MPRI 2–2 Models of programming languages: domains, categories, games

Exercises

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The signs **(*)** and **(**)** try to indicate more difficult and interesting questions. These are of course completely subjective indications!

1. Let E be a coherence space. We use $\mathsf{Cl}(E)$ for the set of all cliques of E. We say that a family of cliques $\vec{x} = (x_i)_{i \in I} \in \mathsf{Cl}(E)^I$ is *summable* if $\forall i, j \in I\ i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$. Let E, F be coherence spaces and $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a function. Prove that f is linear iff for all summable family $\vec{x} \in \mathsf{Cl}(E)^I$, one has:
   - the family $(f(x_i))_{i \in I}$ is summable
   - and $\bigcup_{i \in I} f(x_i) = f(\bigcup_{i \in I} x_i)$.

2. This exercise develops a somehow degenerate model of Linear Logic which does not satisfy $*$-autonomy but satisfies all the other requirements. A *pointed set* is a structure $X = (\underline{X}, 0_X)$ where $\underline{X}$ is a set and $0_X \in \underline{X}$. Given pointed sets X, X_1, X_2 and Y,
   - a morphism of pointed sets from X to Y is a function $f : \underline{X} \to \underline{Y}$ such that $f(0_X) = 0_Y$
   - and a bimorphism of pointed sets from X_1, X_2 to Y is a function $f : \underline{X_1} \times \underline{X_2} \to \underline{Y}$ such that $f(0_{X_1}, x_2) = f(x_1, 0_{X_2}) = 0_Y$ for each $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   (a) Prove that pointed sets together with morphisms of pointed sets form a category $\mathbf{Set}_0$. What are the isos in that category?

   One sets $1 = (\{0_1, *\})$ where $*$ and 0_1 are are distinct chosen elements (for instance 0_1 is the integer 0 and $*$ is the integer 1). Given pointed sets X_1 and X_2 one defines $X_1 \otimes X_2$ as follows:

   $$\underline{X_1 \otimes X_2} = \{(x_1, x_2) \in \underline{X_1} \times \underline{X_2} \mid x_1 = 0_{X_1} \Leftrightarrow x_2 = 0_{X_2}\} \quad \text{and} \quad 0_{X_1 \otimes X_2} = (0_{X_1}, 0_{X_2}).$$

   Given $x_i \in \underline{X_i}$ for $i = 1, 2$, one defines

   $$x_1 \otimes x_2 = \begin{cases} (0_{X_1}, 0_{X_2}) & \text{if } x_1 = 0_{X_1} \text{ or } x_2 = 0_{X_2} \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

   (b) Prove that the function $(x_1, x_2) \mapsto x_1 \otimes x_2$ is a bimorphism from X_1, X_2 to $X_1 \otimes X_2$ which is surjective as a function $\underline{X_1} \times \underline{X_2} \to \underline{X_1 \otimes X_2}$ and that for any bimorphism f from X_1, X_2 to Y there is exactly one morphism $\widetilde{f} \in \mathbf{Set}_0(X_1 \otimes X_2, Y)$ such that $f(x_1, x_2) = \widetilde{f}(x_1 \otimes x_2)$ for all $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   (c) Given $f_i \in \mathbf{Set}_0(X_i, Y_i)$ for $i = 1, 2$, deduce from the above that there is exactly one morphism $f_1 \otimes f_2 \in \mathbf{Set}_0(X_1 \otimes X_2, Y_1 \otimes Y_2)$ such that

   $$\forall x_1 \in \underline{X_1}\, \forall x_2 \in \underline{X_2} \quad (f_1 \otimes f_2)(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2).$$

(d) Using again the universal property of Question (b) prove that the operation on morphisms defined in Question (c) is a functor.

(e) Exhibit isomorphisms $\lambda_X \in \mathbf{Set}_0(1 \otimes X, X)$ and $\alpha_{X_1,X_2,X_3} \in \mathbf{Set}_0((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$.

So $\mathbf{Set}_0$ is an SMC (there is a symmetry iso $\gamma_{X_1,X_2} \in \mathbf{Set}_0(X_1 \otimes X_2, X_2 \otimes X_1)$ such that $\gamma_{X_1,X_2}(x_1 \otimes x_2) = x_2 \otimes x_1$ which is quite easy to define, and the Mac Lane coherence diagrams commute).

(f) One defines $X \multimap Y$ by $\underline{X \multimap Y} = \mathbf{Set}_0(X, Y)$ and for $0_{X \multimap Y}$ we take the function such that $0_{X \multimap Y}(x) = 0_Y$ for all $x \in \underline{X}$. Let $e : \underline{X \multimap Y} \times \underline{X} \to \underline{Y}$ be defined by $e(f, x) = f(x)$. Prove that e is a bimorphism and that the SMC $\mathbf{Set}_0$ is closed.

(g) Prove that there is no object $\perp$ of $\mathbf{Set}_0$ which turns this symmetric monoidal closed category into a $*$-autonomous category.

(h) Given a family $(X_i)_{i\in I}$ of objects of $\mathbf{Set}_0$ we define an object X as follows: $\underline{X} = \prod_{i\in I} \underline{X_i}$ and $0_X = (0_{X_i})_{i\in I} \in \underline{X}$ so that the the projections $\pi_i : \underline{X} \to \underline{X_i}$ are obviously morphisms of $\mathbf{Set}_0$. Prove that X, together with these projections, is the cartesian product of the family $(X_i)_{i\in I}$ that we denote as $\&_{i\in I} X_i$.

Notice that the terminal object (which is the product of an empty family of objects) is $\top = (\{0_\top\}, 0_\top)$.

Contrarily to **Rel**, the category $\mathbf{Set}_0$ has all (projective) limits. It seems rather difficult to build $*$-autonomous categories which are at the same type complete. A noticeable exception is the category of complete semilattices (next exercise).

Given an object X of $\mathbf{Set}_0$, we define $!X$ by $\underline{!X} = \{(0, 0_!)\} \cup \{1\} \times \underline{X}$ where $0_!$ is a chosen element (for instance, a given integer) and $0_{!X} = (0, 0_!)$. Notice that $(1, 0_X) \in \underline{!X}$ but $0_{!X} \neq (1, 0_X)$. So $!X$ is just X to which we have added a new 0-element.

Given $f \in \mathbf{Set}_0(X, Y)$, we define $!f \in \mathbf{Set}_0(!X, !Y)$ by $!f(0_{!X}) = 0_{!Y}$ and $!f(1, x) = (1, f(x))$. This obviously defines a functor $\mathbf{Set}_0 \to \mathbf{Set}_0$.

(i) We define $\mathsf{der}_X : \mathbf{Set}_0(!X, X)$ by $\mathsf{der}_X(0_{!X}) = 0_X$ and $\mathsf{der}_X(1, x) = x$. Prove that this is a natural transformation.

(j) We define $\mathsf{dig}_X \in \mathbf{Set}_0(!X, !!X)$ by $\mathsf{dig}_X(0, 0_!) = (0, 0_!)$, that is $\mathsf{dig}_X(0_{!X}) = 0_{!!X}$, and $\mathsf{dig}_X(1, x) = (1, (1, x))$ which is easily seen to be a natural transformation. Prove that equipped with the natural transformations der and dig the functor $!\_$ is a comonad.

(k) Given two objects X and Y of $\mathbf{Set}_0$, exhibit an isomorphism between $!(X \,\&\, Y)$ and $!X \otimes !Y$.

(l) Prove that the Kleisli category of "!" can be identified with the category whose objects are pointed sets and morphisms are arbitrary functions (not necessarily preserving the 0 element).

3. In this exercise we study a model of linear logic which is based on complete sup-semilattices and linear maps. A complete sup-semilattice (most often we will simply say "complete semilattice" or CSL) is a partially ordered set S (the order relation will always be denoted as $\leq$ or $\leq_S$ if required) such that any subset A of S has a least upper bound $\bigvee A \in S$ (also called "lub", "sup" or "supremum"). Remember that this means

   - $\forall x \in A\ x \leq \bigvee A$
   - $\forall x \in S\ (\forall y \in A\ y \leq x) \Rightarrow \bigvee A \leq x$.

   In particular we have two elements $0 = \bigvee \emptyset$ which is the least element of S and $1 = \bigvee S$ which is the greatest element of S. In particular, a sup-semilattice is never empty.

   A subset A of S is *down-closed* if for all $x \in A$ and all $y \in S$, if $y \leq x$ then $y \in A$. Given $x \in S$ we set $\downarrow x = \{y \in S \mid y \leq x\}$.

   A linear morphism of CSL from S to T is a function $f : S \to T$ such that for all $A \subseteq S$ $f(\bigvee A) = \bigvee f(A)$ where we define as usual $f(A) = \{f(x) \mid x \in A\}$. Notice that this implies that f is monotone: given $x \leq y$ in S we have $f(y) = f(\bigvee\{x, y\}) = f(x) \vee f(y)$, that is $f(x) \leq f(y)$. Let

Csl be the category whose objects are the sup-semilattices and morphisms are the linear maps of sup-semilattices. We set $\perp = \{0 < 1\}$ for the object of **Csl** which has exactly two elements.

It is important to remember that any inf-semilattice, partially ordered set S where each $A \subseteq S$ has a greatest lower bound (also called "glb", "inf" of "infimum") $\bigwedge A$, is also a sup-semilattice: $\bigvee A = \bigwedge\{x \in S \mid \forall y \in A\ y \leq x\} = \bigwedge\{x \in S \mid A \subseteq \downarrow x\}$.

It is easy to check that **Csl** is cartesian. The product of a family $(S_j)_{j\in J}$ of objects of **Csl** is the usual cartesian product $\prod_{j\in J} S_j$ equipped with the product order and projection defined in the usual way. We also use $S = \&_{j\in J} S_j$ for this product and $\pi_j \in \mathbf{Csl}(S, S_j)$ for the projections. The terminal object is $\top = \{0\}$.

(a) Show that the isomorphisms of **Csl** are the linear morphisms which are bijections.

(b) Given a set X we denote as $\mathcal{P}(X)$ its powerset (that is, the set of all of its subsets) ordered under inclusion, so that $\mathcal{P}(X)$ is a sup-semilattice for $\bigvee A = \bigcup A$ for any $A \subseteq \mathcal{P}(X)$. Given $t \in \mathbf{Rel}(X, Y)$ we define $\mathsf{fun}(t) : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\mathsf{fun}(t)(x) = t \cdot x = \{b \in Y \mid \exists a \in x\ (a, b) \in t\}$. Prove that $\mathsf{fun}(t) \in \mathbf{Csl}(\mathcal{P}(X), \mathcal{P}(Y))$ and that, for any $f \in \mathbf{Csl}(\mathcal{P}(X), \mathcal{P}(Y))$ there is exactly one $t = \mathsf{tr}(f) \in \mathbf{Rel}(X, Y)$ such that $f = \mathsf{fun}(t)$. In other words, the functor $L : \mathbf{Rel} \to \mathbf{Csl}$ which maps X to $\mathcal{P}(X)$ and t to $\mathsf{fun}(t)$ is full and faithful. This is the categorical way of saying that **Rel** is a "subcategory" of **Csl**.

(c) Prove that the category **Csl** has all equalizers, in other words: given objects S and T of **Csl** and $f, g \in \mathbf{Csl}(S, T)$ there is an object E of **Csl** and a morphism $e \in \mathbf{Csl}(E, S)$ such that $f\,e = g\,e$ and, for any object V of **Csl** and any morphism $h \in \mathbf{Csl}(V, S)$ such that $f\,h = g\,h$, there is exactly one morphism $h_0 \in \mathbf{Csl}(V, E)$ such that $h = e\,h_0$.

The *Cantor space* is the set $\{0, 1\}^\omega$ of all infinites sequences α of 0 and 1 equipped with the following topology (which is the product topology of the discrete space $\{0, 1\}$): a subset U of $\{0, 1\}^\omega$ is open iff for any $\alpha \in U$ there is a finite prefix w of α such that, for any $\beta \in \{0, 1\}^\omega$, if w is a prefix of β then $\beta \in U$. In other words, a subset F of $\{0, 1\}^\omega$ is closed iff it has the following property: if $\alpha \in \{0, 1\}^\omega$ is such that, for any finite prefix w of α there exists $\beta \in F$ such that w is a prefix of β, then $\alpha \in F$. As in any topological spaces, if $\mathcal{F}$ is a set of closed subsets then $\bigcap \mathcal{F}$ is closed (you are advised to check this directly using the characterization above of closed subsets).

So the set of closed subsets of $\{0, 1\}^\omega$ ordered by inclusions is an inf-CSL (that is, any subset $\mathcal{F}$ has a greatest lower bound, namely $\bigcap \mathcal{F}$) and hence also a CSL: the sup of a set of closed sets is the closure of its union (= the intersection of all closed sets which contain this union).

(d) **(**)** Let $W = \{0, 1\}^*$ be the set of all finite sequences of 0 and 1. If $w = \langle a_1, \dots, a_n\rangle \in W$ is such a sequence and $a \in \{0, 1\}$ let $wa = \langle a_1, \dots, a_n, a\rangle$. Let $\theta = \{(wa, w) \mid w \in W \text{ and } a \in \{0, 1\}\} \in \mathbf{Rel}(W, W)$. Let (C, c) be the equalizer of $\mathsf{Id}, \mathsf{fun}(\theta) \in \mathbf{Csl}(\mathcal{P}(W), \mathcal{P}(X))$ (so that C is a sup-semilattice and $c \in \mathbf{Csl}(C, \mathcal{P}(W))$. Exhibit an order isomorphism between C and the set of all closed subsets of the Cantor space.

Given a CSL S, we say that $x \in S$ is *prime* if

$$\forall A \subseteq S \quad x \leq \bigvee A \Rightarrow \exists y \in A\ x \leq y\,.$$

(e) **(*)** Prove that, for a set X, the prime elements of $\mathcal{P}(X) \in \mathbf{Csl}$ are exactly the singletons. Prove that C, in sharp contrast with the previous case, has no prime elements.

[*Hint:* prove first that if $F \in C$ is prime, it must be a singleton $\{\alpha\}$ and then prove that no such singleton is prime. For this notice that, for a collection $\mathcal{F}$ of closed subsets of $\{0, 1\}^\omega$, the closed set $\bigvee \mathcal{F}$ is the closure of $\bigcup \mathcal{F}$ (the intersection of all closed sets which contain $\bigcup \mathcal{F}$). So consider a set $\mathcal{F}$ of shape $\mathcal{F} = \{\{\alpha(n)\} \mid n \in \mathbb{N}\}$ where $\alpha(n) \to_{n\to\infty} \alpha$ and $\forall n \in \mathbb{N}\ \alpha(n) \neq \alpha$.]

This shows that the category **Rel** is not complete, indeed it has no equalizer for the two maps $\theta, \mathsf{Id} \in \mathbf{Rel}(W, W)$ because the equalizer of $\mathsf{fun}(\theta)$ and Id in **Csl** is not an object of **Rel**. Indeed this equalizer is an infinite CSL which has no prime elements, whereas the only set X such that the CSL $(\mathcal{P}(X), \subseteq)$ has no prime element is $X = \emptyset$.

(f) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S\ f(x) \leq g(x)$), is a sup-CSL. We denote it as $S \multimap T$.

(g) Given $x \in S$ define a function $x^* : S \to \perp$ by

$$x^*(y) = \begin{cases} 1 & \text{if } y \not\leq x \\ 0 & \text{if } y \leq x \end{cases}$$

Prove that $x^* \in S \multimap \perp$.

(h) Given a sup-CSL S, we use S^{op} for the same set S equipped with the reverse order: $x \leq_{S^{\mathsf{op}}} y$ if $y \leq_S x$. Prove that the map $x \mapsto x^*$ is an order isomorphism from the poset S^{op} to $S \multimap \perp$. Warning: one must prove that it is monotone in both directions because a monotone bijection is not necessarily an order isomorphism! Call $\mathsf{k} : (S \multimap \perp) \to S^{\mathsf{op}}$ the inverse isomorphism.

(i) **(*)** Given $f \in (S \multimap T)$ define $f^* : (T \multimap \perp) \to (S \multimap \perp)$ by $f^*(y') = y' f$. Prove that $f^* \in \mathbf{Csl}(T \multimap \perp, S \multimap \perp)$. Let $f^\perp \in \mathbf{Csl}(T^{\mathsf{op}}, S^{\mathsf{op}})$ be the associated morphism (through the iso k defined above, that is $f^\perp(y) = \mathsf{k}(f^*(y^*))$). Prove that

$$\forall x \in S\, \forall y \in T \quad f(x) \leq y \Leftrightarrow x \leq f^\perp(y)\,.$$

One says that f and $f^\perp$ define a Galois connection between S and T. Last prove that $f^{\perp\perp} = f$.

(j) Given sup-CSL S and T we define $S \otimes T$ as the set of all $I \subseteq S \times T$ such that

- I is down-closed
- and, for all $A \subseteq S$ and $B \subseteq T$, if A and B satisfy $A \times B \subseteq I$ then $(\bigvee A, \bigvee B) \in I$.

Prove that $(S \otimes T, \subseteq)$ is an inf-CSL (that is, is closed under arbitrary intersections). As a consequence, it is also a sup-CSL: if $\mathcal{I} \subseteq S \otimes T$ then $\bigvee \mathcal{I} = \bigcap \{I \in S \otimes T \mid \bigcup \mathcal{I} \subseteq I\}$. But notice that in this sup-CSL, the sups are not defined as unions in general.

(k) Prove that the least element of $S \otimes T$ is $0_{S \otimes T} = S \times \{0\} \cup \{0\} \times T$. [*Hint:* Remember that $\bigvee \emptyset = 0$ and that $\emptyset \times B = \emptyset$ for any B.]

(l) We say that a map $f : S \times T \to U$ (where S, T, U are sup-CSLs) is *bilinear* if for all $A \subseteq S$ and $B \subseteq T$ we have $\bigvee f(A \times B) = f(\bigvee (A \times B)) = f(\bigvee A, \bigvee B)$. Prove that this condition is equivalent to the following:

- for all $x \in S$ and $B \subseteq T$, one has $f(x, \bigvee B) = \bigvee_{y \in B} f(x, y)$
- and for all $y \in T$ and $A \subseteq S$, one has $f(\bigvee A, y) = \bigvee_{x \in A} f(x, y)$

that is, f is separately linear in both variables.

(m) **(*)** Given $x \in S$ and $y \in T$ let $x \otimes y = \downarrow(x, y) \cup 0_{S \otimes T} \subseteq S \times T$. Prove that $x \otimes y \in S \otimes T$ and that the function $\tau : (x, y) \mapsto x \otimes y$ is a bilinear map $S \times T \to S \otimes T$.

(n) Let $(S, T) \multimap U$ be the set of all bilinear maps $S \times T \to U$ ordered pointwise (that is $f \leq g$ if $\forall (x, y) \in S \times T\ f(x, y) \leq g(x, y)$). Prove that $(S, T) \multimap U \simeq (S \multimap (T \multimap U))$. Deduce from this fact that $(S, T) \multimap U$ is a sup-CSL.

(o) Given $I \in X \otimes Y$ let $f^I : S \times T \to \perp$ be given by

$$f^I(x, y) = \begin{cases} 0 & \text{if } (x, y) \in I \\ 1 & \text{otherwise.} \end{cases}$$

Prove that f^I is bilinear. Conversely given $f \in (S, T) \multimap \perp$ prove that $\mathsf{ker}_2 f = \{(x, y) \in S \times T \mid f(x, y) = 0\}$ belongs to $S \otimes T$. Prove that these operations define an order isomorphism between $S \otimes T$ and $((S, T) \multimap \perp)^{\mathsf{op}}$.

4. This problem is the sequel of the previous one. We deal now with a class of non-(multi)linear functions. Given two objects S, T of $\mathbf{Csl}$ we define $S \Rightarrow_{\mathsf{s}} T$ as the set of all Scott continuous functions $S \to T$, that is, of all monotone functions $f : S \to T$ such that, for any *directed* $D \subseteq S$ one has $f(\bigvee D) = \bigvee f(D) = \bigvee \{f(x) \mid x \in D\}$. We equip this set with the following order relation: $f \leq g$ is $\forall x \in S\ f(x) \leq g(x)$. We recall that $D \subseteq S$ is directed if D is non-empty and $\forall x, y \in D\, \exists z \in D\ x \leq z$ and $y \leq z$, equivalently: any finite subset of D has an upper bound in D.

(a) Given CSL S, T, U and $f : S \mathbin{\&} T \to U$, prove that f is Scott continuous iff it is separately Scott-continuous, that is: for all $x \in S$ the function $y \mapsto f(x, y)$ is Scott-continuous $T \to U$ and for any $y \in T$ the function $x \mapsto f(x, y)$ is Scott-continuous $S \to U$.

(b) Prove that sup-CSL and Scott-continuous functions form a category, that we will denote as **CslC**. Prove that this category has all products (defined as in **Csl**).

(c) Prove that $S \Rightarrow_{\mathsf{s}} T$ is a sup-CSL.

(d) Prove that the function $\mathsf{Ev} : (S \Rightarrow_{\mathsf{s}} T) \mathbin{\&} S \to T$ which maps (f, x) to $f(x)$ is Scott continuous.

(e) Prove that **CslC** is cartesian closed, with $(S \Rightarrow_{\mathsf{s}} T, \mathsf{Ev})$ as object of morphisms from S to T.

(f) Let S be an object of **Csl**. We define $!_{\mathsf{s}}S$ as the set of all $I \subseteq \mathcal{P}(S)$ (the powerset of S) which are down-closed and such that, for any directed subset D of S, if $D \subseteq I$ then $\bigvee D \in I$. Prove that, equipped with the $\subseteq$ partial order relation, $!_{\mathsf{s}}S$ is an inf-CSL where infima are intersections. Therefore it is also a sup-CSL (but suprema are not unions in general). What is the least element of $!_{\mathsf{s}}S$ (give a proof of your answer)?

(g) Prove that if $\mathcal{I} \subseteq !_{\mathsf{s}}S$ is directed then $\bigvee \mathcal{I} = \bigcup \mathcal{I}$. And prove that if $I \in !_{\mathsf{s}}S$ then $\bigvee\{\downarrow x \mid x \in I\} = \bigcup\{\downarrow x \mid x \in I\} = I$. As a consequence show that if $\varphi \in \mathbf{Csl}(!_{\mathsf{s}}S, T)$ then $\forall I \in !_{\mathsf{s}}S\ \varphi(I) = \bigvee\{\varphi(\downarrow x) \mid s \in I\}$. Show that if $\varphi, \psi \in \mathbf{Csl}(!_{\mathsf{s}}S, T)$ satisfy $\forall x \in S\ \varphi(\downarrow x) = \psi(\downarrow x)$ then $\varphi = \psi$.

(h) Let $\varphi \in \mathbf{Csl}(S, T)$. If $I \in !_{\mathsf{s}}S$ wet set $!_{\mathsf{s}}\varphi(I) = \bigvee\{\downarrow\varphi(x) \mid x \in I\} = \bigcap\{J \in !_{\mathsf{s}}T \mid \varphi(I) \subseteq J\}$. Prove that $!_{\mathsf{s}}\varphi \in \mathbf{Csl}(!_{\mathsf{s}}S, !_{\mathsf{s}}T)$ and that $!_{\mathsf{s}}\_$ is a functor $\mathbf{Csl} \to \mathbf{Csl}$. Notice that $!_{\mathsf{s}}\varphi$ is fully characterized by $\forall x \in S\ !_{\mathsf{s}}\varphi(\downarrow x) = \downarrow\varphi(x)$.

(i) Let $f \in \mathbf{CslC}(S, T)$. For $y \in T$ let

$$\overline{f}(y) = \{x \in S \mid f(x) \leq y\}\,.$$

Prove that $\overline{f} \in \mathbf{Csl}(T^{\mathsf{op}}, (!S)^{\mathsf{op}})$, that is, prove first that $\forall y \in T\ \overline{f}(y) \in !_{\mathsf{s}}S$ and then that, for any $B \subseteq T$, one has $\overline{f}(\bigwedge B) = \bigcap_{y \in B} \overline{f}(y)$.

(j) Let $\mathsf{lin}(f) = \overline{f}^{\perp} \in \mathbf{Csl}(!S, T)$ so that

$$\forall I \in !_{\mathsf{s}}S\, \forall y \in T \quad \mathsf{lin}(f)(I) \leq y \Leftrightarrow I \subseteq \overline{f}(y)\,.$$

Prove that $\mathsf{lin}(f)(I) = \bigwedge\{y \in T \mid f(I) \subseteq \downarrow y\}$ where as usual $f(I) = \{f(x) \mid x \in I\}$. [*Hint:* See Question (i) of Problem 3.]

(k) Let $\mathsf{cnt} : S \to \mathcal{P}(S)$ be defined by $\mathsf{cnt}(x) = \downarrow x$. Prove that $\mathsf{cnt} \in \mathbf{CslC}(S, !_{\mathsf{s}}S)$ and that cnt is never linear [*Hint:* Consider $\mathsf{cnt}(0)$.]. Prove that $\mathsf{lin}(\mathsf{cnt}) = \mathsf{Id}_{!_{\mathsf{s}}S}$.

(l) Prove that the function

$$\begin{aligned} \mathbf{Csl}(!_{\mathsf{s}}S, T) &\to \mathbf{CslC}(S, T) \\ \varphi &\mapsto \varphi \circ \mathsf{cnt} \end{aligned}$$

is the inverse of lin.

(m) We define $\mathsf{der}_S = \mathsf{lin}(\mathsf{Id}_S) \in \mathbf{Csl}(!_{\mathsf{s}}S, S)$ where $\mathsf{Id}_S \in \mathbf{CslC}(S, S)$. Prove that $\mathsf{der}_S(I) = \bigvee I$.

(n) Let $f : S \to !_{\mathsf{s}}!_{\mathsf{s}}S$ be the function given by $f(x) = \downarrow\downarrow x$. Prove that f is Scott continuous. Let $\mathsf{dig}_S = \mathsf{lin}(f) \in \mathbf{Csl}(!_{\mathsf{s}}S, !_{\mathsf{s}}!_{\mathsf{s}}S)$.

(o) Prove that der and dig are natural transformations and that $!_{\mathsf{s}}\_$, equipped with these two natural transformations, is a comonad.

(p) For any objects S, T, U prove that there is a bijection $\mathbf{Csl}(!_{\mathsf{s}}(S \mathbin{\&} T), U) \to \mathbf{Csl}(!_{\mathsf{s}}S \otimes !_{\mathsf{s}}T, U)$.

From this observation, we could deduce with a bit more work that there are also Seely isomorphisms, taking $U = !_{\mathsf{s}}(S \mathbin{\&} T)$ and $U = !_{\mathsf{s}}S \otimes !_{\mathsf{s}}T$.

5. The goal of this exercise is to study the properties of the objects of the Eilenberg Moore category $\mathbf{Rel}^!$ of $\mathbf{Rel}$, the relational model of LL.

Let P be an object of $\mathbf{Rel}^!$ (the category of coalgebras of $!\_$). Remember that $P = (\underline{P}, \mathsf{h}_P)$ where $\underline{P}$ is an object of $\mathbf{Rel}$ (a set) and $\mathsf{h}_P \in \mathbf{Rel}(\underline{P}, !\underline{P})$ satisfies the following commutations:

$$
\begin{tikzcd}
\underline{P} \arrow[r, "\mathsf{h}_P"] \arrow[dr, "\underline{P}"'] & !\underline{P} \arrow[d, "\mathsf{der}_{\underline{P}}"] \\
& \underline{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "\mathsf{h}_P"] \arrow[d, "\mathsf{h}_P"'] & !\underline{P} \arrow[d, "\mathsf{dig}_{\underline{P}}"] \\
!\underline{P} \arrow[r, "!\mathsf{h}_P"] & !!\underline{P}
\end{tikzcd}
$$

(a) Check that these commutations mean:

- for all $a, a' \in \underline{P}$, one has $(a, [a']) \in \mathsf{h}_P$ iff $a = a'$
- and for all $a \in \underline{P}$ and $m_1, \dots, m_k \in !\underline{P}$, one has $(a, m_1 + \cdots + m_k) \in \mathsf{h}_P$ iff there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, m_i) \in \mathsf{h}_P$ for $i = 1, \dots, k$.

Intuitively, $(a, [a_1, \dots, a_k])$ means that a can be decomposed into "$a_1 + \cdots + a_k$" where the "+" is the decomposition operation associated with P.

(b) Prove that if P is an object of $\mathbf{Rel}^!$ such that $\underline{P} \neq \emptyset$ then there is at least one element e of $\underline{P}$ such that $(e, [\,]) \in \mathsf{h}_P$. Explain why such an e could be called a "coneutral element of P".

(c) If P and Q are objects of $\mathbf{Rel}^!$, remember that an $f \in \mathbf{Rel}^!(P, Q)$ (morphism of coalgebras) is an $f \in \mathbf{Rel}(\underline{P}, \underline{Q})$ such that the following diagram commutes

$$
\begin{tikzcd}
\underline{P} \arrow[r, "f"] \arrow[d, "\mathsf{h}_P"'] & \underline{Q} \arrow[d, "\mathsf{h}_Q"] \\
!\underline{P} \arrow[r, "!f"] & !\underline{Q}
\end{tikzcd}
$$

Check that this commutation means that for all $a \in \underline{P}$ and $b_1, \dots, b_k \in \underline{Q}$, the two following properties are equivalent

- there is $b \in \underline{Q}$ such that $(a, b) \in f$ and $(b, [b_1, \dots, b_k]) \in \mathsf{h}_Q$
- there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, b_i) \in f$ for $i = 1, \dots, k$.

(d) Remember that 1 (the set $\{*\}$) can be equipped with a structure of coalgebra (still denoted 1) with $\mathsf{h}_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$. Prove that the elements of $\mathbf{Rel}^!(1, P)$ can be identified with the subsets x of $\underline{P}$ such that: for all $a_1, \dots, a_k \in \underline{P}$, one has $a_1, \dots, a_k \in x$ iff there exists $a \in x$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$. We call *values* of P these subsets of $\underline{P}$ and denote as $\mathsf{val}(P)$ the set of these values.

Prove that an element of $\mathsf{val}(P)$ is never empty and that $\mathsf{val}(P)$, equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to $\subseteq$) is still a value.

(e) Remember that if E is an object of $\mathbf{Rel}$ then $(!E, \mathsf{dig}_E)$ is an object of $\mathbf{Rel}^!$ (the free coalgebra generated by E, that we can identify with an object of the Kleisli category $\mathbf{Rel}_!$). Prove that, as a partially ordered set, $\mathsf{val}(!E, \mathsf{dig}_E)$ is isomorphic to $\mathcal{P}(E)$.

(f) Is it always true that if $x_1, x_2 \in \mathsf{val}(P)$ then $x_1 \cup x_2 \in \mathsf{val}(P)$?

(g) We have seen (without proof) that $\mathbf{Rel}^!$ is cartesian. Remember that the product of P_1 and P_2 is $P_1 \otimes P_2$, the coalgebra defined by $\underline{P_1 \otimes P_2} = \underline{P_1} \otimes \underline{P_2}$ and $\mathsf{h}_{P_1 \otimes P_2}$ is the following composition of morphisms in $\mathbf{Rel}$:

$$
\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2}} !\underline{P_1} \otimes !\underline{P_2} \xrightarrow{\mu^2_{\underline{P_1}, \underline{P_2}}} !(\underline{P_1} \otimes \underline{P_2})
$$

where $\mu^2_{E_1, E_2} \in \mathbf{Rel}(!E_1 \otimes !E_2, !(E_1 \otimes E_2))$ is the lax monoidality natural transformation of $!\_$, remember that in $\mathbf{Rel}$ we have

$$
\begin{aligned}
\mu^2_{E_1, E_2} = \{(([a_1, \dots, a_k], [b_1, \dots, b_k]), [(a_1, b_1), \dots, (a_k, b_k)]) \mid \\
k \in \mathbb{N} \text{ and } (a_1, b_1), \dots, (a_k, b_k) \in E_1 \times E_2\}\,.
\end{aligned}
$$

Concretely, we have simply that $((a_1, a_2), [(a_1^1, a_2^1), \dots, (a_1^k, a_2^k)]) \in \mathsf{h}_{P_1 \otimes P_2}$ iff $(a_i, [a_i^1, \dots, a_i^k]) \in \mathsf{h}_{P_i}$ for $i = 1, 2$.

Prove that $P_1 \otimes P_2$, equipped with suitable projections, is the cartesian product of P_1 and P_2 in $\mathbf{Rel}^!$. Prove also that 1 is the terminal object of $\mathbf{Rel}^!$. Warning: $\mathcal{L}^!$ is always cartesian when $\mathcal{L}$ is a model of LL; I'm not asking for a general proof, just for a verification that this is true in $\mathbf{Rel}^!$.

(h) Check directly that the partially ordered sets $\mathsf{val}(P_1 \otimes P_2)$ and $\mathsf{val}(P_1) \times \mathsf{val}(P_2)$ are isomorphic.

(i) Remember also that we have defined $P_1 \oplus P_2 = (\underline{P_1} \oplus \underline{P_2}, \mathsf{h}_{P_1 \oplus P_2})$ where $\mathsf{h}_{P_1 \oplus P_2}$ is the unique element of $\mathbf{Rel}(\underline{P_1} \oplus \underline{P_2}, !(\underline{P_1} \oplus \underline{P_2}))$ such that, for $i = 1, 2$, the morphism $\mathsf{h}_{P_1 \oplus P_2}\, \overline{\pi}_i$ coincides with the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_i} \xrightarrow{\mathsf{h}_{P_i}} !\underline{P_i} \xrightarrow{!\overline{\pi}_i} !(\underline{P_1} \oplus \underline{P_2})$$

Describe $\mathsf{h}_{P_1 \oplus P_2}$ as simply as possible and prove that, equipped with suitable injections, $P_1 \oplus P_2$ is the coproduct of P_1 and P_2 in $\mathbf{Rel}^!$.

6. The goal of this exercise is to illustrate the fact that **Rel**, the relational model of LL, can be equipped with additional structures of various kinds *without modifying the interpretation of proofs and programs*. As an example we shall study the notion of *non-uniform coherence space* (NUCS). A NUCS is a triple $X = (|X|, \frown_X, \smile_X)$ where

- $|X|$ is a set (the web of X)
- and $\frown_X$ and $\smile_X$ are two symmetric relations on $|X|$ such that $\frown_X \cap \smile_X = \emptyset$. In other words, for any $a, a' \in |X|$, one never has $a \frown_X a'$ and $a \smile_X a'$.

So we can consider an ordinary coherence space (in the sense of the first part of thise series of lectures) as a NUCS X which satisfies moreover:

$$\forall a, a' \in |X| \quad (a \frown_X a' \text{ or } a \smile_X a') \Leftrightarrow a \neq a' \,.$$

It is then possible to introduce three other natural symmetric relations on the elements of $|X|$:

- $a \equiv_X a'$ if it is not true that $a \frown_X a'$ or $a \smile_X a'$.
- $a \coh_X a'$ if $a \frown_X a'$ or $a \equiv_X a'$.
- $a \asymp_X a'$ if $a \smile_X a'$ or $a \equiv_X a'$.

A *clique* of a NUCS X is a subset x of $|X|$ such that $\forall a, a' \in |X|\ a \coh_X a'$, we use $\mathsf{Cl}(X)$ for the set of cliques of X.

We say that a NUCS X satisfies the Boudes' Condition[1] (or simply that X is Boudes) if

$$\forall a, a' \in |X|\ a \equiv_X a' \Rightarrow a = a' \,.$$

We shall show that the class of NUCS's can be turned into a categorical model of LL in such a way that all the operations on objects coincide with the corresponding operations on objects in **Rel**. For instance we shall define $!X$ in such a way that $|!X| = !|X| = \mathcal{M}_{\mathrm{fin}}(|X|)$. Moreover, all the "structure morphisms" of this model *will be defined exactly as in* **Rel**. For instance, the digging morphism from $!X$ to $!!X$ will simply be $\mathsf{dig}_{|X|}$. Important: such definitions are impossible with ordinary coherence spaces. When defining $|!E|$ in ordinary coherence spaces one *needs* to restrict to the finite multisets (or finite sets) of elements of $|E|$ which *are cliques of* E. It is exactly for that reason that, in $NUCS$'s, the relation $\equiv_X$ is not required to coincide with equality. Nevertheless, the weaker Boudes' condition will be preserved by all of our constructions.

(a) Check that a NUCS can be specified by $|X|$ together with any of the following seven pairs of relations.

- Two symmetric relations $\coh_X$ and $\frown_X$ on $|X|$ such that $\frown_X \subseteq \coh_X$. Then setting $\smile_X = (|X| \times |X|) \setminus \coh_X$, the relation $\coh_X$ is the one canonically associated with the NUCS $(|X|, \frown_X, \smile_X)$.
- Two symmetric relations $\asymp_X$ and $\smile_X$ on $|X|$ such that $\smile_X \subseteq \asymp_X$. How should we define $\frown_X$ in that case?
- Two symmetric relations $\coh_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \coh_X$. How should we define $\frown_X$ and $\smile_X$ in that case?
- Two symmetric relations $\asymp_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \asymp_X$. How should we define $\frown_X$ and $\smile_X$ in that case?

[1] From Pierre Boudes who discovered this condition and the nice properties of these objects.

- Two symmetric relations $\frown_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \frown_X = \emptyset$. How should we define $\smile_X$ in that case?
- Two symmetric relations $\smile_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \smile_X = \emptyset$. How should we define $\frown_X$ in that case?
- Two symmetric relation $\supset_X$ and $\asymp_X$ such that $\supset_X \cup \asymp_X = |X| \times |X|$. How should we define $\frown_X$ and $\smile_X$ in that case?

(b) Given *NUCS*'s X and Y, we define a NUCS $X \multimap Y$ by $|X \multimap Y| = |X| \times |Y|$ and

- $(a, b) \equiv_{X \multimap Y} (a', b')$ if $a \equiv_X a'$ and $b \equiv_Y b'$
- and $(a, b) \frown_{X \multimap Y} (a', b')$ if $a \smile_X a'$ or $b \frown_Y b'$.

Check that we have defined in that way a NUCS. Prove that $\mathsf{Id}_{|X|} = \{(a, a) \mid a \in |X|\} \in \mathsf{Cl}(X \multimap X)$. Prove that if X and Y are Boudes then $X \multimap Y$ is Boudes.

(c) Prove that, if $s \in \mathsf{Cl}(X \multimap Y)$ and $t \in \mathsf{Cl}(Y \multimap Z)$ then $t\,s \in \mathsf{Cl}(X \multimap Z)$. So we define a category **Nucs** by taking the NUCS's as object and by setting $\mathbf{Nucs}(X, Y) = \mathsf{Cl}(X \multimap Y)$.

(d) We define $X^\perp$ by $|X^\perp| = |X|$, $\frown_{X^\perp} = \smile_X$ and $\smile_{X^\perp} = \frown_X$. Then we set $X \otimes Y = \left(X \multimap Y^\perp\right)^\perp$. Describe as simply as possible the NUCS structure of $X \otimes Y$. We set $1 = (\{*\}, \emptyset, \emptyset)$ (in other words $* \equiv_1 *$). Prove that if X and Y are Boudes then $X^\perp$ and $X \otimes Y$ is Boudes.

(e) Given $s_i \in \mathbf{Nucs}(X_i, Y_i)$ for $i = 1, 2$, prove that $s_1 \otimes s_2 \in \mathbf{Rel}(|X_1| \otimes |X_2|, |Y_1| \otimes |Y_2|)$ (defined as in **Rel**) does actually belong to $\mathbf{Nucs}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.

(f) Check quickly that **Nucs** (equipped with the $\otimes$ defined above and 1 as tensor unit, and $\perp = 1$ as dualizing object) is a $*$-autonomous category.

(g) Prove that the category **Nucs** is cartesian and cocartesian, with $X = \&_{i \in I} X_i$ given by $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$, and

- $(i, a) \equiv_X (i', a')$ if $i = i'$ and $a \equiv_{X_i} a'$
- $(i, a) \smile_X (i', a')$ if $i = i'$ and $a \smile_{X_i} a'$.

and the associated operations (projections, tupling of morphisms) defined as in **Rel**.

Prove that if all X_i's are Boudes then $\&_{i \in I} X_i$ is Boudes.

(h) We define $!X$ as follows. We take $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and, given $m, m' \in |!X|$

- we have $m \supset_{!X} m'$ if for all $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ one has $a \supset_X a'$
- and $m \equiv_{!X} m'$ if $m \supset_{!X} m'$ and $m = [a_1, \dots, a_k]$, $m' = [a'_1, \dots, a'_k]$ with $a_i \equiv_X a'_i$ for each $i \in \{1, \dots, k\}$.

Notice that $m \smile_{!X} m'$ iff there is $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ such that $a \smile_X a'$. Remember that $\mathsf{supp}(m) = \{a \in |X| \mid m(a) \neq 0\}$.

Let $s \in \mathbf{Nucs}(X, Y)$. Prove that $!s \in \mathbf{Rel}(!|X|, !|Y|)$ actually belongs to $\mathbf{Nucs}(!X, !Y)$.

(i) Prove that $\mathsf{der}_{|X|} = \{([a], a) \mid a \in |X|\}$ belongs to $\mathbf{Nucs}(!X, X)$.

(j) Prove that $\mathsf{dig}_X = \{(m_1 + \cdots + m_k, [m_1, \dots, m_k]) \mid m_1, \dots, m_k \in \mathcal{M}_{\mathrm{fin}}(|X|)\}$ is an element of $\mathbf{Nucs}(!X, !!X)$.

(k) Prove that if X is Boudes then $!X$ is Boudes.

(l) Let $X = 1 \oplus 1$, and let $\mathbf{t}, \mathbf{f}$ be the two elements of $|X|$ (X is the "type of booleans"). Let $s \in \mathbf{Rel}(|X| \otimes |X|, |X|)$ by $s = \{((\mathbf{t}, \mathbf{f}), \mathbf{t}), ((\mathbf{f}, \mathbf{t}), \mathbf{f})\}$. Prove that $s \in \mathbf{Nucs}(X \otimes X, X)$. Let then $t \in \mathbf{Nucs}(!X, X)$ be defined by the following composition of morphisms in **Nucs**:

$$!X \xrightarrow{\mathsf{c}_X} !X \otimes !X \xrightarrow{\mathsf{der}_X \otimes \mathsf{der}_X} X \otimes X \xrightarrow{s} X$$

We recall that contraction

$$\mathsf{c}_X \in \mathbf{Nucs}(!X, !X \otimes !X)$$

is given by $\mathsf{c}_X = \{m_1 + m_2, (m_1, m_2) \mid m_1, m_2 \in !|X|\}$ and dereliction $\mathsf{der}_X \in \mathbf{Nucs}(!X, X)$ is given by $\mathsf{der}_X = \{([a], a) \mid a \in |X|\}$.

Prove that $([\mathbf{t}, \mathbf{f}], \mathbf{t}), ([\mathbf{t}, \mathbf{f}], \mathbf{f}) \in t$. So any notion of coherence on $!|X|$ *must* satisfy $[\mathbf{t}, \mathbf{f}] \smile_{!X} [\mathbf{t}, \mathbf{f}]$ since we have $\mathbf{t} \smile_X \mathbf{f}$ by the definition of the NUCS $1 \oplus 1$ since we *must have* $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \supset_{!X \multimap X}$

$([\mathbf{t},\mathbf{f}],\mathbf{f})$ because t is a clique. In particular it is impossible to endow $!|X|$ with a notion of Girard's coherence space since in such a coherence space we would have $[\mathbf{t},\mathbf{f}] \frown_{!X} [\mathbf{t},\mathbf{f}]$ and hence $([\mathbf{t},\mathbf{f}],\mathbf{t}) \smile_{!X\multimap X} ([\mathbf{t},\mathbf{f}],\mathbf{f})$.

As an illustration of the usefulness of this semantics, consider the language PCF studied during the lectures. Let M be a closed term such that $\vdash M : \iota$. By the Church Rosser Theorem for PCF (of which we have outlined the proof) we know that if $M\ \beta^*\ \underline{n}$ and $M\ \beta^*\ \underline{p}$ then $n = p$. This proof is completely syntactic and not very modular (if we modify the syntax, a lot of work has to be redone). The PCF type ι is interpreted in any model of LL as $\mathsf{N} = 1 \oplus 1 \oplus \cdots$. In **Nucs**, the only cliques of N are $\emptyset$ and the singletons. The semantics of any term is identical in **Rel** and in **Nucs**. Since the semantics of M in **Nucs** is a clique of N, this proves that if $M\ \beta^*\ \underline{n}$ and $M\ \beta^*\ \underline{p}$ then $n = p$.

7. Remember that, in probabilistic coherence spaces, $\mathsf{P}(1) = [0,1]$ (the closed unit interval) and $\mathsf{P}(1\oplus 1)$ can be described as follows

$$\mathsf{P}(1 \oplus 1) = \{(u_0, u_1) \in [0,1]^2 \mid u_0 + u_1 \leq 1\}$$

whose elements are the sub-probability distributions on the booleans. We consider the program $M : \texttt{bool} \to \texttt{bool}$ defined by

$$\begin{aligned}\texttt{let } M(x) = \texttt{if } x \texttt{ then if } x \texttt{ then true else false}\\ \texttt{else if } x \texttt{ then false else true}\end{aligned}$$

and we admit that the semantics of M in the model of probabilistic coherence spaces is the function $f : \mathsf{P}(1 \oplus 1) \to \mathsf{P}(1 \oplus 1)$ given by

$$f(u_0, u_1) = (u_0^2 + u_1^2)\mathsf{e}_0 + 2u_0u_1\mathsf{e}_1$$

where $\mathsf{e}_0 = (1,0) \in \mathsf{P}(1 \oplus 1)$ and $\mathsf{e}_1 = (0,1) \in \mathsf{P}(1 \oplus 1)$. Check that when (u_0, u_1) is a probability distribution (that is $u_0 + u_1 = 1$), so is $f(u_0, u_1)$. Compute the distributions $f(1,0)$, $f(0,1)$, $f(\frac{1}{2}, \frac{1}{2})$, $f(\frac{1}{4}, \frac{3}{4})$ and $f(\frac{2}{3}, \frac{1}{3})$.

8. We consider the program $M : \texttt{bool} \to \texttt{bool}$ defined by

$$\begin{aligned}\texttt{let rec } M(x) = \texttt{if } x \texttt{ then if } x \texttt{ then } M(x) \texttt{ else true}\\ \texttt{else if } x \texttt{ then false else } M(x)\end{aligned}$$

called the *Von Neumann rectifier*. We admit that the semantics of M in the model of probabilistic coherence spaces is a function $f : \mathsf{P}(1) \times \mathsf{P}(1 \oplus 1) \to \mathsf{P}(1)$ which satisfies

$$f(u_0, u_1) = (u_0^2 + u_1^2)f(u_0, u_1) + u_0u_1(\mathsf{e}_0 + \mathsf{e}_1)\,.$$

and is minimal for this property (for the pointwise order on functions: $g \leq g'$ if $g(u_0, u_1) \leq g'(u_0, u_1)$ for all $(u_0, u_1) \in \mathsf{P}(1 \oplus 1)$).

(a) Prove that, assuming that (u_0, u_1) is a probability distribution (that is $u_0 + u_1 = 1$) one has

$$f(u_0, u_1) = \begin{cases} 0 & \text{if } u_0 = 1 \text{ or } u_1 = 1 \\ \frac{1}{2}\mathsf{e}_0 + \frac{1}{2}\mathsf{e}_1 & \text{otherwise.} \end{cases}$$

Explain the name given to this program.

(b) Define the function $g : [0,1] \times [0,1] \to [0,1]$ by

$$g(u,t) = s(f(tu, (1-t)u))$$

where $s : \mathsf{P}(1 \oplus 1) \to [0,1]$ is defined by $s(u_0, u_1) = u_0 + u_1$. Using for instance the function `splot` of `gnuplot`, plot the function g in 3d and observe its behaviour when u is close to 1.

9. We consider the program $M : \texttt{unit} \times \texttt{bool} \to \texttt{unit}$ defined by

$$\texttt{let rec } M(x,y) = \texttt{if } y \texttt{ then } x \texttt{ else } (M(x,y); M(x,y))$$

and we admit that the semantics of M in the model of probabilistic coherence spaces is a function $f : \mathsf{P}(1) \times \mathsf{P}(1 \oplus 1) \to \mathsf{P}(1)$ which satisfies

$$f(w,(u_0,u_1)) = u_0 w + u_1 f(w,(u_0,u_1))^2 \,.$$

(a) Express explicitely $f(w,(u_0,u_1))$ in function of u_0, u_1 and w. For $u_1 \neq 0$ you will have to solve a quadratic equation and hence to choose between "+" and "−" for the square root of the discriminant. Explain why one must choose "−". To this end, remember that there must be coefficients $a_{n,p,q} \in \mathbb{R}_{\geq 0}$ such that $f(w,(u_0,u_1)) = \sum_{n,p,q \in \mathbb{N}} a_{n,p,q} w^n u_0^p u_1^q$ (you are not asked to compute these coefficients). Remember that

$$(1+v)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2} v^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2 \times 3} v^3 + \cdots$$

for $v \in [0,1[$ and $\alpha \in \mathbb{R}_{\geq 0}$ (here $\alpha = \frac{1}{2}$).

(b) Let $g : [0,1] \to [0,1]$ be defined by $g(t) = f(1,(1-t,t))$. Describe this function as simply as possible (observe that there is a critical value for t, namely $t = 1/2$). Draw the graph of g and interpret it computationally.

10. We recall that there is a probabilistic coherence space N such that $|\mathsf{N}| = \mathbb{N}$ and

$$\mathsf{P}(\mathsf{N}) = \{w \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \sum_{n \in \mathbb{N}} w_n \leq 1\} \,.$$

We consider the program $M : \texttt{int} \times \texttt{bool} \to \texttt{unit}$ defined by

$$\texttt{let rec } M(x,y) = \texttt{if } y \texttt{ then } x \texttt{ else } M(x+1,y)$$

and we admit that the semantics of M in the model of probabilistic coherence spaces is a function $f : \mathsf{P}(\mathsf{N}) \times \mathsf{P}(1 \oplus 1) \to \mathsf{P}(\mathsf{N})$ which satisfies

$$f(w,(u_0,u_1)) = u_0 w + u_1 f(s \cdot w,(u_0,u_1)) \,.$$

where $s \in \mathsf{P}(\mathsf{N} \multimap \mathsf{N})$ is given by

$$s_{i,j} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Remember also that, for $i \in \mathbb{N}$, one has $\mathsf{e}_i \in \mathsf{P}(\mathsf{N})$ defined by

$$(\mathsf{e}_i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $w = f(\mathsf{e}_0,(u_0,u_1)) \in \mathsf{P}(\mathsf{N})$ is given by $w_n = u_0 u_1^n$.

11. Let I and J be sets, with $I \subseteq J$. We use $\eta_{I,J}$, or simply η, for the element of $\mathbb{R}_{\geq 0}^{I \times J}$ defined by

$$\eta_{i,j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

and $\rho_{I,J}$, or simply ρ, for the element of $\mathbb{R}_{\geq 0}^{J \times I}$ defined by

$$\rho_{j,i} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

If X and Y are probabilistic coherence spaces, we write $X \leq Y$ if

- $|X| \subseteq |Y|$,
- $\eta_{|X|,|Y|} \in \mathbf{Pcoh}(X,Y)$ (in other words: if $x \in \mathsf{P}(X)$ then $y \in \mathbb{R}_{\geq 0}^{|Y|}$ obtained by extending x with 0's belongs to $\mathsf{P}(Y)$)
- and $\rho_{|X|,|Y|} \in \mathbf{Pcoh}(Y,X)$ (in other words: if $y \in \mathsf{P}(Y)$ then $x \in \mathbb{R}_{\geq 0}^{|X|}$ obtained by restricting y to $|X|$ belongs to $\mathsf{P}(X)$).

When these conditions hold, we use $\eta_{X,Y}$ and $\rho_{X,Y}$ instead of $\eta_{|X|,|Y|}$ and $\rho_{|X|,|Y|}$.

(a) Prove that $X \leq Y \Rightarrow X^{\perp} \leq Y^{\perp}$.

(b) Prove that if $X_1 \leq X_2$ and $Y_1 \leq Y_2$ then $(X_1 \multimap Y_1) \leq (X_2 \multimap Y_2)$. We say that the operation $\_ \multimap \_$ is *increasing*.

(c) Let $(X_n)_{n\in\mathbb{N}}$ be a family of probabilistic coherence spaces which is increasing in the sense that $\forall n \in \mathbb{N}\ X_n \leq X_{n+1}$. Let X be defined as follows: $|X| = \bigcup_{n\in\mathbb{N}} |X_n|$ and $x \in \mathbb{R}_{\geq 0}^{|X|}$ if, for all $n \in \mathbb{N}$, $\rho_{|X_n|,|X|} \cdot x \in \mathsf{P}(X_n)$ (that is, for each $n \in \mathbb{N}$, the restriction of x to $|X_n|$ belongs to $\mathsf{P}(X_n)$). Prove that X is a probabilistic coherence space.

We admit that $X_n \leq X$ for all n, the verification is easy. We set $X = \bigcup_{n\in\mathbb{N}} X_n$.

(d) Prove that X is the colimit of the X_n in the sense that, given any family probabilistic coherence space Y and any family of morphisms $(t(n) \in \mathbf{Pcoh}(X_n, Y))_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ t(n) = t(n+1)\,\eta_{X_n,X_{n+1}}$, there is exactly one $t \in \mathbf{Pcoh}(Y,X)$ such that $\forall n \in \mathbb{N}\ t(n) = t\,\eta_{X_n,X}$.

(e) Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be increasing sequences of probabilistic coherence spaces. Prove that

$$\bigcup_{n\in\mathbb{N}} X_n \multimap \bigcup_{n\in\mathbb{N}} Y_n = \bigcup_{n\in\mathbb{N}} (X_n \multimap Y_n)\,.$$

We will say that the functor $\_ \multimap \_$ is *cocontinuous*.

We admit that the operation $\_ \ \&\ \_$ on probabilistic coherence spaces is similalry increasing and continuous (the proof is very easy). Observe that the probabilistic coherence space 0 whose web is empty is $\leq$ than any probabilistic coherence space.

(f) Prove that if $F(\_)$ is an increasing and continuous operation on probabilistic coherence space, then the sequence $(F^n(0))_{n\in\mathbb{N}}$ is increasing and, setting $\mu F = \bigcup F^n(0)$, one has $\mu F = F(\mu F)$ and μF is minimal with this property (for the order relation $\leq$).

(g) Let F be defined by $F(X) = 1 \ \&\ (X \oplus X)$. Prove that F is increasing and continuous. Prove by induction on $n \in \mathbb{N}$ that $F^n(0)$ can be described as the following probabilistic coherence spaces:
   - $|F^n(0)|$ is the set of all sequences of 0 and 1 of length $\leq n-1$,
   - and $x \in \mathbb{R}_{\geq 0}^{|F^n(0)|}$ belongs to $\mathsf{P}(F^n(0))$ if, for any $u \subseteq |F^n(0)|$, if u satisfies

   $$\forall s, s' \in u \quad s \leq s' \Rightarrow s = s'$$

   (where $\leq$ is the prefix order on finite sequeces; we say that such an u is an *antichain*), then $\sum_{s\in u} x_s \leq 1$.

(h) Prove that μF can be described as follows.
   - $|\mu F| = \{0,1\}^{<\omega}$
   - and $x \in \mathbb{R}_{\geq 0}^{|\mu F|}$ belongs to $\mathsf{P}(\mu F)$ iff, for any $u \subseteq |\mu F|$ which is an antichain, one has $\sum_{s\in u} x_s \leq 1$.

The two last questions require some basic knowledge in measure theory.

(i) Remember that the Cantor space $\mathcal{C}$ is the set $\{0,1\}^{\omega}$ of infinite sequences of 0's and 1's equipped with the coarsest topology such that, for any $s \in \{0,1\}^{<\omega}$ (finite sequence of 0's and 1's) the set $\uparrow s$ of all $\alpha \in \{0,1\}^{\omega}$ such that s is a prefix of α is open.

   Let ν be a sub-probability measure for the Borel σ-algebra of the Cantor space (a measure ν on $\mathcal{C}$ is a sub-probability measure if $\nu(\mathcal{C}) \leq 1$). Define $x \in \mathbb{R}_{\geq 0}^{|\mu F|}$ by $x_s = \nu(\uparrow s)$. Prove that $x \in \mathsf{P}(\mu F)$.

(j) **(*)** Let $x \in \mathsf{P}(\mu F)$. Prove that the two following properties are equivalent:

- there is a sub-probability measure ν for the Borel σ-algebra of the Cantor space such that $x_s = \nu(\uparrow s)$ for all $s \in \{0,1\}^{<\omega}$
- for all $s \in |\mu F|$, one has $x_s = x_{s0} + x_{s1}$

where si is the sequence s to which $i \in \{0,1\}$ has been appended.

12. Let X and Y be probabilistic coherence spaces. Let $B \subseteq \mathbb{R}_{\geq 0}^{|X|}$ be such that $\mathsf{P}(X) = B^{\perp\perp}$. Let $t \in \mathbb{R}_{\geq 0}^{|X| \times |Y|}$. Prove that $t \in \mathsf{P}(X \multimap Y)$ if and only if $\forall x \in B\ t \cdot x \in \mathsf{P}(Y)$.

13. We identify $|1 \oplus 1|$ with $\{0,1\}$ and $|!(1 \oplus 1)|$ with $\mathbb{N} \times \mathbb{N}$.

(a) Justify the second of these identifications.

(b) We identify $|(1 \oplus 1) \multimap 1|$ with $\mathbb{N} \times \mathbb{N}$. Let $s \in \mathbb{R}_{\geq 0}^{|(1 \oplus 1) \multimap 1|}$. Prove that $s \in \mathsf{P}((1 \oplus 1) \multimap 1)$ iff for all $r \in [0,1]$, one has

$$\sum_{n,p \in \mathbb{N}} s_{n,p} r^n (1-r)^p \leq 1\,.$$

(c) Let $s \in \mathbb{R}_{\geq 0}^{\mathbb{N} \times \mathbb{N}}$ be given by

$$s_{n,p} = \begin{cases} 2^n & \text{if } n = p > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $s \in \mathsf{P}((1 \oplus 1) \multimap 1)$. Let $f : \mathsf{P}(1 \oplus 1) \to \mathsf{P}(1) = [0,1]$ be the associated function, that is

$$f(u_0, u_1) = \sum_{n=1}^{\infty} 2^n u_0^n u_1^n$$

Is this function convex, that is, is it true that for all $u, v \in \mathsf{P}(1 \oplus 1)$ and $\lambda \in [0,1]$, one has $f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v)$?