A Categorical Semantics of Constructions

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Abstract

The main object of this work is to propose an abstract framework for the description of the type dependency semantics. Our claim is that the notion of *fibration* introduced by A. Grothendieck in the sixties is perfectly adapted to this goal and provides the greatest simplicity and generality. We extend this semantics to higher order, explaining what a general definition for the semantics of the theory of constructions could be.

Introduction

This paper is based on a common work with P.L. Curien ([8]). We built up together some categorical structures for first order type dependency which we learned while finishing this work to be identical to Cartmell's contextual categories. We present here some more abstract concepts and their extension to the semantics of Constructions where roughly higher order logic is combined with dependency.

Besides defining models for Constructions, our goal is to "modularize" in some sense the semantics of type dependency. Actually two approaches to this semantics have been proposed. The first one by R. Seely (see [20]) uses locally cartesian closed categories. But in this approach every arrow is a type, which is not selective enough in defining types. On the other hand it is a perfectly clean and well known categorical concept. The second approach by J. Cartmell ([2]) introduced ad hoc categorical structures (contextual categories) which match more closely the syntactic nature of type dependency. This approach relies on presheaves of categories. Our main claim is that these notions are best understood in the more abstract framework of fibrations. Contextual categories as well as locally cartesian categories will be instances of this more general notion.

We introduce stepwise categorical structures to interpret first and higher order type dependency. The definition is often directly suggested by the syntax, both by the forms of judgements and the inference rules. The main feature of type dependency as compared with simple Curry types is that types and contexts have to be proven well formed. Recall first the basic categorical interpretation of simply typed λ-calculus. There is only one form of judgement: $\Gamma \vdash M : P$. Types and contexts are interpreted as objects in a category and terms by arrows. Let us keep this in mind and start to examine the syntax of type dependency. Contexts and types are typically introduced by judgements Γ : Context and $\Gamma \vdash A$: Type. The similarity between this last judgement and the one in the simply typed case suggests, as a first attempt, to interpret

- contexts as objects Γ in a category $\mathbf{B}$
- types as arrows $s : \Gamma \to \Omega$ from contexts to a special object Ω associated with "Type".

The obvious drawback of this interpretation is to force a higher order concept into the semantics. So we don't want to be so precise about types. We simply say that they are associated to contexts in some way.

More precisely, we introduce two categories $\mathbf{B}$ and $\mathbf{F}$, and a functor $p : \mathbf{F} \to \mathbf{B}$. $\mathbf{B}$ should be seen as having as objects the proven judgements Γ : Context, and $\mathbf{F}$ proven judgements $\Gamma \vdash P$: Type. Then

$$p(\Gamma \vdash P : \text{Type}) = \Gamma : \text{Context}\,.$$

How can we build new contexts? The basic rule is

$$\frac{\Gamma \vdash P : \text{Type}}{\Gamma(x : P) : \text{Context}}$$

it suggests to introduce a functor $G : \mathbf{F} \to \mathbf{B}$ sending a judgement $\Gamma \vdash M$: Type on the corresponding extended context.

$$G(\Gamma \vdash P : \text{Type}) = \Gamma(x : P) : \text{Context}\,.$$

This operation should generalize in a sense the usual categorical cartesian product, since with simple Curry types the categorical counterpart of the corresponding operation is cartesian product. Hence it seems natural to define it as a right adjoint to a "canonical embedding functor" since the cartesian product is the right adjoint to the diagonal functor. But here $\mathbf{F}$ is not a cartesian product of categories. However as soon as p satisfies the property of being a "fibration" and $\mathbf{F}$ and $\mathbf{B}$ have terminal objects preserved by p, we can define such a canonical embedding I. Intuitively, I sends each judgement Γ : Context on $\Gamma \vdash 1$: Type where 1 is a "terminal" type well formed in any context. This

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requirement is equivalent to the fact that I be a full and faithfull right adjoint to p. Thus we basically require the existence of $p, G : \mathbf{F} \to \mathbf{B}$ and $I : \mathbf{B} \to \mathbf{F}$ with $p \dashv I \dashv G$ and I full and faithfull. Moreover p will be a fibration as we shall see now.

Let us turn to the interpretation of terms, which are introduced by judgements $\Gamma \vdash M : P$. This suggests that types should also be viewed as objects in some "local" categories depending on the contexts and that terms should be interpreted as arrows in those categories. This kind of indexed category structures is already familiar for the interpretation of predicate calculus or higher order logic ([14,19,21,4]). In contrast with those approaches, the basic structure already imposed yields a very natural definition of the local arrows. Namely, $\mu : X \to Y$ in $\mathbf{F}$ will be said to be a "local arrow in context $A \in \mathbf{B}$" if $p(\mu) = \mathrm{Id}_A$. And a judgement $\Gamma \vdash M : P$ will be translated into a morphism

$$\mu : I(\Gamma : \mathrm{Context}) \to (\Gamma \vdash P : \mathrm{Type})$$

such that $p(\mu) = \mathrm{Id}_{(\Gamma:\mathrm{Context})}$. But when we are given a morphism between contexts $f : A \to B$ in $\mathbf{B}$ which could correspond to a structural rule like logical weakening, we should have some way to apply this rule to types (resp. terms) in context B getting types (resp. terms) in context A. This is the essence of type dependency and it suggests that p should be a fibration. Actually it has been shown (see [1]) that the notion of fibration is specially well adapted to the study of "pseudo-functorially" dependent families of categories. And we want to keep these pseudo-functors in the scope of our work because some of the most interesting models of type dependency are of this kind (locally cartesian closed categories).

Then the dependent product is axiomatized as a right adjoint to a weakening functor, a bit like in [19], but in a more economic way since plain fibrations are under consideration, and not presheaves of categories.

By lack of place, we shall not be able to describe the translation of terms into our semantics in this paper because this is quite a technical topic. However to give a more syntactic justification to the use of fibrations, let us explain how we translate variables. Basically, the rule for introducing variables is the following:

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma(x : A) \vdash x : A}$$

The denotation of judgement $\Gamma \vdash A : \mathrm{Type}$ is an object $X \in \mathbf{F}$ which we assume to be built. Since $I \dashv G$ we have a natural transformation $\pi : IG \to \mathrm{Id}$ (the co-unit morphism) which we can understand as containing the projections associated to our operation G of context building. We can guess it will be the basic tool for translating variables. (Remember that for cartesian product which is defined by an adjunction $\Delta \dashv \times$ the co-unit morphism defines both first and second projections.) To simplify a bit assume that $p \circ I = \mathrm{Id}$. We already know that those functors are naturally isomorphic since I is full and faithfull and we shall see that this additional hypothesis is not very strong. Consider $\pi(X) : IG(X) \to X$ in $\mathbf{F}$ and $\mathrm{fst}(X) = p\pi(X) : G(X) \to p(X)$ in $\mathbf{B}$. (We call it $\mathrm{fst}(X)$ because it looks very much like a first projection natural transformation.) Now comes the important point: since p is a fibration we have a "cartesian" morphism $\widehat{\mathrm{fst}(X)}(X) : \mathrm{fst}(X)^*(X) \to X$ (see the precise definition below) that p sends on $\mathrm{fst}(X)$ (we say: "over $\mathrm{fst}(X)$ for p"). Of course the identity is a morphism f in $\mathbf{B}$ such that $\mathrm{fst}(X) \circ f = p\pi(X)$ and thus since $\widehat{\mathrm{fst}(X)}(X)$ is cartesian there exists a unique morphism $\xi : IG(X) \to \mathrm{fst}(X)^*(X)$ such that

$$p(\xi) = \mathrm{Id} \quad \text{and} \quad \widehat{\mathrm{fst}(X)}(X) \circ \xi = \pi(X)$$

The denotation of judgement $\Gamma(x : A) \vdash x : A$ will precisely be ξ.

What about Constructions? (See for instance [3,13] for a description of the Calculus of Constructions of T. Coquand and G. Huet.) Indeed what we have given unto now is a framework where to interpret first order type dependency (that is with predicative product of types). In order to interpret constructions we must be able to

- say that a term is a proposition, and for this we introduce an object $\Omega \in \mathbf{F}$ corresponding to the type Prop of the calculus
- see each proposition as a type, ie. have a "natural" way of sending a morphism $A \to G(\Omega)$ on a type in context A
- impredicatively quantify over propositions, ie. have a "natural" way of sending a morphism $G(X) \to G(\Omega)$ on a morphism $A \to G(\Omega)$ when X is a well formed type in context A.

And those data should satisfy a coherence condition.

Section 3.3 of the paper gives an example of such categorical structures using the ω-sets recently introduced by E. Moggi.

1 Preliminaries

In this section we recall a few categorical notions and constructions which will be useful in the sequel of the paper. Those definitions are bound to the related concepts of fibred and indexed categories.

We assume a basic knowledge of adjunctions. For definitions and general results see for instance [15]. The following result states a kind of symmetry between left and right adjoints to a given functor. It will play a central role in the paper:

Proposition 1 *Let $F : X \to Y$, $G : Y \to X$ and $H : X \to Y$ be functors such that $F \dashv G$ and $G \dashv H$. Then the co-unit of $F \dashv G$ is an iso iff the unit of $G \dashv H$ is.*

Actually, both statements are equivalent to the full- and faithfullness of G.

1.1 Fibrations

The concept of *fibration* has been introduced by A. Grothendieck. We refer to [1] for more detailed definitions and for deep reasons why plain fibrations are to be used rather than indexed categories and split fibrations. In the whole section, $\mathbf{F}$ and $\mathbf{B}$ will be categories, and p will be a functor $\mathbf{F} \to \mathbf{B}$. For each $A \in \mathbf{B}$ we can define the category "over A" which is the subcategory of $\mathbf{F}$ of which morphisms φ satisfy $p(\varphi) = \mathrm{Id}_A$. It will be denoted $p^{-1}(A)$.

Definition 1 *Let $X \in \mathbf{F}$ and $f : B \to A = p(X)$. A morphism $\varphi : Y \to X$ is said to be "cartesian" over f if $p(\varphi) = f$ and if, for each $\psi : Z \to X$ in $\mathbf{F}$ such that there exists a $g : p(Z) \to B$ factorizing $p(\psi)$ in $p(\psi) = f \circ g$, there exists an unique $\chi : Z \to Y$ such that $p(\chi) = g$ and $\psi = \varphi \circ \chi$.*

Next comes the definition of fibrations:

Definition 2 *p is said to be a fibration if for each $A \in \mathbf{B}$, $X \in \mathbf{F}$ and $f : A \to p(X)$ in $\mathbf{B}$ there exists a morphism $Y \to X$ in $\mathbf{F}$ which is cartesian over f for p. We shall sometimes call "basis" of the fibration the category $\mathbf{B}$. The fibration is said to be split if we have done a functorial choice for those cartesian morphisms.*

"Doing a functorial choice of cartesian morphisms" means for each $f : A \to p(X)$ in $\mathbf{B}$ having been able to choose a cartesian morphism $\widehat{f}(X) : f^*(X) \to X$ in such a way that $\widehat{\mathrm{Id}_{p(X)}}(X) = \mathrm{Id}_X$ and if $g : B \to p(Y) = A$ then $(\widehat{f \circ g})(X) = \widehat{f}(X) \circ \widehat{g}(Y)$. Notice already that in many interesting cases such a choice is impossible or at least non canonical.

Thus when p is split, p^{-1} may be extended into a functor $\mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ i.e. an exact (not pseudo-functorial) indexed category. (A pseudo-functorial indexed category is simply a fibration.) Such a functor is also called a "presheaf of categories", and we note $\mathrm{Psh}(\mathbf{B})$ the category of those presheaves with natural transformation as morphisms.

Remark: When $Y \in \mathbf{F}$ we can define a functor

$$p_Y : \mathbf{F}/Y \to \mathbf{B}/p(Y)$$

simply by sending each $F : X \to Y \in \mathbf{F}/Y$ on $p(F) : p(X) \to p(Y)$ and similarly for morphisms. In fact the correspondence $Y \mapsto p_Y$ is natural. We have the following caracterization of fibrations which is just a simple rephrasing of definition 1.

Proposition 2 *A functor $p : \mathbf{F} \to \mathbf{B}$ is a fibration iff for each $Y \in \mathbf{F}$ there exists a functor $\overline{Y} : \mathbf{B}/p(Y) \to \mathbf{F}/Y$ such that $p_Y \circ \overline{Y} = \mathrm{Id}$ and which is a right adjoint to p_Y having the identity as co-unit morphism.*

Particularly, when $\mathbf{F}$ has a terminal object $\mathbf{1}$ that p sends on a terminal object in $\mathbf{B}$, this proposition gives us a functor $I : \mathbf{B} \to \mathbf{F}$ such that $p \circ I = \mathrm{Id}$ and which is a right adjoint to p having the identity as co-unit morphism, since $\mathbf{F}/\mathbf{1}$ and $\mathbf{F}$ are isomorphic.

We have defined the concept of fibration. As usually in category theory, there is a notion of morphism associated with it:

Definition 3 *Let $p : \mathbf{F} \to \mathbf{B}$ and $q : \mathbf{D} \to \mathbf{B}$ be two fibrations over the same basis. A functor $F : \mathbf{F} \to \mathbf{D}$ is said to be cartesian from p to q if it satisfies $q \circ F = p$ and sends each p-cartesian morphism on a q-cartesian one.*

Of course cartesian functors may be composed, yielding cartesian functors and we define in such a way a category $\mathrm{Fib}(\mathbf{B})$ which is a subcategory of $\mathbf{Cat}/\mathbf{B}$.

Given a fibration and a natural transformation between two functors ranging in the basis of the fibration, there is a cartesian functor and a natural transformation which is p-cartesian in some sense, defining a "natural choice of cartesian morphisms".

To be more precise we state first some notations. As usually, $p : \mathbf{F} \to \mathbf{B}$ will be a fibration. Let $F : \mathbf{C} \to \mathbf{B}$ be a functor. We shall adopt the following notations for a chosen pullback of the functors F and p:

$$\begin{array}{ccc} \mathbf{F} \underset{p,F}{\times} \mathbf{C} & \xrightarrow{p_F} & \mathbf{C} \\ {\scriptstyle F_p}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{F} & \xrightarrow{p} & \mathbf{B} \end{array}$$

It is known that p_F defines then a fibration. When no confusion, we shall write F' instead of F_p. With those notations we have the following

Proposition 3 *Let $F, G : \mathbf{C} \to \mathbf{B}$ be two functors and $\sigma : F \to G$ be a natural transformation between them. Then there exists a functor $\sigma^* : \mathbf{F} \underset{p,G}{\times} \mathbf{C} \to \mathbf{F} \underset{p,F}{\times} \mathbf{C}$ which is cartesian from p_G to p_F and a natural transformation $\widehat{\sigma} : F' \circ \sigma^* \to G'$ such that for each $\xi \in \mathbf{F} \underset{p,G}{\times} \mathbf{C}$ the morphism $\widehat{\sigma}(\xi)$ be p-cartesian over $\sigma(p_G(\xi))$.*

Notice that in the split case, when composing a presheaf with a functor one gets a new presheaf, and there is a pullback diagram between the associated Grothendieck's constructions (see section 1.2). But since we deal with plain fibrations the pull-backs are what remains of this figure.

Proof: We make a particular choice of pullback: $\mathbf{F} \underset{p,F}{\times} \mathbf{C}$ will be assumed to be the category of which objects are the pairs (X, A) where $X \in \mathbf{F}$ and $A \in \mathbf{C}$ such that $p(X) = F(A)$ and of which morphisms are also pairs of morphisms that p and F send on the same morphism. Then p_F and F' are the two obvious projection functors.

1) We verify first that as announced p_F is a fibration. Let $(X, A) \in \mathbf{F} \underset{p,F}{\times} \mathbf{C}$ and $f : A' \to A$. Then since $p(X) = F(A)$ there exists a p-cartesian morphism $\varphi : X' \to X$ in $\mathbf{F}$ over $F(f)$. We show that (φ, f) is p_F-cartesian over f. Of course it is a morphism in $\mathbf{F} \underset{p,F}{\times} \mathbf{C}$ over f for p_F. Now let $(\psi, g) : (Z, C) \to (X, A)$ be a morphism in $\mathbf{F} \underset{p,F}{\times} \mathbf{C}$ and assume that there exists an arrow $h : C \to A'$ in $\mathbf{C}$ such that $f \circ h = g$.

Then $F(f \circ h) = F(g)$, i.e. $F(f) \circ F(h) = p(\psi)$ and now since φ is p-cartesian over $F(f)$ there exists a unique $\theta : Z \to X'$ in F such that $\varphi \circ \theta = \psi$ and $p(\theta) = F(h)$. Hence $(\theta, h) : (Z, C) \to (X', A')$ is a morphism in $\mathbf{F} \underset{p,F}{\times} \mathbf{C}$ over h for p_F and which factorizes (ψ, g) in $(\varphi, f) \circ (\theta, h) = (\psi, g)$. The unicity follows from the fact that p_F is the second projection functor. Hence p_F is a fibration.

2) Next we build a functor σ^*. It is defined by a choice of cartesian morphisms, and those cartesian morphisms will turn out to define the natural transformation $\widehat{\sigma}$. More precisely:

- If $(Y, B) \in \mathbf{F} \underset{p,G}{\times} \mathbf{C}$ is an object then $\sigma(B) : F(B) \to G(B) = p(Y)$. Hence there exists a p-cartesian morphism $\widehat{\sigma}(Y, B) : X \to Y$ in $\mathbf{F}$ over it. (Here we use of course very strongly the axiom of choice.) We take $\sigma^*(Y, B) = (X, B)$.

- Now we give the morphism part of σ^*. Let $(\psi, g) : (Y, B) \to (Y', B')$ in $\mathbf{F} \underset{p,G}{\times} \mathbf{C}$. Since σ is a natural transformation the following diagram is commutative:

$$\begin{array}{ccc} F(B) & \xrightarrow{F(g)} & F(B') \\ {\scriptstyle \sigma(B)}\downarrow & & \downarrow{\scriptstyle \sigma(B')} \\ G(B) & \xrightarrow{G(g)} & G(B') \end{array}$$

Thus $p(\psi \circ \widehat{\sigma}(Y, B)) = \sigma(B') \circ F(g)$. Hence since $\widehat{\sigma}(Y', B')$ is p-cartesian over $\sigma(B')$ there exists a *unique* morphism $\varphi : X \to X'$ (X and X' being respectively the sources of $\widehat{\sigma}(Y, B)$ and $\widehat{\sigma}(Y', B')$) over $F(g)$ for p such that the following be commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle \widehat{\sigma}(Y,B)}\downarrow & & \downarrow{\scriptstyle \widehat{\sigma}(Y',B')} \\ Y & \xrightarrow{\psi} & Y' \end{array}$$

and we take $\sigma^*(\psi, g) = (\varphi, g)$.

It is easy to see that σ^* defines a functor using the unicity in the definition of its morphism part and clearly $p_F \circ \sigma^* = p_G$. It is also clear that $\widehat{\sigma}$ is a natural transformation which is "p-cartesian".

3) It remains to show that σ^* sends cartesian morphisms on cartesian morphisms. We show first this property for the cartesian morphisms $(\psi, g) : (Y', B') \to (Y, B)$ in $\mathbf{F} \underset{p,G}{\times} \mathbf{C}$ such that ψ be p-cartesian over $G(g)$ in $\mathbf{F}$. Using the same notations as above 1) we just have to show that φ is p-cartesian over $F(g)$. Let $\rho : Z \to X'$ be any morphism in F and assume that we have in $\mathbf{B}$ a morphism $\alpha : p(Z) \to F(B) = p(X)$ such that $F(g) \circ \alpha = p(\rho)$. Hence we have

$$G(g) \circ \sigma(B) \circ \alpha = \sigma(B') \circ p(\rho)$$

so there is a unique morphism $\lambda : Z \to Y$ such that

$$\psi \circ \lambda = \widehat{\sigma}(Y', B') \quad \text{and} \quad p(\lambda) = \sigma(B) \circ \alpha$$

but furthermore $\widehat{\sigma}(Y, B)$ is p-cartesian over $\sigma(B)$ so there exists a unique morphism $\mu : Z \to X$ such that

$$\widehat{\sigma}(Y, B) \circ \mu = \lambda \quad \text{and} \quad p(\mu) = \alpha$$

Let us verify that $\varphi \circ \mu = \rho$. Actually it results from the cartesianity of $\widehat{\sigma}(Y', B')$. More precisely, we have $p(\varphi \circ \mu) = p(\rho)$ and the following equality is true for $\theta = \rho$ and $\theta = \varphi \circ \mu$:

$$p(\widehat{\sigma}(Y', B') \circ \theta) = p(\psi \circ \widehat{\sigma}(Y, B) \circ \mu)$$

but there is only one morphism $\theta : Z \to X'$ satisfying this requirement and $p(\theta) = p(\rho)$. Hence $\varphi \circ \mu = \rho$. We check that μ is the single morphism $Z \to X$ verifying $p(\mu) = \alpha$ and $\varphi \circ \mu = \rho$ by using successively the cartesianity of ψ and $\widehat{\sigma}(Y, B)$.

4) Last we generalize the result of 3) to all cartesian morphisms. Take $(\psi, g) : (Y', B') \to (Y, B)$ in $\mathbf{F} \underset{p,G}{\times} \mathbf{C}$ a p_G-cartesian morphism over g. Now let $\psi' : Y'' \to Y$ be a p-cartesian morphism over $G(g)$. Then $(\psi', g) : (Y'', B') \to (Y, B)$ is p_G-cartesian over g. So (ψ, g) and (ψ', g) are isomorphic in $\mathbf{F} \underset{p,G}{\times} \mathbf{C}/(Y, B)$ and hence their images by σ^* also are. This completes the proof.

Remark: First, when $\mathbf{C}$ is the category with a unique object and a unique morphism, remark that this proposition introduces a notation for a choice of cartesian morphism over f with codomain X when $f : A \to p(X)$ is given in $\mathbf{B}$, namely $\widehat{f}(X) : f^*(X) \to X$.

Next, in the sequel of the paper, we shall always make the choice of pullback of functors described above. When $q : \mathbf{E} \to \mathbf{B}$ is a fibration and $F : \mathbf{D} \to \mathbf{B}$ is any functor, the projection functor $q_F : \mathbf{E} \underset{q,F}{\times} \mathbf{D} \to \mathbf{D}$ is a fibration for which we make the following choice of cartesian morphisms: let $(X, U) \in \mathbf{E} \underset{q,F}{\times} \mathbf{D}$ and let $f : V \to q_F(X, U) = U$ be a morphism in $\mathbf{D}$. We take

$$f^*(X, U) = (F(f)^*(X), V)$$

and

$$\widehat{f}(X, U) = \left(\widehat{F(f)}(X), f\right) \tag{1}$$

As we have seen the functor σ^* and the natural transformation $\widehat{\sigma}$ defined above correspond to a *choice* of cartesian morphisms over the natural transformation σ. But generally there is no unicity of this choice. But as far as type dependency is concerned this doesn't matter because they are all isomorphic in the following sense:

Proposition 4 *Let $F, G : \mathbf{C} \to \mathbf{B}$ be functors and $\sigma : F \to G$ a natural transformation. Let $S, T : \mathbf{F} \underset{p,G}{\times} \mathbf{C} \to \mathbf{F} \underset{p,F}{\times} \mathbf{C}$ and $s : F' \circ S \to G'$ and $t : F' \circ T \to G'$ be two natural transformations as built in the previous proposition. Then there exists a unique natural isomorphism $i : S \to T$ such that $(p \circ F')i = \mathrm{id}$ and $t \circ F'i = s$.*

The proof is easy and left to the reader. So we shall make such choices and not care about their canonicity. That is why it was justified to introduce notations like σ^* and $\widehat{\sigma}$.

1.2 The Grothendieck's construction

The aim of this construction is to build a fibration from an indexed category. We present it in the simple case where the indexed category is exact (that is not pseudo-functorial) because we don't need more in the paper. We define a functor

$$\mathrm{Gr} : \mathrm{Psh}(\mathbf{B}) \to \mathbf{Cat}/\mathbf{B}$$

- Let $\mathcal{F} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ be an object of $\mathrm{Psh}(\mathbf{B})$. We note $\mathrm{Gr}_0(\mathcal{F})$ the category of which objects are the couples(A, s) where $A \in \mathbf{B}$ and $s \in \mathcal{F}(A)$, and a morphism $(A, s) \to (B, t)$ is a couple (f, μ) where $f \in \mathrm{Hom}_{\mathbf{B}}(A, B)$ and $\mu \in \mathrm{Hom}_{\mathcal{F}(A)}(s, \mathcal{F}(f)(t))$. Of course there is a first projection functor $\mathrm{Gr}_0(\mathcal{F}) \to \mathbf{B}$, and we take $\mathrm{Gr}(\mathcal{F})$ to be this functor.
- Let $\mathcal{T} : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Psh}(\mathbf{B})$. We define $\mathrm{Gr}(\mathcal{T})$ by taking

  $$\mathrm{Gr}(\mathcal{T})(A, s) = (A, \mathcal{T}(A)(s))$$

  and

  $$\mathrm{Gr}(\mathcal{T})(f, \mu) = (f, \mathcal{T}(A)(\mu))$$

  when $(f, \mu) : (A, s) \to (B, t)$ in $\mathrm{Gr}_0(\mathcal{F})$.

Now the important point is the following:

Proposition 5 *Gr defines an equivalence of categories between* $\mathrm{Psh}(\mathbf{B})$ *and the subcategory* $\mathrm{Split}(\mathbf{B})$ *of* $\mathbf{Cat}/\mathrm{B}$ *of split fibrations over* $\mathbf{B}$*. Moreover, when p is a split fibration over* $\mathbf{B}$*, p and* $\mathrm{Gr}(p^{-1})$ *are isomorphic in* $\mathrm{Split}(\mathbf{B})$.

A morphism of $\mathrm{Split}(\mathbf{B})$ is of course a cartesian functor which preserves splittings.

1.3 Adjunctions and fibrations

When dealing with presheaves of categories, one has often to consider "natural adjunctions", i.e. natural transformations between presheaves which at each object of the base category define an adjunction and such that furthermore these adjunctions be natural (this means that for each morphism of the base category, the value of the two presheaves at that morphism define a morphism of adjunctions as defined in [15] chapter IV, section 7). This is the case for instance in [21]. The corresponding notion in the more general framework of fibrations is simpler:

Definition 4 *Let $p : \mathbf{F} \to \mathbf{B}$ and $q : \mathbf{D} \to \mathbf{B}$ be fibrations. Let $F : \mathbf{F} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{F}$ be two cartesian functors between those fibrations. We say that they define a "fibred adjunction" if they are adjoint in the usual sense and if p sends the unit of the adjunction on the identity (or, equivalently, q sends the co-unit on the identity).*

It can be checked that Gr sends natural adjunctions on fibred (and in fact split) adjunctions.

1.4 Fibrations and the Yoneda lemma

Let $q : \mathbf{D} \to \mathbf{B}$ and $p : \mathbf{F} \to \mathbf{B}$ be two fibrations with the same basis. We call $\mathrm{Cart}(q, p)$ the category of which objects are the cartesian functors from q to p and morphisms are the natural transformations of which p sends each component on the identity.

Let $A \in \mathbf{B}$. Let $d_A : \mathbf{B}/A \to \mathbf{B}$ be the domain functor. It defines the split fibration which corresponds to the discrete indexed category which sends each $B \in \mathbf{B}$ on $\mathrm{Hom}_{\mathbf{B}}(B, A)$. For this fibration, all morphism of $\mathbf{B}/A$ is cartesian. The fibred version of the Yoneda lemma is then the following:

Proposition 6 *The category* $\mathrm{Cart}(d_A, p)$ *is equivalent to* $p^{-1}(A)$.

We shall note Yon_A (or simply Yon if no confusion is possible) the direction $p^{-1}(A) \to \mathrm{Cart}(d_A, p)$ of this equivalence.

As the reader can guess, when $X \in p^{-1}(A)$ the functor $\mathrm{Yon}(X) : d_A \to p$ is defined on the object by $\mathrm{Yon}(A)(f) = f^*(X)$ and similarly for morphisms.

2 Categories for dependent types

We introduce in this section a notion of categories where to interpret first order type dependency. In the split case, this notion has strong connections with the "categories with attributes" introduced by J. Cartmell in his thesis (see [2]), but we present it in a simpler, and we claim more natural way. Anyway, the "split" condition is very strong and we shall just assume to have a fibration as suggested by Bénabou's paper [1]. With such assumptions we get more conceptual simplicity and a wider generality including locally closed categories as well as many other categorical and set theoretical constructions as we shall see.

2.1 D-categories

Definition 5 *Let $\mathbf{F}$ and $\mathbf{B}$ be categories. A pre-D-category structure on these categories is given by three functors:*

$$p, G : \mathbf{F} \to \mathbf{B} \quad \textit{and} \quad I : \mathbf{B} \to \mathbf{F}$$

such that $p \dashv I \dashv G$ and the co-unit of $p \dashv I$ be an iso (i.e. I is full and faithfull). We say that (p, I, G) defines a D-category if p is a fibration. When p is split, we say that the D-category is split.

Remark: Two things are to be noticed: first remember that if a fibration p is given and if $\mathbf{F}$ has a terminal object that p sends on a terminal object of $\mathbf{B}$, then there exists a functor $I : \mathbf{B} \to \mathbf{F}$ satisfying the conditions of the definition w.r.t. p (see proposition 2). Next as soon as (p, I, G) defines a pre-D-category the unit of $I \dashv G$ is an iso thanks to proposition 1.

We have an interesting result which shows that each fibre category $p^{-1}(A)$ has a terminal object in a D-category. More precisely:

Proposition 7 *If $p : \mathbf{F} \to \mathbf{B}$ is a fibration and if $I : \mathbf{B} \to \mathbf{F}$ is a right adjoint to p such that the co-unit α of this adjunction be an iso (i.e. a full and faithfull right adjoint), there exists a functor $\mathbf{1} : \mathbf{B} \to \mathbf{F}$ such that $p \circ \mathbf{1} = \mathrm{Id}$, $p \dashv \mathbf{1}$ and the co-unit of this adjunction be the identity. Furthermore $\beta\mathbf{1}$ is a p-cartesian natural transformation over α^{-1} from $\mathbf{1}$ to I. (β denotes the unit of $p \dashv I$.)*

As an easy crollary of this we have that for $A \in \mathbf{B}$, $\mathbf{1}(A)$ is terminal in $p^{-1}(A)$. We give the proof:

Proof: Consider any $A \in \mathbf{B}$. Then $\alpha^{-1}(A) : A \to pI(A)$. Let $\gamma_A : \mathbf{1}(A) \to I(A)$ be a p-cartesian morphism over it. $\mathbf{1}$ defines a functor (as in proposition 3). Now let $X \in \mathbf{F}$ be any object. Then $\beta(X) : X \to I(A)$ where $A = p(X)$. But $\alpha p \circ p\beta = \mathrm{Id}$ and hence $p\beta(X) = \alpha^{-1}(A)$. Whence a unique morphism $!X : X \to \mathbf{1}(A)$ over the identity for p such that $\gamma_A \circ !X = \beta(X)$. To show that $!X$ is the single morphism $X \to \mathbf{1}(A)$ over the identity for p, it is enough to verify $\beta(\mathbf{1}(A)) = \gamma_A$. But this results from the naturality of β which makes the following commute:

$$\begin{array}{ccc} \mathbf{1}(A) & \xrightarrow{\beta\mathbf{1}(A)} & I(A) \\ \gamma_A \downarrow & & \downarrow Ip(\gamma_A) \\ I(A) & \xrightarrow{\beta I(A)} & IpI(A) \end{array}$$

and from $I\alpha \circ \beta I = \mathrm{Id}$. This completes the proof (remaining details are left to the reader).

So we could as well assume that $p \circ I = \mathrm{Id}$ and that $p \dashv I$ with the identity as co-unit. We shall not adopt this point of view for philosophical reasons: we want to speak as few as possible of equality between objects of categories, at least in definitions of basic concepts. Isomorphism is a more categorical concept.

Now we give the "canonical" example of that situation. It has been pointed out to us by J. Bénabou. Consider any category $\mathbf{B}$ and let $\mathbf{F}$ be the category of which objects are the morphisms $f : A \to B$ of $\mathbf{B}$ and the morphisms are the "commutative diagrams", i.e. a morphism $(f : A \to B) \to (f' : A' \to B')$ is a pair (g, h) of morphisms such that the following be commutative:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{h} & B' \end{array}$$

It is the category $\mathbf{B}^{\mathbf{2}}$ where $\mathbf{2}$ is the category $\cdot \to \cdot$. Let $p : \mathbf{F} \to \mathbf{B}$ be the functor which sends each morphism of $\mathbf{B}$ on its goal object and each "diagram" on its second component. For $I : \mathbf{B} \to \mathbf{F}$ we take the functor which sends each object $A \in \mathbf{B}$ on Id_A and each morphism $f : A \to B$ in $\mathbf{B}$ on the pair (f, f). Last we define $G : \mathbf{F} \to \mathbf{B}$ as sending a morphism of $\mathbf{B}$ on its source and a diagram on its first component. It is an easy exercise to verify that $p \dashv I \dashv G$ and we have of course that the co-unit of $p \dashv I$ is the identity. So we have a pre-D-category structure.

Now assume furthermore that $\mathbf{B}$ has all pullbacks. We verify that p is then a fibration. Actually consider an object $f : A \to B$ of $\mathbf{F}$. Let $g : C \to p(f) = B$ be a morphism in $\mathbf{B}$. Then pulling back g along f we get the following limit diagram:

$$\begin{array}{ccc} D & \xrightarrow{g'} & A \\ f' \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

Then (g', g) is a p-cartesian morphism over g. This is the simplest exemple of D-category. And there is usually no way to split it.

Grothendieck's construction also gives rise to a D-category structure which is split (this was actually our initial intuition, whence the choosen notations). Let us precise a bit this point. Take $\mathbf{B} = \mathbf{Cat}$ (the category of small categories) and let $\mathbf{F}$ be the category of presheaves of small categories with terminal objects preserved. (Precisely, an object of $\mathbf{F}$ is given by a small category X and a presheaf $s : X^{\mathrm{op}} \to \mathbf{Cat}$ such that for each $x \in X$ the category $s(x)$ be small and have a terminal object and for each $f : x \to y$ the functor $s(f)$ preserve terminal objects. A morphism $(X, s) \to (Y, t)$ is given by a functor $F : X \to Y$ and a natural transformation $s \to t \circ F$.) Then for p we take the functor which sends each presheaf on its domain category, for I the functor which sends each category on the constant presheaf with value the terminal object of $\mathbf{Cat}$ (the category with one object and one morphism) and for G we take the Grothendieck's construction.

2.2 Split D-categories and categories with attributes

Let us recall the definition by Cartmell (with personal notations):

Definition 6 *A category with attributes over $\mathbf{B}$ is a functor $\mathcal{A} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Set}$ such that*

- *for each $x \in \mathbf{B}$ and each $s \in \mathcal{A}(x)$ there is an object $x \,.\, s \in \mathbf{B}$ and a morphism $\mathrm{fst}_s : x \,.\, s \to x$*
- *for each $\rho : x \to y$ in $\mathbf{B}$ and $s \in \mathcal{A}(y)$ there is a morphism $\rho \wedge s : x \,.\, \mathcal{A}(\rho)(s) \to y \,.\, s$ such that the following be a pullback:*

$$\begin{array}{ccc} x \,.\, \mathcal{A}(\rho)(s) & \xrightarrow{\rho\wedge s} & y \,.\, s \\ \mathrm{fst}_{\mathcal{A}(\rho)(s)} \downarrow & & \downarrow \mathrm{fst}_s \\ x & \xrightarrow{\rho} & y \end{array}$$

  and we furthermore assume that the correspondence $\rho \mapsto \rho \wedge s$ is functorial in a suitable sense
- *for each $x \in \mathbf{B}$ there is an element $1_x \in x^*$ such that fst_{1_x} be an isomorphism.*

Now we can see how the split D-category structure gives rise to a category with attributes: for $\mathcal{A}(x)$ we take $\mathrm{Obj}(p^{-1}(x))$ and for $x \, . \, s$ we take $G(s)$. Next our adjunction yields a natural transformation $\pi : I \circ G \to \mathrm{Id}$ and then, for each $x \in \mathbf{B}$ and $s \in \mathcal{A}(x)$ the first component of $\pi(x,s)$ is a morphism $x \, . \, s \to x$ and we take this morphism for fst_s. It is not very hard to verify that those definitions actually yield a category with attributes. Conversely, being given a category with attributes $\mathcal{A}$ over $\mathbf{B}$, we extend $\mathcal{A}$ into an exact indexed category by taking for morphisms from s to t in a slice $\mathcal{A}(x)$ the morphisms $\mu : x \, . \, s \to x \, . \, t$ of $\mathbf{B}$ such that $\mathrm{fst}_t \circ \mu = \mathrm{fst}_s$ and by defining the morphism part of functors $\mathcal{A}(\rho)$ (for ρ morphism in $\mathbf{B}$) using the pullback axiomatized in the above definition. Then the Grothendieck's construction applied to this indexed category yields a split D-category structure with $\mathbf{F} = \mathrm{Gr}_0(\mathcal{A})$.

2.3 Closed D-categories

Our goal now is to define a notion of categories where it will be possible to interpret the first order dependent type product.

We assume that (p, I, G) defines a D-category. By the adjunction $I \dashv G$ we have a natural transformation $\pi : I \circ G \to \mathrm{Id}$ (the co-unit morphism of the adjunction) and hence $p\pi$ is a natural transformation $p \circ I \circ G \to p$.

Let $\alpha : p \circ I \to \mathrm{Id}$ be the co-unit of $p \dashv I$. It is an iso and we have $\alpha^{-1}G : G \to p \circ I \circ G$. So

$$p\pi \circ \alpha^{-1}G : G \to p$$

We shall call fst this natural transformation.

Now let **Cart** be the subcategory of $\mathbf{F}$ of which objects are those of $\mathbf{F}$ and morphisms are the p-cartesian ones. (Obviously the identity and the composition of two cartesian morphisms are cartesian.) Let $\gamma : \mathbf{Cart} \to \mathbf{F}$ be the obvious injection functor.

Definition 7 *A D-category (p, I, G) is closed if $(\mathrm{fst}\gamma)^*$ has a fibred right adjoint Π.*

This definition makes sense because the existence of a fibred right adjoint doesn't depend on the choice of $\mathrm{fst}\gamma^*$ as easily seen with proposition 4. In this definition, the requirement for Π to be a cartesian functor should be understood as a generalization of the Beck-Chevalley condition in locally cartesian closed categories. (We hope that the example of section 2.4 makes this point clear.) Actually this condition says that the local product functors define a natural transformation when restricted to pull-back squares, that is cartesian morphisms. This is why we have introduce the category **Cart** and restricted our requirement that the adjunction should be fibred to this subcategory of $\mathbf{F}$.

Finally, the *fibred* adjunction corresponds to another level of naturality, syntactically bound to substitution of terms in terms, whereas the above naturality corresponds to substituting terms in types.

We examine first shortly what this definition means in the split case, coming back to Cartmell's categories with attributes. In this framework, using notations of 2.2, **Cart** is the category with objects those of $\mathbf{F}$ and morphisms those of $\mathbf{F}$ having the identity as second component. Then the above definition axiomatizes for each $x \in \mathbf{B}$ and each $s \in \mathcal{A}(x)$ a right adjoint Πs to $\mathrm{fst}_s{}^*$ such that this adjunction be natural over **Cart**. More precisely this last requirement means for instance that $\mathcal{A}(\rho)(\Pi s(\mu)) = \Pi(\mathcal{A}(\rho)(s))(\mathcal{A}(\rho \wedge s)(\mu))$.

2.4 The example of locally cartesian closed categories

We want to extend our "canonical example" to show that a locally cartesian closed category (LCCC for short) is an instance of closed D-category. We first state some notations. If $\mathbf{B}$ has pullbacks and $f : A \to B$ we note f^* the "pullback functor" $\mathbf{B}/B \to \mathbf{B}/A$. When $\mathbf{B}$ is locally closed (what we assume from now on), we note Πf the right adjoint of f^*. (For details about these notions which we assume known, see for instance [20].) These definitions require of course a choice of all pullbacks along each morphism of $\mathbf{B}$ which hasn't to be canonical in any way. We shall sometimes adopt the following notation for the other morphism in the pullback:

$$\begin{array}{ccc} D & \xrightarrow{f^*(g)} & A \\ {\scriptstyle g_f}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

Let us recall the Beck-Chevalley condition which is valid in any locally cartesian closed category:

Proposition 8 *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ A' & \xrightarrow{f'} & B' \end{array}$$

is a pullback then there is a natural isomorphism between the functors $\Pi f \circ h_1{}^*$ and $h_2{}^* \circ \Pi f'$.

In that case, the category **Cart** has objects the morphisms of $\mathbf{B}$ and morphisms its pullback squares. First we make a choice for the categories $\mathbf{F} \underset{p,p\gamma}{\times} \mathbf{Cart}$ and $\mathbf{F} \underset{p,G\gamma}{\times} \mathbf{Cart}$:

- The objects of $\mathbf{F} \underset{p,p\gamma}{\times} \mathbf{Cart}$ are the pairs of morphism $A \xrightarrow{f} B \xleftarrow{g} C$ and a morphism $(A \xrightarrow{f} B \xleftarrow{g} C) \to (A' \xrightarrow{f'} B' \xleftarrow{g'} C')$ is a triple (h_1, h_2, h_3) such that both following squares be commutative

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{g} & C \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_3} \\ A' & \xrightarrow{f'} & B' & \xleftarrow{g'} & C \end{array}$$

and the righthand one be a pullback. Then $p_{p\gamma}$ is defined by $p_{p\gamma}(A \xrightarrow{f} B \xleftarrow{g} C) = g$ and $(p\gamma)'$ by

$(p\gamma)'(A \xrightarrow{f} B \xleftarrow{g} C) = f.$

- The objects of $\mathbf{F} \underset{p,G\gamma}{\times} \mathbf{Cart}$ are the pairs of morphism $A \xrightarrow{f} B \xrightarrow{g} C$ and a morphism $(A \xrightarrow{f} B \xrightarrow{g} C) \to (A' \xrightarrow{f'} B' \xrightarrow{g'} C')$ is a triple (h_1, h_2, h_3) such that both following squares be commutative

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_3} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C \end{array}$$

and the righthand one be a pullback. Then $p_{G\gamma}$ is defined by $p_{G\gamma}(A \xrightarrow{f} B \xrightarrow{g} C) = g$ and $(G\gamma)'$ by $(G\gamma)'(A \xrightarrow{f} B \xrightarrow{g} C) = f$.

Then , since the natural transformation fst is given by $\mathrm{fst}(f) = f$, "the" functor $\mathrm{fst}\gamma^*$ is defined by

$$\mathrm{fst}\gamma^*(A \xrightarrow{f} B \xleftarrow{g} C) = D \xrightarrow{g^*(f)} C \xrightarrow{g} B$$

for objects, and if $(h_1, h_2, h_3) : (A \xrightarrow{f} B \xrightarrow{g} C) \to (A' \xrightarrow{f'} B' \xrightarrow{g'} C')$ we take $\mathrm{fst}\gamma^*(h_1, h_2, h_3) = (h, h_3, h_2)$ where $h : D \to D'$ is the single morphism such that both following diagrams be commutative:

$$\begin{array}{ccc} D & \xrightarrow{g^*(f)} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_3} \\ D' & \xrightarrow{g'^*(f')} & C' \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{f^*(g)} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_1} \\ D' & \xrightarrow{f'^*(g')} & A' \end{array}$$

and the natural transformation $\widehat{\mathrm{fst}\gamma} : (G\gamma)' \circ \mathrm{fst}\gamma^* \to (p\gamma)'$ is given by

$$\widehat{\mathrm{fst}\gamma}(A \xrightarrow{f} B \xleftarrow{g} C) = (f^*(g), g)$$

Let us define a functor $\Pi : \mathbf{F} \underset{p,G\gamma}{\times} \mathbf{Cart} \to \mathbf{F} \underset{p,p\gamma}{\times} \mathbf{Cart}$ which is a fibred right adjoint to $\mathrm{fst}\gamma^*$:

- For objects, we set:

$$\Pi(A \xrightarrow{f} B \xrightarrow{g} C) = E \xrightarrow{\Pi g(f)} C \xleftarrow{g} B$$

- And for morphisms, if $(h_1, h_2, h_3) : (A \xrightarrow{f} B \xrightarrow{g} C) \to (A' \xrightarrow{f'} B' \xrightarrow{g'} C')$ we know that $h_1 : h_2 \circ f \to f'$ in $\mathbf{B}/B'$ so by the adjunction $\mathbf{B}/h_2 \dashv h_2{}^*$ we get a morphism $h_1' : f \to h_2{}^*(f')$ and next we take its image by the functor Πg. But the Beck-Chevalley condition insures us that there exists an iso $e : \Pi g\,(h_2{}^*(f')) \cong h_3{}^*(\Pi g'(f'))$.
Take $h' = e \circ \Pi g\,(h_1') : \Pi g\,(f) \to h_3{}^*(\Pi g'(f'))$ and let $h : h_3 \circ \Pi g\,(f) \to \Pi g'(f')$ obtained from h' through the adjunction $\mathbf{B}/h_3 \dashv \Pi h_3$. We take $\Pi(h_1, h_2, h_3) = (h, h_3, h_2)$.

3 Models for a theory of constructions

We explain now how to interpret a higher order impredicative quantification in the framework described above.

3.1 Basic definition

The definition is as follows:

- We assume that there exists an object $\Omega \in \mathbf{F}$ such that $p(\Omega)$ be terminal in $\mathbf{B}$, and we call d the fibration $d_{G(\Omega)}$, and a $T \in p^{-1}(G(\Omega))$ (or, equivalently thru proposition 6, a cartesian functor $d \to p$). The associated cartesian functor $\mathrm{Yon}(T)$ should be understood as sending each proposition on a type.

- There exists a functor

$$\forall : \mathbf{B}/G(\Omega) \underset{d,G\gamma}{\times} \mathbf{Cart} \to \mathbf{B}/G(\Omega) \underset{d,p\gamma}{\times} \mathbf{Cart}$$

which is cartesian from the fibration $d_{G\gamma} : \mathbf{B}/G(\Omega) \underset{d,G\gamma}{\times} \mathbf{Cart} \to \mathbf{Cart}$ to the fibration $d_{p\gamma} : \mathbf{B}/G(\Omega) \underset{d,p\gamma}{\times} \mathbf{Cart} \to \mathbf{Cart}$.

- The following diagram of functors is commutative:

$$\begin{array}{ccc} \mathbf{B}/G(\Omega) \underset{d,G\gamma}{\times} \mathbf{Cart} & \xrightarrow{\mathrm{Yon}(T)\times\mathbf{Cart}} & \mathbf{F} \underset{p,G\gamma}{\times} \mathbf{Cart} \\ \forall\downarrow & & \downarrow\Pi \\ \mathbf{B}/G(\Omega) \underset{d,p\gamma}{\times} \mathbf{Cart} & \xrightarrow{\mathrm{Yon}(T)\times\mathbf{Cart}} & \mathbf{F} \underset{p,p\gamma}{\times} \mathbf{Cart} \end{array} \qquad (2)$$

3.2 The special case of Locally Cartesian Closed Categories

We have seen that LCCC's are D-categories. Our attempt now is to understand what is a LCCC which is a higher order D-category. It will turn out to be a bit similar to a topos, although much weaker. However any topos will be a category of this kind, trivial in some sense (as a model of constructions).

We consider each of the previous requirement and express its meaning in the framework of locally cartesian closed categories. Remember that $\mathbf{F} = \mathbf{B}^2$.

- There exists a morphism Ω in $\mathbf{B}$ with codomain the terminal object. We shall confuse this morphism with its domain, so we shall consider Ω as a distinguished object of $\mathbf{B}$. And there exists a morphism $T : \Omega' \to \Omega$ in $\mathbf{B}$ corresponding to the operation of sending a proposition on a type. This functorial operation (namely $\mathrm{Yon}(T)$) consists in pulling back along T.

- When given two morphisms $f : X \to \Omega$ and $g : X \to Y$ in $\mathbf{B}$ there exists a morphism $\forall g\,(f) : Y \to \Omega$ corresponding to the higher impredicative product. It is the object part of the functor $\forall$ of which existence is required above. We don't speak of its morphism part since as easily checked it is necessarily trivial. Furthermore $\forall$ is cartesian. This means that for each pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{h'} & Y \end{array}$$

we have $\forall g(f) \circ h' \cong \forall g'(f \circ h)$ with a natural isomorphism. This is a condition similar to the Beck-Chevalley one, but for higher-order product.

- When $f : X \to \Omega$, we shall often note $\widetilde{f} : \widetilde{X} \to X$ the morphism obtained from f by pulling back along T (the associated type). Then if $f : X \to \Omega$ and $g : X \to Y$ are given we require that $\Pi g\left(\widetilde{f}\right)$ and $\widetilde{\forall g(f)}$ be naturally isomorphic in $\mathbf{B}/Y$.

This kind of categories will be called "Higher order locally cartesian closed categories" (HLCCC for short). They present connections with topoi which seem worth studying in details.

As a first example remark that any topos is a HLCCC. Take for T the calssifying morphism $\mathbf{1} \to \mathbf{\Omega}$ and when given $f : X \to \Omega$, take for $\Pi g(f)$ the characteristic morphism $Y \to \Omega$ of $\Pi g\left(\widetilde{f}\right)$ which is a mono since $\widetilde{f}$ is a mono.

It should be noticed that those topos-models of the theory of constructions are trivial in the sense that all proofs are collapsed in $\mathrm{Id}_{\mathbf{1}}$. (We refer to [9] for more details about this topic.)

But let us give an example of such a HLCCC.

3.3 The example of $\omega-$Sets

We propose a model of Constructions using the category ω–**Set** that E. Moggi has introduced as a very simple framework for the semantics of second order λ-calculus. We show that this category gives readily rise to a HLCCC structure. Since Moggi's material is still unpublished, we have to provide the definitions. Here are the components of $\omega-$**Set**:

- Objects: $X = (|X|, \vdash_X)$ where $|X|$ is a set and $\vdash_X \subseteq \omega \times |X|$, called *justification* relation, is such that for all $x \in X$ there exists n such that $n \vdash_X x$.

- Arrows: $\mathrm{Hom}_{\omega\text{-}\mathbf{Set}}(X, Y)$ is the set of $f \in \mathrm{Hom}_{\mathbf{Set}}(|X|, |Y|)$ such that

  $$\exists n, \forall x, m \quad m \vdash_X x \Rightarrow nm \vdash_Y f(x)$$

  Using the fact that $\omega-$**Set** is a cartesian closed category, when we have such an integer n for a function f we write $n \vdash_{X \Rightarrow Y} f$. Such a function will sometimes be called a "justified function".

nm denotes the Kleene application in ω and $\langle\langle n, m\rangle\rangle$ the coding of pairs (n, m).

*The first order structure: $\omega-$***Set** *is a LCCC.* – We check first that we have pullbacks. Consider $f : X \to Y$ and $g : Z \to Y$ two morphisms in $\omega-$**Set**. Let U be defined by: $|U| = |X| \underset{f,g}{\times} |Y|$ and $\langle\langle m, n\rangle\rangle \vdash_U (x, z)$ iff $m \vdash_X x$ and $n \vdash_Z z$. The morphisms $f' : U \to Z$ and $g' : U \to X$ are the second and first projections, justified by the Kleene encodings for projections. This defines a pullback diagram in $\omega-$**Set**:

$$\begin{array}{ccc} U & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{f} & U \end{array}$$

Now we show that $\omega-$**Set** is locally closed. We set first a few notations. Let I be any ω-set and F be a $|I|$-indexed family of ω-sets.

We call $S = \sum_{i \in I} F(i)$ the ω-set of which support is $\sum_{i \in |I|} |F(i)|$ with the justifying relation defined by

$$\langle\langle n, m\rangle\rangle \vdash_S (i, x) \Leftrightarrow n \vdash_I i \quad \text{and} \quad m \vdash_{F(i)} x$$

We call $P = \prod_{i \in I} F(i)$ the ω-set of justified elements of the product set $\prod_{i \in |I|} |F(i)|$ with the following justification relation: if $\alpha \in \prod_{i \in |I|} |F(i)|$,

$$p \vdash_P \alpha \quad \text{iff} \quad \forall i \in |I|, \forall n \in \omega \quad n \vdash_I i \Rightarrow pn \vdash_{F(i)} \alpha(i)$$

If $f : X \to Y$ we shall note f^{-1} the $|Y|$-indexed family of ω-sets such that $|f^{-1}(y)| = f^{-1}(\{y\})$, and the justifying relation be the restriction of $\vdash_X$ to this subset of $|X|$.

We define the local exponential first for local objects. Let $f : X \to Y$ be a local object and $g : Y \to Z$ be a morphism (both are justified morphisms of $\omega-$**Set**, by m and n resp.).

$$\Pi g(f) : \sum_{z \in Z} \prod_{y \in g^{-1}(z)} f^{-1}(y) \to Z$$

will be the first projection, justified by its Kleene encoding. Next consider a local morphism $\varphi : f \to f'$ in $\mathbf{B}/Y$ where $f : X \to X$ and $f' : X' \to Y$. We define $\psi = \Pi g(\varphi)$ as follows: For $z \in \mathbf{Z}$ let $U_z = \prod_{y \in g^{-1}(z)} f^{-1}(y)$ and let $U = \sum_{z \in Z} U_z$. For $(z, \alpha) \in U$ we set $\psi(z, \alpha) = (z, \beta)$ where $\beta \in \prod_{y \in g^{-1}(z)} f'^{-1}(y)$ is defined by

$$\beta(y) = \varphi(\alpha(y))$$

since actually $f'(\beta(y)) = f(\alpha(y)) = y$ and for justification, if (n, p) is such that $\langle\langle n, p\rangle\rangle \vdash_U (z, \alpha)$ (with $n \vdash_Z z$ and $p \vdash_{U_z} \alpha$), β is justified by $\lambda m.r(pm)$ where r justifies φ in $X \Rightarrow X'$. Thus ψ is justified by $\lambda\langle\langle n, p\rangle\rangle.\langle\langle n, \lambda m.r(pm)\rangle\rangle$.

The higher order structure – Now comes the real point: we must interpret higher order quantification. We simply generalize a bit the original idea of E. Moggi. $\omega-$**Set** has a very interesting subcategory of partial equivalence relations, a notion dating back to Scott [18]. They are very good candidates for propositions among types. Partial equivalence relations, also called "modest" ω-sets, are ω-sets X satisfying

$$m \vdash_X x, m \vdash_X y \Rightarrow x = y$$

The class of all modest ω-sets is clearly a set PER, indeed, another presentation of modest ω-sets is as equivalence relations defined on a subset of ω whence the alternative name. It can be endowed with a trivial ω-set structure, using the embedding $\Delta : \mathbf{Set} \to \omega-\mathbf{Set}$ defined by: $\Delta(|X|) = (|X|, \omega \times |X|)$. If $R \in$ PER, the ω-set structure of R is given by

$$|R| = \omega / R \quad \text{and} \quad n \vdash_R C \Leftrightarrow n \in C$$

An important observation is that whenever Y is of the form $\Delta(|Y|)$ we have $\mathrm{Hom}_{\omega\text{-}\mathbf{Set}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(|X|, |Y|)$. Indeed any index of a total recursive function justifies any function $f : |X| \to |Y|$. We have to define an object $\Omega \in \mathbf{B} = \omega-\mathbf{Set}$. We take $\Omega = \Delta(\mathrm{PER})$.

For T we take $\Omega' = \sum_{R\in\Omega} R$ and T is the first projection. Thus $|\Omega'| = \sum_{R\in\mathrm{PER}} \omega/R$ and $n \vdash_{\Omega'} (R, C)$ iff $n \in C$.

Now consider any $\varphi : X \to \Omega$ in $\omega-\mathbf{Set}$. Thus φ may be seen as a $|X|$-indexed family of PER's. One easily checks that $\widetilde{X} = \sum_{x\in X} \varphi(x)$ and $\tilde{\varphi}$ is the projection into X. For the higher order impredicative product if $g : X \to Y$ we take $\forall g(\varphi) : Y \to \Omega$ defined by

$$\forall y \in Y \quad \forall g(\varphi)(y) = \prod_{x \in g^{-1}(y)} \varphi(x)$$

This definition makes sense since each $\psi(y) = \prod_{x\in g^{-1}(y)} \varphi(x)$ is a modest ω-set. Actually let $\alpha, \beta \in \psi(y)$ and let $n \in \omega$ be such that $n \vdash_{\psi(y)} \alpha, \beta$. Let $x \in g^{-1}(y)$ and m be such that $m \vdash_{g^{-1}(y)} x$. We have $nm \vdash_{\varphi(x)} \alpha(x), \beta(x)$ but $\varphi(x)$ is modest. Thus $\alpha = \beta$ and we conclude. The remaining details are routine verifications.

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