(\mathcal{P}(E), C)$ be the unique CSL morphism such that $e = c\, f$ (which exists because (C, c) is the equalizer of θ and Id in **Csl**).
> Given $x \in C$, we can define a function $g_x : \mathcal{P}(1) \to C$ by $g_x(\emptyset) = \emptyset$ and $g(\{*\}) = x$ and it is clear that $g_x \in \mathbf{Csl}(\mathcal{P}(1), C)$. Moreover $\theta\, c\, g_x = c\, g_x$ because $x \in C$ and hence, since $c\, g_x \in \mathbf{Rel}(1, W)$, there is exactly one morphism $h_x \in \mathbf{Rel}(1, E)$ such that $e\, h_x = c\, g_x$. We set $h(x) = h_x(\{*\})$. In that way we have defined a function $h : C \to \mathcal{P}(E)$, which is completely characterized by
>
> $$\forall x \in C \quad e(h(x)) = c(x)\,.$$
>
> We prove that this function h belongs to $\mathbf{Csl}(C, \mathcal{P}(E))$.
> Let $A \subseteq C$. Since $(\mathcal{P}(E), e)$ is an equalizer in **Rel**, e is a monomorphism in **Rel** by Question (h) and hence, considered as a function, it is injective by Question (i). We have, using the definition of c, $e(h(\bigcup A)) = c(\bigcup A) = \bigcup A$ and $e(\bigcup h(A)) = \bigcup e(h(A))$ since e is a morphism in **Csl** and hence $e(\bigcup h(A)) = \bigcup A$. By injectivity of e, we have therefore $h(\bigcup A) = \bigcup_{x \in A} h(x) = \bigcup h(A)$ and hence $h \in \mathbf{Csl}(C, \mathcal{P}(E))$.

We have $e\,h\,f = c\,f = e$ and hence $h\,f = \mathsf{Id}$ since e is a mono. And $c\,f\,h = e\,h = c$ and hence $f\,h = \mathsf{Id}$ since c is a mono.
This shows that f and h define an isomorphism in **Csl** between $\mathcal{P}(E)$ and C which is impossible since C has no prime elements whereas E has $\mathcal{P}(E)$ has prime elements.