A completeness theorem for symmetric product phase spaces

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October 9, 2003

Abstract

In a previous work with Antonio Bucciarelli, we introduced *indexed linear logic* as a tool for studying and enlarging the denotational semantics of linear logic. In particular, we showed how to define new denotational models of linear logic using *symmetric product phase models* (truth-value models) of indexed linear logic. We present here a strict extension of indexed linear logic for which symmetric product phase spaces provide a complete semantics. We study the connection between this new system and indexed linear logic.

Keywords: linear logic, phase semantics, completeness, denotational semantics.

Introduction

In [BE00, BE01], we have proposed a sequent calculus of *indexed linear logic* in which each formula A has a *domain* $d(A)$, which is a subset of a global infinite set of indexes I. A formula A in this system is an ordinary formula of linear logic, decorated as follows: the multiplicative constants are equipped with a subset of I, and the exponentials are equipped with an *almost injective* function[1] from the domain of the sub-formula whose exponential is taken to the domain of the formula itself (if A is a formula of domain J and $u : J \to K$ is almost injective, then $!_u A$ and $?_u A$ are formulae of domain K). So, to such a formula A, it is possible to associate an underlying formula of ordinary linear logic, denoted by $\underline{A}$. The basic idea of indexed linear logic is that a formula A in this system represents a $d(A)$-indexed family of elements of $|\underline{A}|$, the set associated to the formula $\underline{A}$ in the standard denotational semantics of linear logic in the model of sets and relations (called *pure relational model* in the sequel). In this particularly simple denotational model, additive constants are interpreted as the empty set, additive operations are interpreted as the disjoint sum of sets, multiplicative constants are interpreted as the singleton set, multiplicative operations are interpreted as the cartesian product of sets, and last, exponentials are interpreted as the operation which consists in taking the set of all finite multi-subsets of a set. A proof of a formula is then interpreted as a subset of the set associated to this formula by the semantics. The sequent calculus of indexed linear logic presented in [BE01] is such that a formula A is provable iff the family of

[1] That is, a function which preserves finite sets under inverse image

elements of $|\underline{A}|$ associated to A is contained in the interpretation of some proof of $\underline{A}$ in ordinary linear logic.

We have developed a phase semantics for this indexed linear logic. A *symmetric product phase space* is a pair $M = (P_0^I, \perp)$ where P_0 is a commutative monoid which has a zero absorbing element, and $\perp$ is a subset of P_0^I (the I-product of P_0, endowed with the product monoid structure). This subset $\perp$ must be non-empty and is subject to two closure properties (closure under restrictions, i.e.: if $p \in \perp$, then any element of P_0^I obtained from p by replacing some components of p by 0 must also belong to $\perp$, and symmetry, that is, roughly, closure under the obvious action of the permutations of I over P_0^I). Then any partial power monoid P_0^J (for $J \subseteq I$) is naturally endowed with a structure of symmetric product phase space, and we call *local phase space at J induced by M* this phase space. A formula A of indexed linear logic of domain J is interpreted as a fact of the local phase space at J. We proved a soundness theorem for this semantics: if a formula A of domain J is provable, its interpretation is *true*, that is contains the unit of P_0^J. We have also shown how to define a denotational model when a symmetric product phase space is given. One obtains in that way a denotational model of linear logic where each formula is interpreted as a set (the *web*, which is identical to the interpretation of this formula in the pure relational model) equipped with a notion of coherence taking as values facts of the various local phase spaces[2]. Considering a particular symmetric product phase space, we have obtained in that way a quite surprising *non-uniform* version of the coherence space semantics of linear logic. This semantics is described thoroughly in [BE01].

However, for this phase semantics of indexed linear logic, completeness (which would imply denotational completeness with respect to ordinary linear logic for the derived denotational model) does not hold, as any symmetric product phase space admits the two following principles, which are not admissible in indexed linear logic.

- Any formula of empty domain is true, whereas if a formula A is provable in indexed linear logic, then $\underline{A}$ is provable in ordinary linear logic. This principle is a *partiality* principle: if S is a formula of linear logic and A is a formula of indexed linear logic with $\underline{A} = S$ and $d(A) = \emptyset$ (actually, there is only one such formula A for a given S), A represents the completely undefined element of S. Accepting A as provable in a syntax is similar to the introduction of the symbol Ω in λ-calculus for representing the completely undefined λ-term (typically in the theory of Böhm trees).

- For any formula A, the formula $A \multimap\, !_{\mathrm{Id}} A$ is valid, whereas in general not provable in indexed linear logic. This principle is more difficult to understand. It corresponds to the fact that in many natural denotational models of linear logic, there is a canonical embedding of any space X into the space $!X$. This embedding allows to "linearize" any morphism f from X to Y in the co-Kleisli category[3] of the model, by simply pre-composing f with the mentioned embedding of X into $!X$: in that way, one extracts from f a "linear component" $\varphi : X \to Y$ (a kind of "derivative at 0" of f), which, by the way, can perfectly be empty, even if f is not. So it would not make sense to admit this linearization principle without accepting the above partiality principle.

[2]This construction presents some similarities with the work of Lamarche [Lam95], see the introductions of [BE00] and [BE01] for discussions about the possible relations between indexed linear logic and previous works.

[3]That is, the associated cartesian closed intuitionistic category, through the usual Girard translation of intuitionistic logic into linear logic.

This linearization principle suggests to extend linear logic (or the lambda-calculus) with differential primitives, leading e.g. to the *differential lambda-calculus* developed by the author and Laurent Regnier ([ER03]). This extension of linear logic is however much more drastic than the one we consider here for indexed linear logic and in particular, in the differential lambda-calculus, proofs of the same formula can freely be *added*. Therefore the two additive connectives & and $\oplus$ are identified. This possibility of freely adding proofs is essential because, in the differential setting, we allow derivatives to be evaluated *at any point* (and not only at 0) in order to compute higher derivatives, whereas the extension of indexed linear logic under consideration in the present paper allows differentiation only at 0.

Our aim is to develop an extension of the sequent calculus of indexed linear logic for which symmetric product phase spaces provide a complete semantics. In order to admit the partiality and linearization principles, we extend the sequent calculus of indexed linear logic in two directions. First, we admit as provable the empty sequent of empty domain. This will ensure the provability of all sequents of empty domain. The second extension consists in modifying the notion of sequent itself and in slightly simplifying the notion of formula. Remember that the usual unilateral sequents of linear logic are obtained by juxtaposing formulae separated by "," signs whose (phase and denotational) semantics is the same as the semantics of the "$\parr$" connective. We introduce another operation for building sequents which is unary, and similarly corresponds to the "$?_u$" logical connectives of indexed linear logic. So a sequent will be an expression of the shape $\vdash_J A_1^{u_1}, \dots, A_n^{u_n}$ where J is a subset of I, $A_1, \dots, A_n$ are formulae of indexed of linear logic, and, for each i, u_i is an almost injective function from the domain of A_i to J ($A_i^{u_i}$ is just a notation for the formal pair (A_i, u_i)). This new setting allows to express the structural and exponential rules of indexed linear logic in a more liberal way.

- The structural rules are no more restricted to formulae of the shape $?_u A$, like in indexed linear logic. For instance, if A is a formula of empty domain, one is allowed to deduce from a sequent $\vdash_J \Gamma$ the sequent $\vdash_J \Gamma, A^u$, where u is the empty function from the domain of A to J. There is a similar way of expressing the contraction rule.

- The promotion rule is no more restricted to contexts of formulae of the shape $?_u A$, but can be performed in any context and modifies the exponents of the context.

- Dereliction becomes a rule which allows to replace, in a context, an expression of the from A^u by the formula $?_u A$, with the identity exponent (a natural convention, that we adopt systematically, is to write simply B instead of B^{Id} when the exponent of B in a context is the identity).

- Last, the logical rules for the $\otimes$ and $\parr$ connectives, as well as the rule for $!_u$ (promotion) can be applied only to formulae having the identity as exponent.

A surprising effect of this extension is that the additive connectives become superfluous: they can be defined in terms of the multiplicative and exponential connectives. Specifically, if A and B are formulae of disjoint domains L and R, the formula $A \,\&\, B$, of domain $L + R$, is represented by the formula $!_l A \otimes !_r B$ where l and r are the obvious injections of L and R into $L + R$. In the present setting, the so-called "additive contraction" becomes an ordinary structural contraction. As to the constants, only one, of empty domain, is needed. We denote it by Ω, and the usual constants of indexed linear logic will be defined in terms of it, using the exponentials.

We first show in Section 2 that this new system (called $\mathsf{LL}^+(I)$ in the present paper) is essentially equivalent to the extension of the sequent calculus of indexed linear logic $\mathsf{LL}(I)$ presented in [BE01] by axioms corresponding to the logical principles mentioned above.

Then in Section 3 we establish the completeness of symmetric product phase spaces for $\mathsf{LL}^+(I)$ (soundness is straightforward). For this purpose, we proceed in the usual way, which consists in defining a syntactical space and showing that a formula which is true in this particular model is provable. This is done in two steps: first, one defines a commutative monoid P, and then one defines a subset $\perp$ of P_0^I which satisfies the closure conditions mentioned above (P_0 is obtained by adding an absorbing element 0 to P). The construction of this model is similar to Girard's construction for proving completeness in [Gir99]; his trick of introducing *labels* for distinguishing between the various occurrences of a formula in a context will play a crucial rôle here. The completeness proof follows a global pattern first introduced in linear logic by Okada ([Oka94, Oka99]) which allows to obtain moreover a cut elimination theorem for free (this is standard in classical logic and Okada observed that the same can be done in linear logic).

We consider this completeness result as important because it precisely delimits the expressive power of the class of symmetric product phase spaces and of the new class of denotational models of linear logic they give rise to, showing that the denotational incompleteness of this class of models with respect to linear logic is entirely contained in two principles: *partiality* and *linearization*. The first of these principles is easy to understand whereas the second is much more mysterious, and it was not obvious *a priori* that it could be expressed in a *logical* way, as we shall do introducing the well-behaved sequent calculus $\mathsf{LL}^+(I)$. The main still open problem here is to find a *non-indexed* system of linear logic which would play with respect to $\mathsf{LL}^+(I)$ the same rôle as the rôle played by ordinary linear logic with respect to $\mathsf{LL}(I)$.

This paper is concerned only with the *purely first-order propositional* fragment of linear logic, the fragment which contains all connectives and constants of linear logic, but no propositional variables and *a fortiori* no quantifiers. Dealing in a completely satisfactory manner with propositional variables would need to extend our approach to second order linear logic, introducing a pure relational interpretation of that system. Such an extension has been studied by Alexandra Bruasse-Bac in her PhD dissertation [BB01], leading to a system of second order indexed linear logic whose properties require further investigations.

Since the material on indexed linear logic needed for understanding the present work is rather new, we have decided to give a detailed summary of the theory. This is the object of the unusually long preliminary Section 1. This paper requires from the reader a reasonable knowledge of the denotational semantics and of the phase space semantics of linear logic. Many good texts on these topics are available, see for instance [Gir87, Gir95, AC98].

1 A summary of indexed linear logic

In this section, we recall the main ideas, definitions and properties of indexed linear logic and symmetric product phase spaces. The material summarized here is presented in full details in [BE01].

1.1 Notations

We first fix some terminology and notations.

If E is a set, we denote by $\#E$ its cardinality.

If $f : E \to F$ is a function, and if $E' \subseteq E$, we denote by $f|_{E'}$ the restriction of f to E', that we consider as a function from E' to F.

If E and F are disjoint sets, we denote by $E + F$ their union. If E and F are not necessarily disjoint, we also denote by $E + F$ their *disjoint sum*, which can be for instance defined by $E + F = (\{1\} \times E) \cup (\{2\} \times F)$. If $f : E \to G$ and $g : F \to G$ are functions, we denote by $f + g$ the function defined by cases, from $E + F$ to G.

A multi-set over E is a map $\mu : E \to \mathbf{N}$. The domain of μ is the set of all the elements a of E such that $\mu(a) \neq 0$. A multi-set is finite if its domain is finite. If $\alpha \in E^J$ is a finite family of elements of E, indexed by a finite set J, we denote by $\mathsf{m}(\alpha)$ the multi-set of elements of E defined by $\mathsf{m}(\alpha)(a) = \#\alpha^{-1}(a)$ (for each $a \in E$). When α is given explicitly ($\alpha = (a_1, \ldots, a_n)$), one writes often $[a_1, \ldots, a_n]$ instead of $\mathsf{m}(\alpha)$. We denote by $\mathcal{M}_{\mathrm{fin}}(E)$ the set of all the finite multi-sets whose support is included in E.

1.2 Linear logic and its pure relational semantics

The fragment of linear logic under consideration is the first order and purely propositional one, so that we consider no other atomic formulae than the additive and multiplicative constants. The formulae are built as follows (each construction is given with its dual construction).

- 0 and $\top$ are formulae (additive constants).
- If R and S are formulae, so are $R \oplus S$ and $R \mathbin{\&} S$ (additive connectives).
- 1 and $\perp$ are formulae (multiplicative constants).
- If R and S are formulae, so are $R \otimes S$ and $R \parr S$ (multiplicative connectives).
- If R is a formula, so are $!R$ and $?R$ (exponential modalities).

The *linear negation* $R^\perp$ of a formula R is defined by induction on R, by replacing each constant or connective occurring in R by its dual constant or connective. Girard defined a sequent calculus for linear logic, the rules of this calculus can be found in many papers on linear logic, see for instance [Gir95].

In the pure relational denotational semantics of linear logic, one associates to each formula R of linear logic a set $|R|$, called the *web* of R, by induction:

- $|0| = |\top| = \emptyset$.
- $|R \oplus S| = |R \mathbin{\&} S| = |R| + |S|$, the disjoint sum of $|R|$ and $|S|$.
- $|1| = |\perp| = \{*\}$ where $*$ is arbitrary.
- $|R \otimes S| = |R \parr S| = |R| \times |S|$.
- $|!R| = |?R| = \mathcal{M}_{\mathrm{fin}}(|R|)$.

Then to each proof π of a formula R of linear logic, one associates by induction a subset π^* of $|R|$: the denotational semantics of π. This is explained in the appendix (one will also find there a reminder of the rules of the sequent calculus of linear logic). Linear logic enjoys the fundamental *cut elimination* property: there are *normalizing* reduction rules on proofs for which all normal proofs are cut-free. The main property of denotational semantics is that it is invariant under these conversion rules on proofs: if π and ρ are two proofs of the same formula which are equivalent under these conversion rules, then $\pi^* = \rho^*$. From a categorical viewpoint, this results from the fact that the category whose objects are sets and whose morphisms are binary relations between sets is a model of linear logic, and more precisely, a *Lafont category* in the sense of [Bie95]. This model is degenerate in the sense that the category of sets and relations is isomorphic to its opposite, and that this isomorphism commutes to all the categorical operations needed for interpreting linear logic.

1.3 Indexed linear logic

Due to this degeneracy, the pure relational semantics of linear logic is not informative at all. We develop a method for endowing the webs associated to formulae with a kind of *coherence relation*, generalizing the well known structures of coherence spaces or hypercoherences.

For this purpose, the basic idea is to consider the *families* of elements of the webs associated to the formulae of linear logic, aiming at associating to these families a kind of *coherence value*. We assume fixed once and for all a set I, which is infinite (and denumerable): all the index sets we shall consider will be subsets of I. We start from the following observations (J being an arbitrary subset of I):

- $\emptyset^J$ is non-empty iff J is empty, and then $\emptyset^J$ is a singleton.
- There is a bijective correspondence between $(X+Y)^J$ and $\sum_{L+R=J}(X^L \times Y^R)$; if L and R are disjoint sets such that $L+R=J$, and if $\alpha \in X^L$ and $\beta \in Y^R$, we denote by $\alpha+\beta$ the corresponding element of $(X+Y)^J$.
- $\{*\}^J$ is always a singleton.
- There is a bijective correspondence between $(X \times Y)^J$ and $X^J \times Y^J$; if $\alpha \in X^J$ and $\beta \in Y^J$, we denote by (α, β) the corresponding element of $(X \times Y)^J$.

So within the multiplicative-additive fragment of linear logic, we have a bijective correspondence between the elements of $|R|^J$ and the formulae A of the multiplicative additive fragment of *indexed linear logic* which satisfy $\underline{A} = R$ and $d(A) = J$. We define the formulae of this new system below, as well as, for each formula A, its underlying formula of linear logic $\underline{A}$, its domain $d(A) \subseteq I$ and the corresponding element $\langle A \rangle$ of $|\underline{A}|^{d(A)}$.

- 0 and $\top$ are formulae, and $\underline{0} = 0$, $\underline{\top} = \top$ and $d(0) = d(\top) = \emptyset$. Both $\langle 0 \rangle$ and $\langle \top \rangle$ are defined as the empty family.
- If A and B are formulae, with $d(A) \cap d(B) = \emptyset$, then $A \oplus B$ and $A \mathbin{\&} B$ are formulae, and $\underline{A \oplus B} = \underline{A} \oplus \underline{B}$, $\underline{A \mathbin{\&} B} = \underline{A} \mathbin{\&} \underline{B}$ and $d(A \oplus B) = d(A \mathbin{\&} B) = d(A) + d(B)$. And one sets $\langle A \oplus B \rangle = \langle A \mathbin{\&} B \rangle = \langle A \rangle + \langle B \rangle$.
- If $J \subseteq I$ then 1_J and $\perp_J$ are formulae with $\underline{1_J} = 1$, $\underline{\perp_J} = \perp$ and $d(1_J) = d(\perp_J) = J$. Both families $\langle 1_J \rangle$ and $\langle \perp_J \rangle$ are the J-indexed family constantly equal to $*$.

- If A and B are formulae with $d(A) = d(B)$, then $A \otimes B$ and $A \parr B$ are formulae, and $\underline{A \otimes B} = \underline{A} \otimes \underline{B}$, $\underline{A \parr B} = \underline{A} \parr \underline{B}$ and $d(A \otimes B) = d(A \parr B) = d(A) = d(B)$. And one sets $\langle A \otimes B\rangle = \langle A \parr B\rangle = (\langle A\rangle, \langle B\rangle)$.

When exponentials come in, we renounce to this bijective correspondence between families and formulae of indexed linear logic: a surjective mapping from formulae to families will be sufficient for our purpose. For defining this mapping we observe that an element ξ of $\mathcal{M}_{\mathrm{fin}}(X)^J$ can be described by the following data: a set K, an element α of X^K, and a function $u : K \to J$ which is *almost injective* (we mean by that that u preserve finite sets under inverse image), such that ξ_j is the multi-set of all the elements of the family $\alpha|_{u^{-1}(j)}$, taking repetitions into account, in other words, $\xi_j = \mathsf{m}(\alpha|_{u^{-1}(j)})$.

Therefore, we extend the syntax of formulae of indexed linear logic as follows: if A is a formula and if $u : d(A) \to J$ is an almost injective function, then $!_u A$ as well as $?_u A$ are formulae, $\underline{!_u A} = !\underline{A}$, $\underline{?_u A} = ?\underline{A}$ and $d(!_u A) = d(?_u A) = J$. The corresponding J-indexed families $\langle !_u A\rangle$ and $\langle ?_u A\rangle$ are defined by $\langle !_u A\rangle_j = \langle ?_u A\rangle_j = \mathsf{m}(\langle A\rangle|_{u^{-1}(j)})$.

The *linear negation* $A^\perp$ of a formula A of domain J is the formula of domain J obtained by applying recursively the usual De Morgan laws between dual connectives, for instance $(!_u A)^\perp = ?_u(A^\perp)$.

For these formulae of indexed linear logic, we have defined a sequent calculus, that we have called $\mathsf{LL}(I)$. A *sequent* of $\mathsf{LL}(I)$ is an expression of the shape $\vdash_J \Gamma$ where J is a subset of I and Γ is a (possibly empty) sequence $(A_1, \ldots, A_n)$ of formulae of $\mathsf{LL}(I)$ such that each A_i has domain J (a sequence Γ of formulae satisfying this condition will sometimes be called a *context*, and we shall denote by $d(\Gamma)$ the common domain of the elements of Γ, when Γ is not empty). Before giving the rules of this sequent calculus, we need to define the operations of *restriction* and *re-indexing* on formulae.

Restriction. If A is a formula of $\mathsf{LL}(I)$ with $d(A) = J$, and if $K \subseteq I$, we define the restriction of A by K, denoted by $A|_K$, which is a formula of $\mathsf{LL}(I)$ with domain $J \cap K$, as follows:

- $\top|_K = \top$ and $0|_K = 0$.
- $\perp_J|_K = \perp_{J\cap K}$ and $1_J|_K = 1_{J\cap K}$.
- $(A \otimes B)|_K = A|_K \otimes B|_K$, $(A \parr B)|_K = A|_K \parr B|_K$, $(A \oplus B)|_K = A|_K \oplus B|_K$ and $(A \,\&\, B)|_K = A|_K \,\&\, B|_K$.
- $(!_u A)|_K = !_v(A|_{u^{-1}(K\cap J)})$ where $v : u^{-1}(K \cap J) \to K \cap J$ is obtained by restricting u. The definition of $(?_u A)|_K$ is similar.

If $\Gamma = (A_1, \ldots, A_n)$ is a context, one defines $\Gamma|_K = (A_1|_K, \ldots, A_n|_K)$ so that again, $d(\Gamma|_K) = d(\Gamma) \cap K$. Last, observe that trivially $A^\perp|_K = (A|_K)^\perp$. It is also easily checked that $(A|_K)|_L = A|_{K\cap L}$ and that $\langle A|_K\rangle$ is the restriction of the family $\langle A\rangle$ to $d(A) \cap K$.

Re-indexing. When $\varphi : J \to K$ is a bijection, one can define, for each formula A of domain J, a formula $\varphi_* A$ of domain K, as follows:

- $\varphi_* \top = \top$ and $\varphi_* 0 = 0$ (in that case, $J = K = \emptyset$).
- $\varphi_* \perp_J = \perp_K$ and $\varphi_* 1_J = 1_K$.
- $\varphi_*(A \otimes B) = \varphi_* A \otimes \varphi_* B$ and $\varphi_*(A \parr B) = \varphi_* A \parr \varphi_* B$.

- $\varphi_*(A \oplus B) = \psi_* A \oplus \chi_* B$ where $\psi : d(A) \to \varphi(d(A))$ and $\chi : d(B) \to \varphi(d(B))$ are obtained by restricting φ (observe that $K = \varphi(d(A)) + \varphi(d(B))$ as φ is bijective). The formula $\varphi_*(A \mathbin{\&} B)$ is defined in a similar way.

- $\varphi_*(!_v A) = !_{\varphi \circ v} A$ and $\varphi_*(?_v A) = ?_{\varphi \circ v} A$.

Then one checks easily that $\langle \varphi_* A \rangle$ is the element β of $|\underline{A}|^K$ defined by $\beta_{\varphi(j)} = \langle A \rangle_j$ (for each $j \in J$).

We recall now the rules of $\mathsf{LL}(I)$.

We have the following axioms:

$$\frac{}{\vdash_J 1_J}$$

and

$$\frac{}{\vdash_\emptyset \Gamma, \top}$$

this latter making sense only under the assumption that Γ is empty, or has empty domain.

Multiplicative rules:

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, \bot_J}$$

$$\frac{\vdash_J \Gamma, A \qquad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash_J \Gamma, A, B}{\vdash_J \Gamma, A \mathbin{⅋} B}$$

Additive rules. In the introduction rules for $\oplus$, observe that B must have an empty domain.

$$\frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, A \oplus B} \qquad \frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, B \oplus A}$$

Assume that $d(A) = L$, $d(B) = R$ with $L \cap R = \emptyset$, and that $d(\Gamma) = L + R$.

$$\frac{\vdash_L \Gamma|_L, A \qquad \vdash_R \Gamma|_R, B}{\vdash_{L+R} \Gamma, A \mathbin{\&} B}$$

We give now the exponential rules. For A a formula of empty domain, 0_J denoting the empty function from $\emptyset$ to J, the weakening rule is the following:

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, ?_{0_J} A}$$

For A a formula of domain K, u an almost injective function from K to J, K_1 and K_2 two subsets of K such that $K = K_1 + K_2$, u_i (for $i = 1, 2$) the almost injective function $K_i \to J$ obtained by restricting u to K_i, the contraction rule is the following:

$$\frac{\vdash_J \Gamma, ?_{u_1}(A|_{K_1}), ?_{u_2}(A|_{K_2})}{\vdash_J \Gamma, ?_u A}$$

Let $\varphi : J \to K$ be a *bijection*. The dereliction rule is the following:

$$\frac{\vdash_K \Gamma, \varphi_* A}{\vdash_K \Gamma, ?_\varphi A}$$

Let $(A_i)_{i=1,\dots,n}$ be a family of formulae and let K_i be the domain of A_i. Let J be a set and, for each $i = 1, \dots, n$, let u_i be an almost injective function from K_i to J. Let A be a formula of domain J and let $v : J \to L$ be an almost injective function. The promotion rule is the following:

$$\frac{\vdash_J ?_{u_1} A_1, \dots, ?_{u_n} A_n, A}{\vdash_L ?_{v \circ u_1} A_1, \dots, ?_{v \circ u_n} A_n, !_v A}$$

The only structural rule is the exchange rule, which is

$$\frac{\vdash_J A_1, \dots, A_n}{\vdash_J A_{\sigma(1)}, \dots, A_{\sigma(n)}}$$

where σ is any permutation of $\{1, \dots, n\}$.

Last, the cut rule:

$$\frac{\vdash_J \Gamma, A \qquad \vdash_J \Delta, A^\perp}{\vdash_J \Gamma, \Delta}$$

With respect to the denotational semantics of linear logic presented above, the main property of $\mathsf{LL}(I)$ is the following.

Proposition 1 *Let R be a formula of linear logic, let J be a subset of I and let $\alpha \in |R|^J$. The following properties are equivalent.*

- *There exists a proof π of $\vdash R$ in linear logic such that $\alpha \in (\pi^*)^J$ (that is π^* contains the range of α).*
- *There exists a formula A of $\mathsf{LL}(I)$ such that $\underline{A} = R$ and $\langle A \rangle = \alpha$, and such that the sequent $\vdash_J A$ is provable in $\mathsf{LL}(I)$.*
- *For each formula A of $\mathsf{LL}(I)$ such that $\underline{A} = R$ and $\langle A \rangle = \alpha$, the sequent $\vdash_J A$ is provable in $\mathsf{LL}(I)$.*

1.4 Symmetric product phase spaces

Remember that *phase spaces* have been introduced by Girard in [Gir87] for giving a *truth-value semantics* to linear logic. As this is standard material, we just recall here the basic definitions and a few easy properties that we use implicitly in the sequel. A phase space is a pair $(Q, \perp)$ where Q is a commutative monoid and $\perp$ is a subset of Q on which no special requirement is made. If $U \subseteq Q$, one defines $U^\perp = \{p \in Q \mid \forall q \in U \ pq \in \perp\}$. A *fact* is a subset U of Q such that $U = U^{\perp\perp}$ (the inclusion $U \subseteq U^{\perp\perp}$ always holds, obviously). The following basic properties are useful (U and V being subsets of Q):

- $U \subseteq V \Rightarrow V^\perp \subseteq U^\perp$;
- $U^{\perp\perp\perp} = U^\perp$;

- $(U \cup V)^\perp = U^\perp \cap V^\perp$;
- $(UV^{\perp\perp})^\perp = (UV)^\perp$, where $UV = \{pq \mid p \in U \text{ and } q \in V\}$.

In particular, $U^\perp$ is always a fact, and for showing that $U^{\perp\perp} \subseteq F$ (when F is a fact), it suffices to show that $U \subseteq F$.

Phase semantics consists in interpreting each formula of linear logic as a fact, to be considered as a kind of "truth value". A fact is *true* when it contains the unit of Q, and indeed one shows that any provable formula is interpreted in any phase space as a true fact (soundness). Moreover, this semantics of linear logic is complete (the proof of this uses a particular phase space $(Q, \perp)$ where Q is the monoid of all contexts of linear logic, and where $\perp$ is the set of all provable contexts).

We develop now a phase semantics for indexed linear logic. In that way, through the correspondence above between families and formulae, we shall be able to define a fact-valued notion of coherence on webs.

Let $\mathcal{I}(I)$ be the category whose objects are the subsets of I and whose morphisms are the *injective* functions between them.

Given a set E, we denote by Fam_E the contravariant functor from **Set** to **Set** which to J associates the set of all J-indexed families of elements of E: $\mathrm{Fam}_E(J) = E^J$ and if $f : J \to K$, $\mathrm{Fam}_E(f)(\alpha) = \alpha \circ f$, for any $\alpha \in E^K$. In the sequel, we shall only consider the restriction of this functor to $\mathcal{I}(I)$ (a small sub-category of **Set**), and we shall denote $\mathrm{Fam}_E(f)$ by f^*, according to a well established tradition. In particular, when $K \subseteq J \subseteq I$, we denote by $\pi_K : E^J \to E^K$ the projection function given by $\pi_K = u^*$ where $u : K \to J$ is the inclusion.

If Q is a monoid, and if we consider Q^J (for $J \subseteq I$) as equipped with its structure of product monoid induced by the monoid structure of Q, then the functor Fam_Q becomes a contravariant functor from the category $\mathcal{I}(I)$ to the category of monoids.

A *product phase space* is a pair $(P_0^I, \perp)$ where P_0 is a commutative monoid possessing an absorbing element 0 and $\perp \subseteq P_0^I$ is non-empty and satisfies $\varepsilon_J \perp \subseteq \perp$ for each $J \subseteq I$ (where $\varepsilon_J \in P_0^I$ is defined by $(\varepsilon_J)_i = 1$ if $i \in J$ and $(\varepsilon_J)_i = 0$ if $i \notin J$).

If $M = (P_0^I, \perp)$ is a product phase space, then for any $J \subseteq I$, we define a product phase space $M(J) = (P_0^J, \perp(J))$, where $\perp(J) = \pi_J(\perp)$. We denote by 1^J the unit of the monoid P_0^J and by $\mathcal{F}_M(J)$ the set of all facts of the *local phase space* $M(J)$.

A product phase space $(P_0^I, \perp)$ is *symmetric*, intuitively, if $\perp$ is invariant under all permutations of I. However, this condition is too weak for our purpose. The precise definition is as follows.

Definition 2 A product phase space $(P_0^I, \perp)$ is *symmetric* if for any $J, K \subseteq I$ and any bijection $\varphi : J \to K$, one has $\perp(J) = \varphi^* \perp(K)$.

One checks easily that this condition is equivalent to the following: for any $J \subseteq I$, if $u, v : J \to I$ are injections, then $u^*\perp = v^*\perp$. When M is symmetric, so are all of the local spaces $M(J)$. Moreover the local spaces $M(J)$ and $M(K)$ are isomorphic as soon as J and K have the same cardinality.

Lemma 3 *Let $M = (P_0^I, \perp)$ be a symmetric product phase space. Let $J, K \subseteq I$ and let $u : J \to K$ be injective. If $F \in \mathcal{F}_M(K)$, then $u^*F \in \mathcal{F}_M(J)$ and $u^*(F^\perp) = (u^*F)^\perp$. Moreover, if $F, G \in \mathcal{F}_M(K)$, then $u^*(F \otimes G) = (u^*F) \otimes (u^*G)$.*

The proof is easy, and can be found in [BE01]. This turns the operation $J \mapsto \mathcal{F}_M(J)$ into a contravariant functor from $\mathcal{I}(I)$ to **Set**.

We explain now how the exponentials are interpreted in such a phase space. Let $u : J \to K$ be almost injective. For any commutative monoid Q, there is a functorial covariant way of defining a monoid morphism $u_* : Q^J \to Q^K$ as follows:

$$(u_*(p))_k = \prod_{j \in u^{-1}(k)} p_j \,.$$

In particular, when φ is a bijection, $\varphi_* = (\varphi^*)^{-1}$.

If $F \in \mathcal{F}_M(J)$, one sets $!_u F = (u_* F)^{\perp\perp} \in \mathcal{F}_M(K)$ and $?_u F = \left(u_*(F^\perp)\right)^\perp \in \mathcal{F}_M(K)$, two operations which are obviously De Morgan dual of each other.

Lemma 4 *If $\varphi : J \to K$ is a bijection and if $F \in \mathcal{F}_M(J)$, then*

$$!_\varphi F = ?_\varphi F = \varphi_* F \,.$$

The behavior of these operations under injective re-indexing is easy to understand, as soon as one observes the following simple fact.

Lemma 5 *Let Q be a commutative monoid. If the following diagram in* **Set** *is a pull-back*

$$\begin{tikzcd} K \arrow[r, "u"] & J \\ K' \arrow[u, "v'"] \arrow[r, "u'"] & J' \arrow[u, "v"'] \end{tikzcd}$$

where $K, J, K', J' \subseteq I$, u is almost injective and v is injective, then v' is injective and u' is almost injective, and moreover, the following diagram is commutative[4]*.*

$$\begin{tikzcd} Q^K \arrow[r, "u_*"] \arrow[d, "v'^*"'] & Q^J \arrow[d, "v^*"] \\ Q^{K'} \arrow[r, "{u'}_*"] & Q^{J'} \end{tikzcd}$$

In particular, if $Q = P_0$ and if $F \in \mathcal{F}_M(K)$, then one has

$$v^*(!_u F) = !_{u'}(v'^* F) \,.$$

The interpretation of a formula A of domain J as a fact $A^\bullet \in \mathcal{F}_M(J)$ is based on these logical constructions on facts, together with the following interpretation for the constants: $0^\bullet = \top^\bullet = \{0\}$ (the only fact of $\mathcal{F}_M(\emptyset)$) and ${\perp_J}^\bullet = \perp(J)$, ${1_J}^\bullet = \perp(J)^\perp$. If $\Gamma = (A_1, \dots, A_n)$ is a context of domain J, it will be interpreted as the element $\Gamma^\bullet \in \mathcal{F}_M(J)$ given by $\Gamma^\bullet = {A_1}^\bullet \parr \cdots \parr {A_n}^\bullet$. Then the soundness theorem is expressed as follows.

[4]Actually this commutation also holds if v is not injective.

Theorem 6 *Let M be a symmetric product phase space and Γ be a context of domain J. If the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$, then, in M, one has $1^J \in A^\bullet$.*

In [BE01], we showed how to associate to any symmetric product phase space $M = (P_0^I, \perp)$ a categorical denotational model of linear logic $\mathcal{C}(M)$ (a *new-Seely category*, in the sense of [Bie95]). An object of this category is a pair $X = (|X|, \widehat{X})$ where $|X|$ is a set (the web of X) and $\widehat{X}$ is a natural transformation from the functor $\mathrm{Fam}_{|X|}$ to the functor $\mathcal{F}_M$ (both considered as contravariant functors from $\mathcal{I}(I)$ to **Set**). Then a subset x of $|X|$ is an *M-clique* if $1^J \in \widehat{X}_J(\alpha)$, for any $J \subseteq I$ and any $\alpha \in x^J$ (that is, any J-indexed family α whose range is included in x). As far as the webs are concerned, the logical constructions on these *M-spaces* are identical to their analogues in the category of sets and relations. For a formula R of linear logic, let us denote R^*_M the associated M-space, we have therefore $|R^*_M| = |R|$. Moreover, the model $\mathcal{C}(M)$ is defined in such a way that, for any $\mathsf{LL}(I)$ formula A,

$$\widehat{(\underline{A}^*_M)}_J(\langle A\rangle) = A^\bullet$$

where J is the domain of A.

2 An extension of indexed linear logic

In any symmetric product phase space $M = (P_0^I, \perp)$, one has $A^\bullet = \{0\}$ for any $\mathsf{LL}(I)$-formula A such that $d(A) = \emptyset$, since $\{0\}$ is the only fact of the (trivial) local space $M(\emptyset)$, but $1^\emptyset = 0$ and hence $1^{d(A)} \in A^\bullet$ as soon as $d(A) = \emptyset$. Moreover, if A is any $\mathsf{LL}(I)$-formula, we have $(!_{\mathrm{Id}}A)^\bullet = (\mathrm{Id}_* A^\bullet)^{\perp\perp} = (A^\bullet)^{\perp\perp} = A^\bullet$ and therefore, $1^{d(A)} \in (A \multimap !_{\mathrm{Id}}A)^\bullet$.

We define an extension $\mathsf{LL}^+(I)$ of $\mathsf{LL}(I)$ which admits these additional principles. The formulae of this new system $\mathsf{LL}^+(I)$ of indexed linear logic are built like those of $\mathsf{LL}(I)$ with the following differences: there is only one additive constant, Ω, there are no additive connectives, and no multiplicative constants (they will be definable). As in $\mathsf{LL}(I)$, each formula A has an underlying formula of linear logic $\underline{A}$ and a domain $d(A) \subseteq I$.

- There is only one constant Ω. Its domain is empty: $d(\Omega) = \emptyset$.
- If A and B are formulae with $d(A) = d(B) = J \subseteq I$, then $A \otimes B$ and $A \parr B$ are formulae, with $d(A \otimes B) = d(A \parr B) = J$.
- If A is a formula with domain $d(A) = J \subseteq I$ and if $K \subseteq I$ and $u : J \to K$ is an almost injective function, then $!_u A$ and $?_u B$ are formulae, with $d(!_u A) = d(?_u A) = K$.

If A is a formula, $A^\perp$ is the formula defined as usual, using the De Morgan rules extended by the equation $\Omega^\perp = \Omega$. There is nothing special to say about the restriction and re-indexing operations on these formulae: they are defined like in $\mathsf{LL}(I)$, and behave in the same way.

Let $J \subseteq I$. A *context of domain J* is a sequence $(A_1^{u_1}, \dots, A_n^{u_n})$ where, for each $i = 1, \dots, n$, A_i is a formula and $u_i : d(A_i) \to J$ is an almost injective function. The expression "$A_i^{u_i}$" denotes the formal pair (A_i, u_i). A *sequent* is an expression of the form $\vdash_J \Gamma$ where $J \subseteq I$ and Γ is a context of domain J. If A is a formula of domain J, the pair A^{Id_J} will be simply written A. If $\Gamma = (A_1^{u_1}, \dots, A_n^{u_n})$ is a context of domain J and if $u : J \to K$ is an almost injective function, we define Γ^u as the context $(A_1^{u \circ u_1}, \dots, A_n^{u \circ u_n})$ of domain K.

We give a sequent calculus for these formulae. Our system has only one axiom:

$$\frac{}{\vdash_{\emptyset}}$$

The multiplicative rules are given by

$$\frac{\vdash_J \Gamma, A \qquad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash_J \Gamma, A, B}{\vdash_J \Gamma, A \parr B}$$

Observe that in the premises of these rules, both occurrences of formulae A and B must have the identity function as exponent.

We give next the exponential logical rules. Dereliction:

$$\frac{\vdash_J \Gamma, A^u}{\vdash_J \Gamma, ?_u A}$$

as soon as $u : d(A) \to J$ is almost injective. Promotion is given by

$$\frac{\vdash_J \Gamma, A}{\vdash_K \Gamma^v, !_v A}$$

as soon as $v : d(A) = J \to K$ is an almost injective function. Observe that in the premise, A must have an identity function exponent.

There are four kinds of structural rules: weakening, contraction, re-indexing and exchange. Weakening is given by

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, A^{0_J}}$$

when $d(A) = \emptyset$ (0_J denotes the empty function from $\emptyset$ to J).

Contraction is given by

$$\frac{\vdash_J \Gamma, (A|_L)^l, (A|_R)^r}{\vdash_J \Gamma, A^{l+r}}$$

where $d(A) = L + R$ and $l : L \to J$ and $r : R \to J$ are two almost injective functions and $l + r : L + R \to J$ is the "co-pairing" of l and r, which is almost injective. Our exponential notations for contexts are motivated by these two rules.

There are two re-indexing structural rules. The first rule allows to globally re-localize a context. If $\varphi : J \to K$ is a bijection, we have the rule

$$\frac{\vdash_J \Gamma}{\vdash_K \Gamma^{\varphi}}$$

The second rule allows to extract a bijective factor from an exponent and will be called factorization in the sequel. If φ is a bijection, then

$$\frac{\vdash_J \Gamma, A^{u \circ \varphi}}{\vdash_J \Gamma, (\varphi_* A)^u}$$

Exchange is given by

$$\frac{\vdash_J A_1^{u_1}, \dots, A_n^{u_n}}{\vdash_J A_{\sigma(1)}^{u_{\sigma(1)}}, \dots, A_{\sigma(n)}^{u_{\sigma(n)}}}$$

where σ is a permutation on $\{1, \dots, n\}$.

Last the cut rule is

$$\frac{\vdash_K \Gamma, A^u \qquad \vdash_J \Delta, A^\perp}{\vdash_K \Gamma, \Delta^u}$$

This rule may seem weird because it is not symmetrical, but it is apparently the best version of the cut rule available in this system. The version with an exponent on both sides would not be sound in our phase semantics, and, with an identity exponent on both sides, it would be impossible to eliminate the cut rule in general: consider the case where the last rule on the left is a !-rule and the last rule on the right is a ?-rule.

Observe that there is no specific rule concerning Ω, and so this formula can only be introduced by weakening.

We say that two formulae A and B with $d(A) = d(B) = J$ are provably equivalent in $\mathsf{LL}^+(I)$ if both sequents $\vdash_J A^\perp, B$ and $\vdash_J A, B^\perp$ are provable in $\mathsf{LL}^+(I)$.

Lemma 7 *Any formula of $\mathsf{LL}^+(I)$ is provably equivalent to itself.*

The proof is by induction on the formula.

Lemma 8 *Let A be a formula of $\mathsf{LL}^+(I)$, let $\varphi : J = d(A) \to L$ be bijective and let $u : L \to R$ be almost injective. Then the formulae $!_{u \circ \varphi} A$ and $!_u \varphi_* A$ are provably equivalent in $\mathsf{LL}^+(I)$.*

Proof: We check just one of the two directions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi \\ \vdash_J A^\perp, A\end{array}}{\vdash_J A^\perp, (\varphi_* A)^{\varphi^{-1}}}\,\text{(Factorization)}}{\vdash_L (A^\perp)^\varphi, \varphi_* A}\,\text{(Re-localization)}}{\vdash_R (A^\perp)^{u \circ \varphi}, !_u \varphi_* A}\,\text{(Promotion)}}{\vdash_R ?_{u \circ \varphi} A^\perp, !_u \varphi_* A}\,\text{(Dereliction)}$$

where π is a proof whose existence results from Lemma 7. ■

We explain now in what sense the present system is an extension of $\mathsf{LL}(I)$. For this purpose, we explain first how a formula A of $\mathsf{LL}(I)$ can be represented as a formula $\widetilde{A}$ of $\mathsf{LL}^+(I)$ satisfying $d(\widetilde{A}) = d(A)$. This mapping $A \mapsto \widetilde{A}$ is surjective but clearly not injective.

- $\widetilde{\top} = \widetilde{0} = \Omega$.
- $\widetilde{A \mathbin{\&} B} = !_l \widetilde{A} \otimes !_r \widetilde{B}$ and $\widetilde{A \oplus B} = ?_l \widetilde{A} \parr ?_r \widetilde{B}$, where A and B are formulae of domain L and R respectively, with $L \cap R = \emptyset$, and l and r are the injections of L and R in $L + R$.
- $\widetilde{1_J} = !_{0_J} \Omega$ and $\widetilde{\perp_J} = ?_{0_J} \Omega$ where 0_J denotes the empty function from $\emptyset$ to J.

- $\widetilde{A \parr B} = \widetilde{A} \parr \widetilde{B}$ and $\widetilde{A \otimes B} = \widetilde{A} \otimes \widetilde{B}$.
- $\widetilde{?_u A} = ?_u \widetilde{A}$ and $\widetilde{!_u A} = !_u \widetilde{A}$.

When A contains no additive connectives and no multiplicative constants, $\widetilde{A}$ is simply obtained by replacing in A all occurrences of $\top$ and 0 by Ω.

By the first statement of the next lemma, $\widetilde{A}$ is always a well-formed formula of $\mathsf{LL}^+(I)$.

Lemma 9 *Let A be a formula of $\mathsf{LL}(I)$. One has $d(\widetilde{A}) = d(A)$ and $\widetilde{A^\perp} = \widetilde{A}^\perp$. Let $K \subseteq I$, then $\widetilde{A}|_K = \widetilde{A|_K}$. Let $\varphi : d(A) \to K$ be a bijection. If A contains no additive connectives and no multiplicative constants then the formulae $\varphi_* \widetilde{A}$ and $\widetilde{\varphi_* A}$ are identical. In general, they are provably equivalent in $\mathsf{LL}^+(I)$.*

The only statement which is not completely straightforward is the last one. It is proved by induction on A and the only interesting cases are when $A = B \,\&\, C$ and when $A = B \oplus C$. The main tool for dealing with these cases is Lemma 8.

If Γ is a sequence $(A_1, \dots, A_n)$ of formulae of $\mathsf{LL}(I)$, we denote by $\widetilde{\Gamma}$ the sequence $(\widetilde{A_1}, \dots, \widetilde{A_n})$.

We call $\mathsf{LL}^{\mathrm{ext}}(I)$ the sequent calculus $\mathsf{LL}(I)$ extended with the axioms

$$\frac{}{\vdash_\emptyset}$$

called *partiality* and

$$\frac{}{\vdash_J A^\perp, !_{\mathrm{Id}} A}$$

(for each $\mathsf{LL}(I)$ formula A of domain J) called *linearization*. This sequent calculus does apparently not enjoy cut elimination, otherwise there would be no point[5] in introducing the new system $\mathsf{LL}^+(I)$: we shall show that the systems $\mathsf{LL}^{\mathrm{ext}}(I)$ and $\mathsf{LL}^+(I)$ are equivalent.

Lemma 10 *Let Γ be a context of domain $J \subseteq I$ in $\mathsf{LL}(I)$. If the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, then the sequent $\vdash_J \widetilde{\Gamma}$ is provable in $\mathsf{LL}^+(I)$.*

Proof: It suffices to show that the additional axioms of $\mathsf{LL}^{\mathrm{ext}}(I)$ are provable in $\mathsf{LL}^+(I)$ and that, if the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$, then the sequent $\vdash_J \widetilde{\Gamma}$ is provable in $\mathsf{LL}^+(I)$.

The partiality axiom of $\mathsf{LL}^{\mathrm{ext}}(I)$ is an axiom of $\mathsf{LL}^+(I)$. Next, let A be a formula of domain J of $\mathsf{LL}^+(I)$, we must show that the sequent $\vdash_J A^\perp, !_{\mathrm{Id}} A$ is provable in $\mathsf{LL}^+(I)$. By Lemma 7, the sequent $\vdash_J A^\perp, A$ is provable. A promotion rule in $\mathsf{LL}^+(I)$ yields directly $\vdash_J A^\perp, !_{\mathrm{Id}} A$.

By induction on the proof π of $\vdash_J \Gamma$ in $\mathsf{LL}(I)$, we define now a proof π' of $\vdash_J \widetilde{\Gamma}$ in $\mathsf{LL}^+(I)$.

If π is the proof

$$\frac{}{\vdash_J 1_J}$$

then π' is the proof

$$\cfrac{\cfrac{\cfrac{}{\vdash_\emptyset}}{\vdash_J \Omega^{0_J}}}{\vdash_J !_{0_J} \Omega}$$

[5] Apart from the fact that proving the Completeness theorem 24 for $\mathsf{LL}^{\mathrm{ext}}(I)$ seems to be an intractable task.

If π is the proof

$$\frac{}{\vdash_\emptyset \Gamma, \top}$$

then π' is the proof

$$\dfrac{\dfrac{}{\vdash_\emptyset}}{\vdash_\emptyset \widetilde{\Gamma}, \Omega}$$

applying several times the weakening rule, as indeed in that case we have $d(\Gamma) = \emptyset$ and thus $d(\widetilde{\Gamma}) = \emptyset$.

If π is a tensor or a par rule, the construction of π' is straightforward.

Assume now that π ends with a left plus rule (the case of a right plus rule is of course similar):

$$\dfrac{\begin{array}{c}\vdots\,\rho\\ \vdash_J \Gamma, A\end{array}}{\vdash_J \Gamma, A \oplus B}$$

then by inductive hypothesis we have a proof ρ' in $\mathsf{LL}^+(I)$ of the sequent $\vdash_J \widetilde{\Gamma}, \widetilde{A}$. Applying a weakening rule, we get $\vdash_J \widetilde{\Gamma}, \widetilde{A}, \widetilde{B}^{0_J}$, since we must have $d(B) = \emptyset$. Two dereliction rules lead to $\vdash_J \widetilde{\Gamma}, ?_{\mathrm{Id}}\widetilde{A}, ?_{0_J}\widetilde{B}$, and then, by a par rule, we get a proof π' of $\vdash_J \widetilde{\Gamma}, (?_{\mathrm{Id}}\widetilde{A}) \parr (?_{0_J}\widetilde{B})$ as announced.

Assume that π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\lambda\\ \vdash_L \Gamma|_L, A\end{array} \qquad \begin{array}{c}\vdots\,\rho\\ \vdash_R \Gamma|_R, B\end{array}}{\vdash_{L+R} \Gamma, A \,\&\, B}$$

where L and R are two disjoint subsets of I which are respectively the domains of A and B. By inductive hypothesis, we have a proof λ' of $\vdash_L \widetilde{\Gamma|_L}, \widetilde{A}$. Applying a promotion rule in $\mathsf{LL}^+(I)$, we obtain a proof of $\vdash_{L+R} \widetilde{\Gamma|_L}^{l}, !_l\widetilde{A}$, where l is the inclusion of L into the disjoint union $L + R$. Similarly, we obtain a proof of $\vdash_{L+R} \widetilde{\Gamma|_R}^{r}, !_r\widetilde{B}$ where r is the inclusion of R into $L + R$. Now applying a tensor rule, we obtain a proof of the sequent

$$\vdash_{L+R} \widetilde{\Gamma|_L}^{l}, \widetilde{\Gamma|_R}^{r}, !_l\widetilde{A} \otimes !_r\widetilde{B}$$

The context Γ is a sequence of formulae $(C_1, \dots, C_n)$ of domain $L + R$, and by Lemma 9, we have $\widetilde{\Gamma|_L}^{l} = ((\widetilde{C_1|_L})^l, \dots, (\widetilde{C_n|_L})^l)$ and similarly $\widetilde{\Gamma|_R}^{r} = ((\widetilde{C_1|_R})^r, \dots, (\widetilde{C_n|_R})^r)$. So applying several exchange rules and n contraction rules, we get a proof of the sequent $\vdash_{L+R} \widetilde{\Gamma}, !_l\widetilde{A} \otimes !_r\widetilde{B}$ as required.

Assume that π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\rho\\ \vdash_J \Gamma\end{array}}{\vdash_J \Gamma, ?_{0_J}A}$$

where A is a formula of empty domain. By inductive hypothesis, we have a proof in $\mathsf{LL}^+(I)$ of $\vdash_J \widetilde{\Gamma}$. Applying a weakening rule, we obtain a proof of $\vdash_J \widetilde{\Gamma}, \widetilde{A}^{0_J}$, and then by a dereliction rule, we get a proof of $\vdash_J \widetilde{\Gamma}, ?_{0_J}\widetilde{A}$, as required.

Assume that π is the proof

$$\dfrac{\vdots \atop \vdash_J \Gamma, ?_l A|_L, ?_r A|_R}{\vdash_J \Gamma, ?_{l+r} A}$$

where L and R are disjoint subsets of I, A is a formula of domain $L+R$, l is an almost injective function from L to J and r is an almost injective function from R to J. Then by inductive hypothesis, the sequent $\vdash_J \widetilde{\Gamma}, ?_l \widetilde{A}|_L, ?_r \widetilde{A}|_R$ is provable in $\mathsf{LL}^+(I)$ (we also apply here Lemma 9). By Lemma 7, the sequent $\vdash_L \widetilde{A}|_L, \widetilde{A}|_L{}^{\perp}$ is provable in $\mathsf{LL}^+(I)$. Applying a promotion rule, we obtain a proof of the sequent $\vdash_J \widetilde{A}|_L{}^{l}, !_l(\widetilde{A}|_L{}^{\perp})$. Similarly, the sequent $\vdash_J \widetilde{A}|_R{}^{r}, !_r(\widetilde{A}|_R{}^{\perp})$ is provable in $\mathsf{LL}^+(I)$. So, applying two cut rules (and some exchange rules) in $\mathsf{LL}^+(I)$, we get a proof of the sequent $\vdash_J \widetilde{\Gamma}, \widetilde{A}|_L{}^{l}, \widetilde{A}|_R{}^{r}$ in $\mathsf{LL}^+(I)$. We conclude, applying a contraction rule, and then a dereliction rule in $\mathsf{LL}^+(I)$.

Assume that π is the proof

$$\dfrac{\vdots \atop \vdash_J \Gamma, \varphi_* A}{\vdash_J \Gamma, ?_\varphi A}$$

where A is a formula of domain K and φ is a bijection from K to J. By inductive hypothesis, the sequent $\vdash_J \widetilde{\Gamma}, \widetilde{\varphi_* A}$ is provable in $\mathsf{LL}^+(I)$. By Lemma 9 (using a cut rule), the sequent $\vdash_J \widetilde{\Gamma}, \varphi_* \widetilde{A}$ is provable in $\mathsf{LL}^+(I)$. Applying a re-localization rule, we get a proof of $\vdash_K \widetilde{\Gamma}^{\varphi^{-1}}, (\varphi_* \widetilde{A})^{\varphi^{-1}}$. By a factorization rule, we get a proof of $\vdash_K \widetilde{\Gamma}^{\varphi^{-1}}, \widetilde{A}$. Re-localizing again, we obtain a proof of $\vdash_J \widetilde{\Gamma}, \widetilde{A}^{\varphi}$ and we conclude applying a dereliction rule in $\mathsf{LL}^+(I)$.

Assume that π is the proof

$$\dfrac{\vdots \atop \vdash_J ?_{u_1} C_1, \ldots, ?_{u_n} C_n, A}{\vdash_K ?_{v \circ u_1} C_1, \ldots, ?_{v \circ u_n} C_n, !_v A}$$

where each u_i is an almost injective function from $d(A_i)$ to J and v is an almost injective function from J to K. Then, by inductive hypothesis, and applying the same kind of cuts as in the case of a contraction rule above, we obtain a proof in $\mathsf{LL}^+(I)$ of the sequent $\vdash_J \widetilde{C_1}^{u_1}, \ldots, \widetilde{C_n}^{u_n}, \widetilde{A}$. We obtain a proof of $\vdash_K \widetilde{C_1}^{v \circ u_1}, \ldots, \widetilde{C_n}^{v \circ u_n}, !_v \widetilde{A}$ by a promotion rule in $\mathsf{LL}^+(I)$ and then conclude, applying n times the dereliction rule.

If π ends with an exchange rule, one concludes straightforwardly, and also if π ends with a cut rule. ∎

The next lemma is trivial and will be implicitly used in the sequel.

Lemma 11 *Let A be a formula of $\mathsf{LL}(I)$ which contains no additive connectives and no multiplicative constants.*

- *If $\widetilde{A} = \Omega$ then $A = \top$ or $A = 0$.*
- *If $\widetilde{A}$ is of the shape $B' \otimes C'$, then A is of the shape $A = B \otimes C$ with $\widetilde{B} = B'$ and $\widetilde{C} = C'$, and similarly if $\widetilde{A}$ is of the shape $B' \parr C'$.*

- *If $\widetilde{A}$ is of the shape $!_u B'$ then A is of the shape $A = !_u B$ with $\widetilde{B} = B'$, and similarly if $\widetilde{A}$ is of the shape $?_u B'$.*

Lemma 12 *Let Γ be a context of formulae of $\mathsf{LL}(I)$ of domain J and let $\varphi : J \to K$ be a bijection. If $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$ (resp. in $\mathsf{LL}^{\mathrm{ext}}(I)$), then $\vdash_K \varphi_* \Gamma$ is provable in $\mathsf{LL}(I)$ (resp. in $\mathsf{LL}^{\mathrm{ext}}(I)$).*

Proof: Straightforward induction. Let us just consider the case of the additional linearization axiom. Let A be a $\mathsf{LL}(I)$-formula of domain J, we must prove the sequent $\vdash_K (\varphi_* A)^\perp, !_\varphi A$.

$$\dfrac{\dfrac{}{\vdash_K (\varphi_* A)^\perp, !_{\mathrm{Id}}(\varphi_* A)} \qquad \dfrac{\dfrac{\dfrac{\vdots\, \rho}{\vdash_J {\varphi^{-1}}_* \varphi_* A^\perp, A}}{\vdash_J ?_{\varphi^{-1}}(\varphi_* A^\perp), A}}{\vdash_K ?_{\mathrm{Id}}(\varphi_* A^\perp), !_\varphi A}}{\vdash_K (\varphi_* A)^\perp, !_\varphi A}$$

is a proof in $\mathsf{LL}^{\mathrm{ext}}(I)$, where ρ is a proof in $\mathsf{LL}(I)$ of $\vdash_J A^\perp, A$ (the analogue of Lemma 7 holds of course for $\mathsf{LL}(I)$).

■

Lemma 13 *Let $A_1, \ldots, A_n$ be formulae of $\mathsf{LL}(I)$ which contain no additive connectives and no multiplicative constants, let J be a subset of I and, for each $i = 1, \ldots, n$, let u_i be an almost injective function from $d(A_i)$ to J. If the sequent $\vdash_J \widetilde{A_1}^{u_1}, \ldots, \widetilde{A_n}^{u_n}$ is provable in $\mathsf{LL}^+(I)$, then the sequent $\vdash_J ?_{u_1} A_1, \ldots, ?_{u_n} A_n$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$.*

Proof: By induction on the proof π in $\mathsf{LL}^+(I)$ of the sequent $\vdash_J \widetilde{A_1}^{u_1}, \ldots, \widetilde{A_n}^{u_n}$.

If π consists of an axiom $\vdash_\emptyset$, then $n = 0$ and we conclude immediately because $\vdash_\emptyset$ is an axiom of $\mathsf{LL}^{\mathrm{ext}}(I)$.

If π ends with a tensor rule, then, by Lemma 11, π must be of the shape

$$\dfrac{\dfrac{\vdots}{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, \widetilde{B}} \qquad \dfrac{\vdots}{\vdash_J \widetilde{C_1}^{w_1}, \ldots, \widetilde{C_q}^{w_q}, \widetilde{C}}}{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, \widetilde{C_1}^{w_1}, \ldots, \widetilde{C_q}^{w_q}, \widetilde{B \otimes C}}$$

By inductive hypothesis, the sequent $\vdash_J ?_{v_1} B_1, \ldots, ?_{v_p} B_p, ?_{\mathrm{Id}} B$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, and so (applying a cut rule with a linearization axiom), the sequent $\vdash_J ?_{v_1} B_1, \ldots, ?_{v_p} B_p, B$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, and similarly for the sequent $\vdash_J ?_{w_1} C_1, \ldots, ?_{w_p} C_q, C$. Therefore, the sequent $\vdash_J ?_{v_1} B_1, \ldots, ?_{v_p} B_p, ?_{w_1} C_1, \ldots, ?_{w_p} C_q, B \otimes C$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, and, applying a dereliction rule, we obtain that the sequent $\vdash_J ?_{v_1} B_1, \ldots, ?_{v_p} B_p, ?_{w_1} C_1, \ldots, ?_{w_p} C_q, ?_{\mathrm{Id}}(B \otimes C)$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, as required.

When π ends with a par rule, one proceeds in a similar way.

Assume now that π ends with a promotion rule, so that, by Lemma 11, π must be of the shape

$$\dfrac{\vdots}{\dfrac{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, \widetilde{B}}{\vdash_K \widetilde{B_1}^{v \circ v_1}, \ldots, \widetilde{B_p}^{v \circ v_p}, \widetilde{!_v B}}}$$

and so, by inductive hypothesis, the sequent $\vdash_J ?_{v_1}B_1, \ldots, ?_{v_p}B_p, ?_{\mathrm{Id}}B$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, and therefore (applying a cut rule with a linearization axiom), the sequent $\vdash_J ?_{v_1}B_1, \ldots, ?_{v_p}B_p, B$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. We are in position of applying a promotion rule in $\mathsf{LL}(I)$, obtaining a proof in $\mathsf{LL}^{\mathrm{ext}}(I)$ of the sequent $\vdash_K ?_{v \circ v_1}B_1, \ldots, ?_{v \circ v_p}B_p, !_v B$, and we conclude as above, with a dereliction rule.

When π ends with a dereliction rule, one concludes straightforwardly.

Assume that π ends with a contraction rule (the case of a weakening rule is similar, and simpler). Then, by Lemma 9, π is of the shape

$$\dfrac{\vdots}{\dfrac{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, \widetilde{B|_L}^{l}, \widetilde{B|_R}^{r}}{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, \widetilde{B}^{l+r}}}$$

where B is a formula of $\mathsf{LL}(I)$ of domain $L + R$ and $l : L \to J$ and $r : R \to J$ are almost injective functions. By inductive hypothesis, the sequent $\vdash_J ?_{v_1}B_1, \ldots, ?_{v_p}B_p, ?_l(B|_L), ?_r(B|_R)$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, and one concludes, applying a contraction rule of $\mathsf{LL}(I)$.

Assume that π ends with a factorization rule

$$\dfrac{\vdots}{\dfrac{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, C^{v \circ \varphi}}{\vdash_J \widetilde{B_1}^{v_1}, \ldots, \widetilde{B_p}^{v_p}, (\varphi_* C)^{v}}}$$

with C a formula of $\mathsf{LL}^{+}(I)$ and $\varphi_* C = \widetilde{B}$ for a formula B of $\mathsf{LL}(I)$ which contains no additive connectives and no multiplicative constants. Then $C = \widetilde{D}$ where $D = {\varphi^{-1}}_* B$, since $\widetilde{{\varphi^{-1}}_* B} = {\varphi^{-1}}_* \widetilde{B}$ by our assumption that B contains no additive connectives and no multiplicative constants. By inductive hypothesis, the sequent $\vdash_J ?_{v_1}B_1, \ldots, ?_{v_p}B_p, ?_{v \circ \varphi}D$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. But the sequent $\vdash_J !_{v \circ \varphi}(D^{\perp}), ?_v(\varphi_* D)$ is provable in $\mathsf{LL}(I)$ as follows

$$\dfrac{\dfrac{\dfrac{\vdots\, \rho}{\vdash_{d(D)} D^{\perp}, {\varphi^{-1}}_*(\varphi_* D)}}{\vdash_{d(D)} D^{\perp}, ?_{\varphi^{-1}}\varphi_* D}}{\vdash_J !_{v \circ \varphi}(D^{\perp}), ?_v(\varphi_* D)}$$

where ρ is the "identity" proof, and so, applying a cut rule, we obtain that the sequent $\vdash_J ?_{v_1}B_1, \ldots, ?_{v_p}B_p, ?_v(\varphi_* D)$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, as required.

If π ends with a re-localization rule, one applies Lemma 12. The case where π ends with an exchange rule is trivial.

If π ends with a cut rule, it is of the shape

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_K \widetilde{B_1}^{v_1},\ldots,\widetilde{B_p}^{v_p},D^v\end{array} \qquad \begin{array}{c}\vdots\\ \vdash_J \widetilde{C_1}^{w_1},\ldots,\widetilde{C_q}^{w_q},D^\perp\end{array}}{\vdash_K \widetilde{B_1}^{v_1},\ldots,\widetilde{B_p}^{v_p},\widetilde{C_1}^{v\circ w_1},\ldots,\widetilde{C_q}^{v\circ w_q}}$$

where D is a $\mathsf{LL}^+(I)$ formula of domain J. Let E be any $\mathsf{LL}(I)$-formula which does not contain additive connectives and satisfies $\widetilde{E}=D$ (it is easy to see that such a formula E exists). Then by inductive hypothesis, the sequents $\vdash_K ?_{v_1}B_1,\ldots,?_{v_p}B_p,?_vE$ and $\vdash_J ?_{w_1}C_1,\ldots,?_{w_q}C_q,?_{\mathrm{Id}}(E^\perp)$ are provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. So, by a cut rule with a linearization axiom of $\mathsf{LL}^{\mathrm{ext}}(I)$, the sequent $\vdash_J ?_{w_1}C_1,\ldots,?_{w_q}C_q,E^\perp$ is provable. We apply a promotion rule, and obtain a proof in $\mathsf{LL}^{\mathrm{ext}}(I)$ of the sequent $\vdash_K ?_{v\circ w_1}C_1,\ldots,?_{v\circ w_q}C_q,!_v(E^\perp)$ and we conclude with a cut rule in $\mathsf{LL}(I)$. ∎

As a corollary of these lemmas, we obtain the following result, which states that the systems $\mathsf{LL}^+(I)$ and $\mathsf{LL}^{\mathrm{ext}}(I)$ are equivalent.

Proposition 14 *Let A be a $\mathsf{LL}(I)$ formula of domain $J\subseteq I$. The sequent $\vdash_J A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$ iff the sequent $\vdash_J \widetilde{A}$ is provable in $\mathsf{LL}^+(I)$.*

Proof: The "only if" part is just an application of Lemma 10. For the "if" part, consider first the case where A contains no additive connectives and no multiplicative constants. Then $\vdash_J ?_{\mathrm{Id}}A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$ by Lemma 13. With a cut rule on a linearization axiom we obtain that $\vdash_J A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$.

Observe now that, when B and C are $\mathsf{LL}(I)$ formulae of disjoint domains L and R, the sequents $\vdash_{L+R} ?_lB \parr ?_rC, B^\perp \mathbin{\&} C^\perp$ and $\vdash_{L+R} !_lB\otimes !_rC, B^\perp\oplus C^\perp$ are provable in $\mathsf{LL}^{\mathrm{ext}}(I)$, where $l:L\to L+R$ and $r:R\to L+R$ are the injections. The first of these sequents is indeed already provable in $\mathsf{LL}(I)$. We give a proof of the second. The sequent $\vdash_L B,B^\perp$ is provable in $\mathsf{LL}(I)$, hence also the sequent $\vdash_L B, B^\perp\oplus C|_L{}^\perp$ since $d(C|_L)=\emptyset$. Applying a dereliction rule, and then a promotion rule, we obtain a proof of $\vdash_{L+R} !_lB, ?_l(B^\perp\oplus C|_L{}^\perp)$ in $\mathsf{LL}(I)$. Similarly, we obtain a proof in $\mathsf{LL}(I)$ of $\vdash_{L+R} !_rC, ?_r(B|_R{}^\perp\oplus C^\perp)$ in $\mathsf{LL}(I)$. Applying a tensor rule on these sequents, we obtain a proof in $\mathsf{LL}(I)$ of the sequent $\vdash_{L+R} !_lB\otimes !_rC, ?_l(B^\perp\oplus C|_L{}^\perp), ?_r(B|_R{}^\perp\oplus C^\perp)$. Now a contraction rule leads to $\vdash_{L+R} !_lB\otimes !_rC, ?_{\mathrm{Id}}(B^\perp\oplus C^\perp)$ and then a cut rule on a linearization axiom yields a proof in $\mathsf{LL}^{\mathrm{ext}}(I)$ of the required sequent. One proves similarly in $\mathsf{LL}^{\mathrm{ext}}(I)$ (in $\mathsf{LL}(I)$ indeed) the two sequents $\vdash_J ?_{0_J}0, 1_J$ and $\vdash_J \bot_J, !_{0_J}\top$.

To any formula A of domain J of $\mathsf{LL}(I)$, we can associate a formula A' of domain J of $\mathsf{LL}(I)$ which contains no additive connectives and no multiplicative constants, mimicking the translation $A\mapsto\widetilde{A}$: $(B \mathbin{\&} C)' = !_lB'\otimes !_rC'$, $1'_J = ?_{0_J}0$, $(B\otimes C)'=B'\otimes C'$, $(!_uB)' = !_uB'$, $0'=0$ and dually for the other cases. Then, using the observation above, one proves easily, by induction on A, that the sequent $\vdash_J A'^\perp, A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. Moreover, it is clear from the definition of A' that $\widetilde{A'}=\widetilde{A}$.

Let A be a $\mathsf{LL}(I)$ formula of domain J, and assume that $\vdash_J \widetilde{A}$ is provable in $\mathsf{LL}^+(I)$, that is $\vdash_J \widetilde{A'}$ is provable in $\mathsf{LL}^+(I)$. Since A' contains no additive connectives and no multiplicative constants, $\vdash_J A'$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. But we have seen that $\vdash_J A'^\perp, A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$ and we conclude with a cut rule that $\vdash_J A$ is provable in $\mathsf{LL}^{\mathrm{ext}}(I)$. ∎

Until the end of the paper, the only system of indexed linear logic under consideration is $\mathsf{LL}^+(I)$. We describe the phase semantics of this system and state a soundness theorem for this semantics.

Let $M = (P_0^I, \perp)$ be a symmetric product phase space. To each formula A of domain $J \subseteq I$, we associate a fact $A^\bullet \in M(J)$ exactly as we did for $\mathsf{LL}(I)$ in Section 1.4. Of course, $\Omega^\bullet$ is the unique fact of $M(\emptyset)$. If $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ is a context of domain J, one defines $\Gamma^\bullet = ?_{u_1}(A_1{}^\bullet) \parr \cdots \parr ?_{u_n}(A_n{}^\bullet)$.

Theorem 15 *If the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}^+(I)$, then $1^J \in \Gamma^\bullet$.*

The proof is completely similar to the soundness proof in [BE01].

3 The completeness theorem

For a formula A of $\mathsf{LL}^+(I)$, one defines $\underline{A}$ as the underlying formula of A in linear logic, deciding for instance that $\underline{\Omega} = \top$ (one can also take $\underline{\Omega} = 0$). For any formula A, one also defines $\langle A \rangle \in |\underline{A}|^{d(A)}$, like in $\mathsf{LL}(I)$ (see Section 1.3). Given two formulae A and B, we say that they are *denotationally equivalent*, and we write $A \sim B$, if $d(A) = d(B)$, $\underline{A} = \underline{B}$, and last, $\langle A \rangle = \langle B \rangle$. The three next lemmas immediately result from this definition.

Lemma 16 *Let B be a formula. If $\Omega \sim B$, then $B = \Omega$.*

Lemma 17 *Let A_1 and A_2 be two formulae of domain J. Let B be a formula. If $A_1 \otimes A_2 \sim B$, then $B = B_1 \otimes B_2$ for two formulae B_1 and B_2 such that $A_i \sim B_i$ for $i = 1, 2$. Similarly for $A_1 \parr A_2$.*

Lemma 18 *Let A be a formula and let $u : d(A) \to J$ be an almost injective function. Let B be a formula. If $!_u A \sim B$, then $B = !_v C$ where C is a formula, $v : d(C) \to J$ is an almost injective function, and moreover, there exists a bijection $\varphi : d(C) \to d(A)$ such that $u \circ \varphi = v$ and $\varphi_* C \sim A$.*

Proof: B must be of the shape $!_v C$ where C is a formula and $v : d(C) \to J$ is an almost injective function. Then we must have $\underline{A} = \underline{C}$. Moreover, we recall that $\langle !_v C \rangle_j = \mathsf{m}(\langle C \rangle|_{v^{-1}(j)})$ for each $j \in J$. So for each $j \in J$, there is a bijection $\varphi_j : v^{-1}(j) \to u^{-1}(j)$ such that, for all $k \in v^{-1}(j)$, one has $\langle C \rangle_k = \langle A \rangle_{\varphi_j(k)}$. Then, defining $\varphi : d(C) \to d(A)$ by $\varphi(k) = \varphi_{v(k)}(k)$, we define a bijection satisfying the announced conditions. ∎

Let $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ be a context of domain J and let $\Delta = (B_1^{v_1}, \ldots, B_m^{v_m})$ be a context of domain K. We say that Γ and Δ are denotationally equivalent and write $\Gamma \sim \Delta$ if $J = K$, $n = m$, and for $i = 1, \ldots, n$, there is a bijection $\varphi_i : d(A_i) \to d(B_i)$ such that $\varphi_{i*} A_i \sim B_i$ and $v_i \circ \varphi_i = u_i$.

Two contexts which are denotationally equivalent differ only by the indexes used in the exponential constructs they contain, but have exactly the same structure. Therefore, they behave in the same way with respect to (cut-free) provability, and this will be essential in the forthcoming completeness proof.

Lemma 19 *Let Γ' be a context of domain J and assume that $\vdash_J \Gamma'$ is cut-free provable. Let Δ be a context such that $\Delta \sim \Gamma'$. Then $\vdash_J \Delta$ is cut-free provable.*

Proof: We proceed by induction on the cut-free proof σ of $\vdash_J \Gamma'$.

If σ consists just of an axiom, then $J = \emptyset$ and $\Delta = \Gamma'$ is the empty context.

If σ ends with a par rule, then $\Gamma' = (\Gamma, A_1 \parr A_2)$ and σ is of the shape

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_J \Gamma, A_1, A_2\end{array}}{\vdash_J \Gamma, A_1 \parr A_2}$$

and then Δ must be of the shape $\Delta = (\Lambda, B^v)$ with $\Lambda \sim \Gamma$, and there must exist a bijection $\varphi : J \to d(B)$ such that $v \circ \varphi = \mathrm{Id}_J$ (that is, $v = \varphi^{-1}$), and $\varphi_*(A_1 \parr A_2) \sim B$. Therefore, by Lemma 17, B is of the shape $B = B_1 \parr B_2$ with $\varphi_* A_1 \sim B_1$ and $\varphi_* A_2 \sim B_2$, hence $(\Gamma, A_1, A_1) \sim (\Lambda, B_1^v, B_2^v)$ and so, by inductive hypothesis, the sequent $\vdash_J \Lambda, B_1^v, B_2^v$ is cut-free provable. Re-localizing, we obtain that the sequent $\vdash_{d(B)} \Lambda^\varphi, B_1, B_2$ is cut-free provable. We conclude, applying a par rule and then a re-localization rule again (with $v = \varphi^{-1}$ as bijection). The case where σ ends with a tensor rule is similar.

Assume now that the last rule of σ is a dereliction rule, that is, σ is of the shape

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_J \Gamma, A^u\end{array}}{\vdash_J \Gamma, ?_u A}$$

where $u : d(A) \to J$ is an almost injective function. Then Δ must be of the shape $\Delta = (\Lambda, (?_v B)^w)$, with $\Lambda \sim \Gamma$, and there must exist a bijection $\varphi : J \to K$ (where K denotes the co-domain of v, which is also the domain of w) such that $w \circ \varphi = \mathrm{Id}_J$ (that is $w = \varphi^{-1}$) and $?_{\varphi \circ u} A \sim ?_v B$. Therefore, by Lemma 18 there is a bijection $\psi : d(A) \to d(B)$ such that $v \circ \psi = \varphi \circ u$ and $\psi_* A \sim B$. Then we have $(\Lambda, B^{\varphi^{-1} \circ v}) \sim (\Gamma, A^u)$. So by inductive hypothesis, the sequent $\vdash_J \Lambda, B^{\varphi^{-1} \circ v}$ is cut-free provable, and hence so is the sequent $\vdash_K \Lambda^\varphi, B^v$ (re-localization), and then so is $\vdash_K \Lambda^\varphi, ?_v B$ (dereliction), and last, so is $\vdash_J \Lambda, (?_v B)^w$ (re-localization again).

Assume next that σ is of the shape

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_J \Gamma, A\end{array}}{\vdash_K \Gamma^u, !_u A}$$

Then Δ must be of the shape $\Delta = (\Lambda, (!_v B)^w)$, with $\Lambda \sim \Gamma^u$, and there must exist a bijection $\varphi : K \to L$ (where L denotes the co-domain of v, which is also the domain of w) such that $w \circ \varphi = \mathrm{Id}_K$ (that is $w = \varphi^{-1}$) and $!_{\varphi \circ u} A \sim !_v B$. Therefore, by Lemma 18 there is a bijection $\psi : d(A) \to d(B)$ such that $v \circ \psi = \varphi \circ u$ and $\psi_* A \sim B$. The context Γ is a sequence $(C_1^{u_1}, \dots, C_n^{u_n})$, and Λ is a sequence of the same length, $\Lambda = (D_1^{v_1}, \dots, D_n^{v_n})$, and we know that, for $i = 1, \dots, n$, there is a bijection $\theta_i : d(C_i) \to d(D_i)$ such that $\theta_{i*} C_i \sim D_i$ and $v_i \circ \theta_i = u \circ u_i$. Let us set $t_i = u_i \circ \theta_i{}^{-1}$, so that $v_i = u \circ t_i$. Then we define a new context $\Lambda' = (D_1^{t_1}, \dots, D_n^{t_n})$ which satisfies $\Lambda = (\Lambda')^u$ and $\Lambda' \sim \Gamma$ (since $t_i \circ \theta_i = u_i$). We have therefore $(\Gamma, A) \sim (\Lambda', B^{\psi^{-1}})$ and so, by inductive hypothesis, the sequent $\vdash_J \Lambda', B^{\psi^{-1}}$ is cut-free provable. Hence, so is $\vdash_{d(B)} \Lambda'^\psi, B$ (re-localization), and then so is $\vdash_L \Lambda'^{v \circ \psi}, !_v B$ (promotion), and, therefore, so is $\vdash_K \Lambda'^{\varphi^{-1} \circ v \circ \psi}, (!_v B)^{\varphi^{-1}}$ (re-localization), but this latter sequent is identical to $\vdash_K \Delta$ since $\varphi^{-1} \circ v \circ \psi = u$ and $u \circ t_i = v_i$.

Assume now that σ ends with a contraction rule (the case of a weakening rule is trivial), so σ is of the shape

$$\frac{\vdots \atop \vdash_J \Gamma, (A|_L)^l, (A|_R)^r}{\vdash_J \Gamma, A^{l+r}}$$

Then Δ must be of the shape $\Delta = (\Lambda, B^u)$ with $\Lambda \sim \Gamma$ and there is a bijection $\varphi : d(A) = L + R \to d(B)$ such that $\varphi_* A \sim B$ and $u = (l+r) \circ \varphi^{-1}$. Let $L' = \varphi(L)$ and $R' = \varphi(R)$, so that $d(B) = L' + R'$ and let $\lambda : L \to L'$ be the bijection obtained by restricting φ, define similarly a bijection $\rho : R \to R'$. Last, let $l' : L' \to J$ and $r' : R' \to J$ be the restrictions of u to L' and R', so that $u = l' + r'$. Then $(\Lambda, (B|_{L'})^{l'}, (B|_{R'})^{r'}) \sim (\Gamma, (A|_L)^l, (A|_R)^r)$. Indeed, we have $\varphi_* A \sim B$ and therefore $(\varphi_* A)|_{L'} \sim B|_{L'}$, but it is easily checked (induction on formulae) that $(\varphi_* A)|_{L'} = \lambda_*(A|_L)$. Moreover, we have $l' \circ \lambda = l$. Applying the inductive hypothesis, and then a contraction rule, we deduce that $\vdash_J \Lambda, B^{l'+r'}$ is cut-free provable, as required.

Assume next that σ ends with a re-localization rule

$$\frac{\vdots \atop \vdash_J \Gamma}{\vdash_K \Gamma^\varphi}$$

where φ is bijection. Then since $\Delta \sim \Gamma^\varphi$, we have $\Delta^{\varphi^{-1}} \sim \Gamma$ and so, by inductive hypothesis, the sequent $\vdash_J \Delta^{\varphi^{-1}}$ is cut-free provable. Applying a re-localization rule, we conclude that the sequent $\vdash_K \Delta$ is cut-free provable.

Assume last that σ ends with a factorization rule (the case of an exchange rule is trivial)

$$\frac{\vdots \atop \vdash_J \Gamma, A^{u \circ \varphi}}{\vdash_J \Gamma, (\varphi_* A)^u}$$

where φ is a bijection $d(A) \to K$. Then Δ is of the shape $\Delta = (\Lambda, B^v)$ with $\Lambda \sim \Gamma$ and there is a bijection $\psi : K \to d(B)$ such that $\psi_* \varphi_* A \sim B$ and $v \circ \psi = u$. That is $(\psi \circ \varphi)_* A \sim B$ and $v \circ (\psi \circ \varphi) = u \circ \varphi$, and so $(\Gamma, A^{u \circ \varphi}) \sim \Lambda, B^v$. Hence, by inductive hypothesis, $\vdash_J \Lambda, B^v$ is cut-free provable. ∎

We define now the syntactic commutative monoid P which will be used in our completeness proof.

Let $\mathcal{L}$ be a fixed infinite denumerable set whose elements will be called *labels*. Let P be the set of all finite multi-sets of the shape $[(l_1, R_1, a_1), \ldots, (l_n, R_n, a_n)]$ where, for each $i = 1, \ldots, n$, $l_i \in \mathcal{L}$, R_i is a formula of linear logic and $a_i \in |R_i|$. We consider P as a commutative monoid (the free commutative monoid generated by the triples (l, R, a)). We use multiplicative notations for denoting the operation of this monoid. When A is a formula and $l \in \mathcal{L}$, we define $\langle A, l \rangle \in P^{d(A)}$ by $\langle A, l \rangle_i = (l, \underline{A}, \langle A \rangle_i)$ for each $i \in d(A)$. If $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ is a context of domain J and if $\lambda = (l_1, \ldots, l_n)$ is a vector of labels of the same length, we define $\langle \Gamma, \lambda, J \rangle \in P^J$ as follows (for

$j \in J$):

$$\begin{aligned}\langle\Gamma, \lambda, J\rangle_j &= \prod_{i=1}^{n}(u_{i*}\langle A_i, l_i\rangle)_j \\ &= \prod_{i=1}^{n} \prod_{\substack{s\in d(A_i)\\ u_i(s)=j}} (l_i, \underline{A_i}, \langle A_i\rangle_s)\end{aligned}$$

The presence of J in $\langle\Gamma, \lambda, J\rangle$ is usually superfluous, it is useful only when Γ is the empty context, and in that case, $\langle\Gamma, \lambda, J\rangle$ is the unity of the monoid P^J. So we tend to omit this additional information.

For $J \subseteq I$, we denote by $\mathcal{A}_J$ the set of all the pairs (Γ, λ) where Γ is a context of domain J and λ is a sequence of *pairwise distinct* labels having the same length as Γ. These pairs will be sometimes called *adapted pairs*. Before proving the Main lemma 23, we first prove three decomposition lemmas. It is only in the second of these lemmas that we use the labels and the definition above of an adapted pair in a crucial way.

Lemma 20 *Let $J \subseteq I$ and let $p, q \in P^J$. Let $(\Gamma, \lambda) \in \mathcal{A}_J$, and assume that $\langle\Gamma, \lambda\rangle = pq$. Then there exist Δ and Λ such that $(\Delta, \lambda), (\Lambda, \lambda) \in \mathcal{A}_J$, $\langle\Delta, \lambda\rangle = p$, $\langle\Lambda, \lambda\rangle = q$, and, moreover, for any context Φ of domain J, if the sequent $\vdash_J \Delta, \Lambda, \Phi$ is cut-free provable, then the sequent $\vdash_J \Gamma, \Phi$ is cut-free provable.*

Proof: Writing $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ and $\lambda = (l_1, \ldots, l_n)$, we have, for each $j \in J$:

$$p_j q_j = \prod_{i=1}^{n} \prod_{\substack{s\in d(A_i)\\ u_i(s)=j}} (l_i, \underline{A_i}, \langle A_i\rangle_s)$$

Let $j \in J$. For each $i = 1, \ldots, n$, we can write $u_i^{-1}(j)$ as the disjoint union of two sets, $u_i^{-1}(j) = S_{i,j} + T_{i,j}$, in such a way that

$$p_j = \prod_{i=1}^{n} \prod_{s\in S_{i,j}} (l_i, \underline{A_i}, \langle A_i\rangle_s)$$

and

$$q_j = \prod_{i=1}^{n} \prod_{s\in T_{i,j}} (l_i, \underline{A_i}, \langle A_i\rangle_s)\,.$$

For each fixed $i = 1, \ldots, n$, the sets $S_{i,j}$ (for $j \in J$) are pairwise disjoint, and we set $S_i = \sum_{j\in J} S_{i,j}$. Similarly we set $T_i = \sum_{j\in J} T_{i,j}$. Moreover, for $j, j' \in J$, $S_{i,j}$ and $T_{i,j'}$ are disjoint, so that S_i and T_i are disjoint, and clearly $S_i + T_i = d(A_i)$. We define $v_i : S_i \to J$ as the restriction of u_i to S_i and $w_i : T_i \to J$ as the restriction of u_i to T_i, so that $u_i = v_i + w_i$. We define $\Delta = ((A_1|_{S_1})^{v_1}, \ldots, (A_n|_{S_n})^{v_n})$ and $\Lambda = ((A_1|_{T_1})^{w_1}, \ldots, (A_n|_{T_n})^{w_n})$. Then of course $(\Delta, \lambda) \in \mathcal{A}_J$ and $(\Lambda, \lambda) \in \mathcal{A}_J$. Moreover, for

$j \in J$, we have

$$\begin{aligned}\langle\Delta,\lambda\rangle_j &= \prod_{i=1}^{n}\prod_{\substack{s\in d(A_i|_{S_i})\\ v_i(s)=j}}(l_i,\underline{A_i},\langle A_i\rangle_s)\\ &= \prod_{i=1}^{n}\prod_{\substack{s\in S_i\\ u_i(s)=j}}(l_i,\underline{A_i},\langle A_i\rangle_s)\\ &= p_j\end{aligned}$$

as $S_{i,j} = S_i \cap u_i^{-1}(j)$. So $\langle\Delta,\lambda\rangle = p$ and similarly $\langle\Lambda,\lambda\rangle = q$. Now let Φ be a context of domain J and assume that $\vdash_J \Delta,\Lambda,\Phi$ is cut-free provable. Applying some exchange rules and then n times the contraction rule (between the occurrence $(A_i|_{S_i})^{v_i}$ in Δ and the corresponding occurrence $(A_i|_{T_i})^{w_i}$ in Λ), we get a cut-free proof of $\vdash_J \Gamma,\Phi$, since $S_i + T_i = d(A_i)$ and $v_i + w_i = u_i$. ∎

The next lemma will be of constant use in the proof of Lemma 23. It uses heavily our particular definition of P as well as the notion of denotational equivalence and Lemma 19.

Lemma 21 *Let $J \subseteq I$ and let $p \in P^J$. Let $(\Gamma,\lambda) \in \mathcal{A}_J$ and let A be a formula of domain J. Let $l \in \mathcal{L}$ and assume that $\langle\Gamma,\lambda\rangle = p\langle A,l\rangle$. Then there exists a context Δ of domain J such that $(\Delta,\lambda) \in \mathcal{A}_J$, $\langle\Delta,\lambda\rangle = p$ and such that, moreover, if the sequent $\vdash_J \Delta, A$ is cut-free provable, then the sequent $\vdash_J \Gamma$ is cut-free provable.*

Proof: By Lemma 20, there are two contexts Δ and Λ such that $(\Delta,\lambda),(\Lambda,\lambda) \in \mathcal{A}_J$, $\langle\Delta,\lambda\rangle = p$, $\langle\Lambda,\lambda\rangle = \langle A,l\rangle$ and such that, if $\vdash_J \Delta,\Lambda$ is cut-free provable, then so is $\vdash_J \Gamma$. So we assume that $\vdash_J \Delta, A$ is cut-free provable, and show that so is $\vdash_J \Delta,\Lambda$, and this will prove the lemma.

The context Λ can be written $\Lambda = (B_1^{v_1},\dots,B_n^{v_n})$ where n is the common length of the sequences Γ and $\lambda = (l_1,\dots,l_n)$. For $j \in J$, we have

$$\prod_{i=1}^{n}\prod_{\substack{s\in d(B_i)\\ v_i(s)=j}}(l_i,\underline{B_i},\langle B_i\rangle_s) = (l,\underline{A},\langle A\rangle_j)$$

and therefore, there is exactly one pair $(i(j),s(j))$ with $i(j) \in \{1,\dots,n\}$ and $s(j) \in {v_{i(j)}}^{-1}(j)$, and for that pair $(i(j),s(j))$, we have $(l_{i(j)},\underline{B_{i(j)}},\langle B_{i(j)}\rangle_{s(j)}) = (l,\underline{A},\langle A\rangle_j)$. But the pair (Γ,λ) is adapted, so the elements of λ are pairwise distinct[6], so the index $i(j)$ does not depend on j, and we can assume, without loss of generality, that $i(j) = 1$ for all $j \in J$, and then $l_1 = l$. Now we have ${v_1}^{-1}(j) = \{s(j)\}$ for all $j \in J$, so that v_1 is a bijection $\varphi : d(B_1) \to J$. For $i \geq 2$, we have ${v_i}^{-1}(j) = \emptyset$ for all $j \in J$ so that $d(B_i) = \emptyset$ and v_i is the empty function 0_J. Moreover, we clearly have $\underline{B_1} = \underline{A}$ and for each $j \in J$, $\langle B_1\rangle_{\varphi^{-1}(j)} = \langle A\rangle_j$, so that $\varphi_* B_1 \sim A$. So by Lemma 19, the sequent $\vdash_J \Delta, \varphi_* B_1$ is cut-free provable. Applying a re-localization rule, we get that the sequent $\vdash_J \Delta^{\varphi^{-1}}, (\varphi_* B_1)^{\varphi^{-1}}$ also is cut-free provable. By a factorization rule, $\vdash_J \Delta^{\varphi^{-1}}, B_1$ is cut-free provable, and so, re-localizing again, the sequent $\vdash_J \Delta, B_1^{\varphi}$ is cut-free provable. Applying next $n-1$ weakening rules (and some exchange rules, but we shall not mention them anymore), we get that the sequent $\vdash_J \Delta,\Lambda$ is cut-free provable, as required. ∎

[6] This is the only place of the completeness proof where we really use the labels, but their rôle is crucial.

Lemma 22 *Let $J, K \subseteq I$, let $p \in P^J$ and let $u : J \to K$ be an almost injective function. Let $(\Gamma, \lambda) \in \mathcal{A}_K$ and assume that $\langle \Gamma, \lambda \rangle = u_* p$. Then there is a context Δ such that $(\Delta, \lambda) \in \mathcal{A}_J$, $\langle \Delta, \lambda \rangle = p$ and $\Delta^u = \Gamma$.*

Proof: We write $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ and $\lambda = (l_1, \ldots, l_n)$. For each $k \in K$, we have

$$\langle \Gamma, \lambda \rangle_k = \prod_{i=1}^{n} \prod_{\substack{s \in d(A_i) \\ u_i(s) = k}} (l_i, \underline{A_i}, \langle A_i \rangle_s) = \prod_{\substack{j \in J \\ u(j) = k}} p_j \, .$$

Let k be a fixed element of K. For each $j \in u^{-1}(k)$ and each $i \in \{1, \ldots, n\}$, we can find a subset $S_{i,j}^k$ of ${u_i}^{-1}(k) \subseteq d(A_i)$ in such a way that:

- for each fixed $i \in \{1, \ldots, n\}$, the sets $S_{i,j}^k$ are pairwise disjoint (for $j \in u^{-1}(k)$), and moreover ${u_i}^{-1}(k) = \sum_{j \in u^{-1}(k)} S_{i,j}^k$,
- and $p_j = \prod_{i=1}^n \prod_{s \in S_{i,j}^k} (l_i, \underline{A_i}, \langle A_i \rangle_s)$ for each $j \in u^{-1}(k)$.

Observe also that, when $i \in \{1, \ldots, n\}$ is fixed, and when $k, k' \in K$ are distinct, the sets $S_{i,j}^k$ and $S_{i,j'}^{k'}$ (when $j \in u^{-1}(k)$ and $j' \in u^{-1}(k')$) are disjoint as they are respectively subsets of ${u_i}^{-1}(k)$ and ${u_i}^{-1}(k')$. Therefore $d(A_i) = \sum_{j \in J} S_{i,j}^{u(j)}$. We denote by $S_{i,j}$ the index set $S_{i,j}^{u(j)}$ and, for each $s \in d(A_i)$, we denote by $v_i(s)$ the unique element of J such that $s \in S_{i,v_i(s)} = S_{i,v_i(s)}^{u(v_i(s))} \subseteq {u_i}^{-1}(u(v_i(s)))$, defining a function $v_i : d(A_i) \to J$ which satisfies $u \circ v_i = u_i$ and is almost injective.

We define the context Δ of domain J as $\Delta = (A_1^{v_1}, \ldots, A_n^{v_n})$. By construction, we have $\Delta^u = \Gamma$ and $\langle \Delta, \lambda \rangle = p$. The second equation results from the fact that, for each $i \in \{1, \ldots, n\}$ and each $j \in J$, ${v_i}^{-1}(j) = S_{i,j}$ (by definition of v_i, for each $s \in d(A_i)$, one has $v_i(s) = j$ iff $s \in S_{i,j}$). ∎

Now we are in position of proving the main lemma for establishing completeness. We consider a particular product phase space, whose underlying monoid is P. The support monoid is P_0^I, where P_0 is obtained by adding a zero element to P. But we prefer to consider this product monoid as the sum $\sum_{J \subseteq I} P^J$ with neutral element and product defined in the natural way (through the obvious bijection between the sets P_0^I and $\sum_{J \subseteq I} P^J$). We keep denoting this support monoid by P_0^I. The subset $\perp$ of P_0^I defining our phase space is described as follows. Let $J \subseteq I$ and let $p \in P^J$. We decide that $p \in \perp$ if:

for each $K \subseteq J$ and each $(\Gamma, \lambda) \in \mathcal{A}_K$, if $\langle \Gamma, \lambda \rangle = p|_K$,
then the sequent $\vdash_K \Gamma$ is cut-free provable.

We first prove that we have defined in that way a symmetric product phase space $M = (P_0^I, \perp)$.

First, $\perp$ is not empty, as it contains 0, the unique element of $P^\emptyset$. Indeed, let $(\Gamma, \lambda) \in \mathcal{A}_\emptyset$ be such that $\langle \Gamma, \lambda \rangle = 0$. So $\Gamma = (A_1^{u_1}, \ldots, A_n^{u_n})$ is a context of empty domain, so each of the formulae A_i has empty domain (since $u_i : d(A_i) \to \emptyset$), and $u_i = 0_\emptyset$ for $i = 1, \ldots, n$. Now the sequent $\vdash_\emptyset$ is cut-free provable, and so the sequent $\vdash_\emptyset \Gamma$ is also cut-free provable (apply n times the weakening rule).

Next we check that $\perp$ is closed under restriction. Let $J \subseteq I$ and let $p \in \perp \cap P^J$. Let $K \subseteq J$, we must prove that $p|_K \in \perp \cap P^K$. So let $L \subseteq K$ and let $(\Gamma, \lambda) \in P^L$ be such that $\langle \Gamma, \lambda \rangle = (p|_K)|_L = p|_L$. The sequent $\vdash_L \Gamma$ is cut-free provable, since $p \in \perp$ and $L \subseteq J$.

As to symmetry, we take two subsets J and K of I and a bijection $\varphi : J \to K$, and we must prove that $\varphi_*(\perp(J)) \subseteq \perp(K)$. So let $L \subseteq J$ and let $p \in P^L$ be such that $p \in \perp(J)$, that is, $p \in \perp$. Let $R \subseteq \varphi(L)$ and let (Γ, λ) be such that $\langle \Gamma, \lambda \rangle = (\varphi_* p)|_R$, we have to show that $\vdash_R \Gamma$ is cut-free provable. Let $S = \varphi^{-1}(R) \subseteq L$, so that $(\varphi_* p)|_R = \varphi_*(p|_S)$. Then we have $\langle \Gamma^{\varphi^{-1}}, \lambda \rangle = p|_S$ so that the sequent $\vdash_S \Gamma^{\varphi^{-1}}$ is cut-free provable, since $p \in \perp$. Now, applying a re-localization rule, we get that the sequent $\vdash_R \Gamma$ is cut-free provable, as required.

So the pair $M = (P_0^I, \perp)$ defined above is a symmetric product phase space. The next lemma is the key lemma for proving completeness. Following [Oka94, Oka99], the main idea is to prove an *inclusion* and not an equality between the two facts involved, as for proving this inclusion the cut rule is not needed. In that way, one can derive a cut elimination theorem from the completeness theorem, combined with the soundness theorem.

Lemma 23 *Let $J \subseteq I$ and let A be a formula of domain J. Then, in the model M defined above, one has $A^\bullet \subseteq \{\langle A, l \rangle \mid l \in \mathcal{L}\}^\perp$, that is $A^\bullet \subseteq \bigcap_{l \in \mathcal{L}} \{\langle A, l \rangle\}^\perp$.*

Proof: By induction on A. We denote by A° the fact $\{\langle A, l \rangle \mid l \in \mathcal{L}\}^\perp$ of the local phase space $M(J)$.

Assume $A = \Omega$ so that $J = \emptyset$. Then both $A^\bullet$ and A°, being facts, must be equal to $\{0\}$, the unique fact of $M(\emptyset)$.

Assume $A = B \parr C$, so that $d(B) = d(C) = J$. We know by inductive hypothesis that $B^\bullet \subseteq B^\circ$ and $C^\bullet \subseteq C^\circ$, so that $A^\bullet \subseteq B^\circ \parr C^\circ$ and so it will be sufficient to prove that $B^\circ \parr C^\circ \subseteq (B \parr C)^\circ$. We have $B^\circ \parr C^\circ = \{\langle B, l \rangle \langle C, m \rangle \mid l, m \in \mathcal{L}\}^\perp$. Let $K \subseteq J$ and let $p \in P^K$ be such that $p \in B^\circ \parr C^\circ$. We want to prove that $p \in (B \parr C)^\circ$, so let $l \in \mathcal{L}$, and let us show that $p\langle B \parr C, l \rangle \in \perp$. Let $L \subseteq K$ and let $(\Gamma, \lambda) \in \mathcal{A}_L$ be such that $\langle \Gamma, \lambda \rangle = (p\langle B \parr C, l \rangle)|_L = p|_L \langle B|_L \parr C|_L, l \rangle$. By Lemma 21, there exists Δ such that $(\Delta, \lambda) \in \mathcal{A}_L$, $\langle \Delta, \lambda \rangle = p|_L$ and such that, if $\vdash_L \Delta, B|_L \parr C|_L$ is cut-free provable, then $\vdash_L \Gamma$ is cut-free provable. Now let $k, m \in \mathcal{L}$ be distinct, and distinct from all the elements of λ. We know that $p \in \{\langle B, k \rangle \langle C, m \rangle\}^\perp$ and hence $p|_L \langle B|_L, k \rangle \langle C|_L, m \rangle \in \perp$. But from our assumptions about k and m, we have $(\Delta B|_L C|_L, \lambda k m) \in \mathcal{A}_L$, and moreover, $\langle \Delta B|_L C|_L, \lambda k m \rangle = p|_L \langle B|_L, k \rangle \langle C|_L, m \rangle$, so that, by definition of $\perp$, the sequent $\vdash_L \Delta, B|_L, C|_L$ is cut-free provable. Hence (applying a par rule), the sequent $\vdash_L \Delta, B|_L \parr C|_L$ is cut-free provable. Hence $\vdash_L \Gamma$ is cut-free provable, as required.

Assume $A = B \otimes C$, so that $d(B) = d(C) = J$. We know by inductive hypothesis that $B^\bullet \subseteq B^\circ$ and $C^\bullet \subseteq C^\circ$, so that $A^\bullet \subseteq B^\circ \otimes C^\circ$ and so it will be sufficient to prove that $B^\circ \otimes C^\circ \subseteq (B \otimes C)^\circ$, i.e. $B^\circ C^\circ \subseteq (B \otimes C)^\circ$. So let $p \in B^\circ$ and $q \in C^\circ$. Without loss of generality, we assume that, for some $K \subseteq J$, one has $p, q \in P^K$. Let $L \subseteq K$ and let $(\Gamma, \lambda) \in \mathcal{A}_L$ be such that $\langle \Gamma, \lambda \rangle = p|_L q|_L \langle B|_L \otimes C|_L, l \rangle$, we have to show that $\vdash_L \Gamma$ is cut-free provable. By Lemma 21, there exists Γ' such that $(\Gamma', \lambda) \in \mathcal{A}_L$, $\langle \Gamma', \lambda \rangle = p|_L q|_L$ and, moreover, if $\vdash_L \Gamma', B|_L \otimes C|_L$ is cut-free provable, so is $\vdash_L \Gamma$. By Lemma 20, there exist Δ and Λ such that $(\Delta, \lambda), (\Lambda, \lambda) \in \mathcal{A}_L$, $\langle \Delta, \lambda \rangle = p|_L$, $\langle \Lambda, \lambda \rangle = q|_L$ and moreover, if $\vdash_L \Delta, \Lambda, B|_L \otimes C|_L$ is cut-free provable, so is $\vdash_L \Gamma', B|_L \otimes C|_L$. Let $m \in \mathcal{L}$ be different from all the elements of λ. We know that $p\langle B, m \rangle \in \perp$ since we have taken p in B°. But $(\Delta B|_L, \lambda m) \in \mathcal{A}_L$ and $\langle \Delta B|_L, \lambda m \rangle = (p\langle B, m \rangle)|_L$ since $\langle \Delta, \lambda \rangle = p|_L$, and hence $\vdash_L \Delta, B|_L$ is cut-free provable. Similarly, $\vdash_L \Lambda, C|_L$ is cut-free provable, so applying a tensor rule, we get that the sequent $\vdash_L \Delta, \Lambda, B|_L \otimes C|_L$ is cut-free provable and we conclude that $\vdash_L \Gamma$ is cut-free provable.

Assume now that $A = ?_u B$ with $d(B) = K \subseteq I$ and $u : K \to J$ almost injective. Then, as before, it will be sufficient to show that $?_u(B^\circ) \subseteq (?_u B)^\circ$. For any $m \in \mathcal{L}$, we have $\langle B, m \rangle \in B^{\circ\perp}$, so $u_*\langle B,m\rangle \in u_*(B^{\circ\perp})$ and therefore $?_u(B^\circ) \subseteq \{u_*\langle B,m\rangle\}^\perp$. Hence it is sufficient to show that, for any $l \in \mathcal{L}$, one has

$$\bigcap_{m\in\mathcal{L}} (\{u_*\langle B,m\rangle\}^\perp) \subseteq \{\langle ?_u B, l\rangle\}^\perp .$$

So let $L \subseteq J$ and let $p \in P^L$ be such that $p(u_*\langle B,m\rangle)|_L \in \perp$ for each $m \in \mathcal{L}$. Let $l \in \mathcal{L}$, we show that $p\langle ?_u B, l\rangle \in \perp$. So let $R \subseteq L$ and let $(\Gamma,\lambda) \in \mathcal{A}_R$ be such that $\langle\Gamma,\lambda\rangle = p|_R\langle ?_u B, l\rangle|_R$. According to Lemma 21, we can find Δ such that $(\Delta,\lambda) \in \mathcal{A}_R$, $\langle\Delta,\lambda\rangle = p|_R$ and moreover, if $\vdash_R \Delta, (?_u B)|_R$ is cut free provable, so is $\vdash_R \Gamma$. Now let $m \in \mathcal{L}$ be different from all the elements of λ. We know that $p(u_*\langle B,m\rangle)|_L \in \perp$ by our assumption about p. Let $S = u^{-1}(R) \subseteq K$, and let $v : S \to R$ be the restriction of u to S, so that the diagram

$$\begin{tikzcd} K \arrow[r, "u"] & J \\ S \arrow[u, "\sigma"] \arrow[r, "v"] & R \arrow[u, "\rho"'] \end{tikzcd}$$

(where ρ and σ are the inclusions) is a pull-back in **Set**. Then we have $(\Delta(B|_S)^v, \lambda m) \in \mathcal{A}_R$ and $\langle\Delta(B|_S)^v, \lambda m\rangle = \langle\Delta,\lambda\rangle v_*\langle B|_S, m\rangle = p|_R(u_*\langle B,m\rangle)|_R$ by Lemma 5. Therefore $\vdash_R \Delta, (B|_S)^v$ is cut-free provable, and so, applying a dereliction rule, the sequent $\vdash_R \Delta, ?_v(B|_S)$ is cut-free provable, i.e. the sequent $\vdash_R \Delta, (?_u B)|_R$ is cut-free provable, and therefore, so is $\vdash_R \Gamma$, as required.

Assume last that $A = !_u B$ with $d(B) = K \subseteq I$ and $u : K \to J$ almost injective. Then, as before, it will be sufficient to show that $!_u(B^\circ) \subseteq (!_u B)^\circ$. For this, it is enough to show that $u_*(B^\circ) \subseteq (!_u B)^\circ$. Let $L \subseteq K$ and $p \in B^\circ \cap P^L$. Then, by definition of $u_* : P_0^K \to P_0^J$, we have $u_* p \in P^R$, where $R \subseteq J$ is

$$R = \{j \in J \mid u^{-1}(j) \subseteq L\} .$$

Let $S = u^{-1}(R) \subseteq L$ and let $v : S \to R$ be the restriction of u to S, so that the diagram

$$\begin{tikzcd} K \arrow[r, "u"] & J \\ S \arrow[u, "\sigma"] \arrow[r, "v"] & R \arrow[u, "\rho"'] \end{tikzcd}$$

(where ρ and σ are the inclusions) is a pull-back in **Set**, therefore $u_* p = (u_* p)|_R = v_*(p|_S)$ by Lemma 5. We must show that $v_*(p|_S) \in (!_u B)^\circ$. Let $l \in \mathcal{L}$, we have to show that $v_*(p|_S)\langle !_u B, l\rangle \in \perp$. So let $T \subseteq R$ and let $(\Gamma,\lambda) \in \mathcal{A}_T$ be such that $\langle\Gamma,\lambda\rangle = (v_*(p|_S)\langle !_u B, l\rangle)|_T = w_*(p|_U)\langle !_w(B|_U), l\rangle$ where $U = v^{-1}(T) = u^{-1}(T) \subseteq S$ and $w : U \to T$ is the restriction of u (or of v: these restrictions are equal). By Lemma 21, there exists Δ such that $(\Delta,\lambda) \in \mathcal{A}_T$, $\langle\Delta,\lambda\rangle = w_*(p|_U)$ and, if $\vdash_T \Delta, !_w(B|_U)$ is cut-free provable, then so is $\vdash_T \Gamma$. By Lemma 22, there exists Λ such that $(\Lambda,\lambda) \in \mathcal{A}_U$, $\langle\Lambda,\lambda\rangle = p|_U$ and $\Lambda^w = \Delta$. Let $m \in \mathcal{L}$ be a label not occurring in λ, so that $(\Lambda B|_U, \lambda m) \in \mathcal{A}_U$ and $\langle\Lambda B|_U, \lambda m\rangle = p|_U\langle B,m\rangle|_U \in \perp$ since $p \in B^\circ$. Therefore, $\vdash_U \Lambda, B|_U$ is cut-free provable. Applying a promotion rule, we get that $\vdash_T \Lambda^w, !_w(B|_U)$ i.e. $\vdash_T \Delta, !_w(B|_U)$ is cut-free provable and hence so is $\vdash_T \Gamma$ as required. ∎

Now we can prove the completeness theorem.

Theorem 24 *Let A be a formula of indexed linear logic with domain $J \subseteq I$. If, in each symmetric product phase space $M = (P_0^I, \perp)$, one has $1^J \in A^\bullet$, then $\vdash_J A$ is cut-free provable.*

Proof: Take for M the particular symmetric product phase space defined above. By Lemma 23, we have $1^J \in A^\circ$. So let l be any element of $\mathcal{L}$. We have $\langle A, l\rangle \in \perp$. Setting $\Gamma = (A)$ and $\lambda = (l)$, we obviously have $(\Gamma, \lambda) \in \mathcal{A}_J$ and $\langle \Gamma, \lambda\rangle = \langle A, l\rangle$ and so, by definition of $\perp$, the sequent $\vdash_J \Gamma$ is cut-free provable. ∎

Combining the completeness and soundness theorems, we derive easily the cut elimination theorem.

Theorem 25 *Let A be a formula of indexed linear logic with domain $J \subseteq I$. If the sequent $\vdash_J A$ is provable, it is cut-free provable.*

Remark: In these two theorems, we can also replace the formula A by an arbitrary context Γ of domain J. Indeed, the rules for the par and for the interrogation mark connectives being reversible, if $\vdash_J ?_{u_1} A_1 \parr \cdots \parr ?_{u_n} A_n$ is cut-free provable, so is $\vdash_J A_1^{u_1}, \ldots, A_n^{u_n}$.

Remark: This cut elimination theorem does not provide any procedure for "locally" eliminating the cut rule, in the style of Gentzen cut elimination theorem for LK. It expresses precisely that the rules given in Section 2, cut rule excluded, are sufficient for proving the formulae which are true in all models: the system is *complete* in the sense that there is no point in adding any new rule to it. In particular, the cut rule we have given in Section 2 is superfluous, as would be any other deduction rule. For instance, a weaker cut rule (with both A and $A^\perp$ having the identity as exponent) would be superfluous as well, although it would clearly not give rise to a local elimination procedure: consider a cut with main formulae of the shape $!_u A$ and $(!_u A)^\perp$. Nevertheless, it can be proved that the cut rule we have given admits a local elimination procedure, a result which is stronger than Theorem 25.

4 Appendix: the interpretation of proofs in the category of sets and relations

To each proof π of a sequent in first order propositional linear logic $\vdash \Phi$, we associate a subset of π^* of the set $|\Phi|$ defined in Section 1.2, by induction on π.

Tensor unit: if the proof π is

$$\frac{}{\vdash 1}$$

then $\pi^* = \{*\}$.

With unit: if the proof π is

$$\frac{}{\vdash \Phi, \top}$$

then $\pi^* = \emptyset$.

With: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}\qquad\qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Psi, T\end{array}}{\vdash \Psi, S \,\&\, T}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in {\pi_1}^*)\} \cup \{(c,(2,b)) \mid (c,b) \in {\pi_2}^*)\}$.

Left plus: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}}{\vdash \Psi, S \oplus T}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in {\pi_1}^*)\}$. And similarly if π ends with a right plus rule.

Par unit: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi\end{array}}{\vdash \Psi, \bot}$$

then $\pi^* = \{(c,*) \mid c \in {\pi_1}^*\}$.

Par: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S, T\end{array}}{\vdash \Psi, S \parr T}$$

then $\pi^* = \{(c,(a,b)) \mid (c,a,b) \in {\pi_1}^*\}$.

Tensor: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}\qquad\qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Theta, T\end{array}}{\vdash \Psi, \Theta, S \otimes T}$$

then $\pi^* = \{(c,d,(a,b)) \mid (c,a) \in {\pi_1}^*)$ and $(d,b) \in {\pi_2}^*\}$.

Weakening: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c,[]) \mid c \in {\pi_1}^*\}$.

Contraction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, ?S, ?S\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c,x+y) \mid (c,x,y) \in {\pi_1}^*\}$ where $x+y$ denotes the sum of the multi-sets x and y.

Dereliction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash \Psi, S\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c, [a]) \mid (c, a) \in {\pi_1}^*\}$.

Promotion: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash ?S^1, \ldots, ?S^k, S\end{array}}{\vdash ?S^1, \ldots, ?S^k, !S}$$

then π^* is the set of all $k+1$-tuples of the shape $(\sum_{j=1}^n x_j^1, \ldots, \sum_{j=1}^n x_j^k, [a_1, \ldots, a_n])$ where $((x_j^1, \ldots, x_j^k, a_j))_{j=1,\ldots,n}$ is any finite family of elements of ${\pi_1}^*$.

Exchange: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash S^1, \ldots, S^k\end{array}}{\vdash S^{\sigma(1)}, \ldots, S^{\sigma(k)}}$$

where σ is a permutation of $\{1, \ldots, k\}$, then $\pi^* = \{(a_{\sigma(1)}, \ldots, a_{\sigma(k)}) \mid (a_1, \ldots, a_k) \in {\pi_1}^*\}$.

Cut: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash \Psi, S\end{array} \qquad \begin{array}{c}\vdots\ \pi_2\\ \vdash \Theta, S^\perp\end{array}}{\vdash \Psi, \Theta}$$

then $\pi^* = \{(c, d) \mid \exists a\, (c, a) \in {\pi_1}^* \text{ and } (d, a) \in {\pi_2}^*\}$.

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