Correction du devoir # 2

Ex 1 (4 pts)

1) Soient $A = (a_{ij})_{1 \leq i,j \leq n}$ et $B = (b_{ij})_{1 \leq i,j \leq n}$

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$
$$= \operatorname{tr}(BA)$$

2) $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr} A$

Soit $\operatorname{tr} f = \operatorname{tr} \mathcal{M}_{\mathcal{B}}(f)$ pour un certain choix de $\mathcal{B}$ base de V

Comme $\operatorname{tr} \mathcal{M}_{\mathcal{B}'}(f) = \operatorname{tr}(P^{-1}\mathcal{M}_{\mathcal{B}}(f)P) = \operatorname{tr} \mathcal{M}_{\mathcal{B}}(f)$,

le choix de $\mathcal{B}$ n'importe pas dans la définition de $\operatorname{tr} f$.

Ex 2 (4 pts)

$\operatorname{tr} : \mathcal{M}_n(K) \longrightarrow K$

tr est linéaire : $\operatorname{tr}(\lambda A + \mu B) = \lambda \operatorname{tr} A + \mu \operatorname{tr} B$

donc $\operatorname{tr} \in (\mathcal{M}_n(K))^*$

$$T^* : (\mathcal{M}_n(K))^* \longrightarrow (\mathcal{M}_n(K))^*$$
$$\varphi \longmapsto \varphi \circ T$$

$T^*(\operatorname{tr})(A) = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0 \quad \forall A$

donc $T^*(\operatorname{tr}) = 0$

Si $\underline{n > 1}$ $\det \notin (\mathcal{M}_n(K))^*$ car det pas linéaire :

$$\det\left(\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\right) = 1$$

<u>mais</u> $\det\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = 0 + 0 = 0$

Si $\underline{n = 1}$ $\det \in (\mathcal{M}_1(K))^*$

Ex 3 (6 pts)

W^0 sev de V^* : $*\ \forall x \in W,\ 0(x) = 0$ donc $0 \in W^0$

$*\ \forall \varphi, \psi \in W^0, \forall x \in W, (\varphi + \psi)(x) = 0$

donc $\varphi + \psi \in W^0$

$*\ \forall \varphi \in W^0, \forall \lambda \in K, \forall x \in W, (\lambda\varphi)(x) = 0$

donc $\lambda \varphi \in W^0$

On suppose pour la suite que $\mathbb{K}$ est de caractéristique 0

~~[illegible]~~ (ie $n.1 = \underbrace{1 + 1 + \ldots + 1}_{n \text{ fois}} \neq 0 \quad \forall n \in \mathbb{N}$)

et on note $\mathbb{K}_n[X]$ l'espace vectoriel de polynômes de $\mathbb{K}[X]$ de degré inférieur ou égal à n

_ $\operatorname{Ker} D = \{ P \in \mathbb{K}_n[X] \,/\, \frac{dP}{dX} = 0 \} = \mathbb{K}$

Base de Ker D : $\{1\}$ $\Rightarrow$ dimension : 1

_ $\operatorname{Ker} D^* = \{ \varphi \in (\mathbb{K}_n[X])^* \,/\, \forall P \in \mathbb{K}_n[X],\ \varphi(\frac{dP}{dX}) = 0 \}$

$= \{ \lambda_0 e_0^* + \ldots + \lambda_n e_n^* \,/\, \forall P = \sum_{k=0}^{n} a_k X^k,$

$\lambda_0 a_1 + 2\lambda_1 a_2 + \ldots + n\lambda_{n-1} a_n = 0 \}$

où $\{e_0^*, \ldots, e_n^*\}$ est la base duale de $\{e_0, \ldots, e_n\}$

avec $e_0 = 1, e_1 = X, \ldots, e_n = X^n$

donc $\operatorname{Ker} D^* = \{ \lambda_n e_n^* \,/\, \lambda_n \in \mathbb{K} \}$

Base de Ker D^* : $\{e_n^*\}$ $\Rightarrow$ dimension : 1

_ $\operatorname{Im} D = \{ P \in \mathbb{K}_n[X] \,/\, \exists Q \in \mathbb{K}_n[X],\ P = \frac{dQ}{dX} \} = \mathbb{K}_{n-1}[X]$

Base de Im D : $\{1, X, \ldots, X^{n-1}\}$ $\Rightarrow$ dimension : n

$- \operatorname{Im} D^* = \{ \varphi \ / \ \exists \psi = \lambda_0 e_0^* + \dots + \lambda_n e_n^*, \ \varphi = \psi \circ D \}$

$= \{ \varphi \ / \ \exists \lambda_0, \dots, \lambda_n, \ \forall P \in \mathbb{K}_n[X], \ P = \sum_{i=0}^{n} a_i X^i$

$\varphi(P) = \lambda_0 a_1 + 2\lambda_1 a_2 + \dots + n\lambda_n a_{n-1} \}$

$= \{ \varphi \ / \ \exists \mu_0, \dots, \mu_n, \ \forall P \in \mathbb{K}_n[X],$

$\varphi(P) = \mu_0 e_0^*(P) + \mu_1 e_2^*(P) + \dots + \mu_n e_n^*(P) \}$

Base de $\operatorname{Im} D^*$: $\{ e_0^*, \dots, e_n^* \} \Rightarrow$ dimension : n

$- (\operatorname{Im} D)^\circ = \{ \varphi \ / \ \forall P \in \mathbb{K}_{n-1}[X], \ \varphi(P) = 0 \}$

$= \{ \lambda_0 e_0^* + \dots + \lambda_n e_n^* \ / \ \forall a_0 \dots a_{n-1},$

$\lambda_0 a_0 + \dots + \lambda_{n-1} a_{n-1} = 0 \}$

$= \{ \lambda_n e_n^* \ / \ \lambda_n \in \mathbb{K} \}$

Base de $(\operatorname{Im} D)^\circ$: $\{ e_n^* \} \Rightarrow$ dimension : 1

$- (\operatorname{Ker} D)^\circ = \{ \varphi \ / \ \forall P \in \mathbb{K}, \ \varphi(P) = 0 \}$

$= \{ \lambda_0 e_0^* + \dots + \lambda_n e_n^* \ / \ \forall a_0, \ \lambda_0 a_0 = 0 \}$

Base de $(\operatorname{Ker} D)^\circ$: ~~[illegible]~~ $\{ e_1^*, \dots, e_n^* \} \Rightarrow$ dimension : n

$\operatorname{Ker} D^*$ et $(\operatorname{Im} D)^\circ$ ont la même base $\Rightarrow \operatorname{Ker} D^* = (\operatorname{Im} D)^\circ$

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<u>Ex 4</u> (6 pts)

(i) $\mathcal{B} = \{ 1, x, \dots, x^n \}$ $\qquad V = \operatorname{Vect}(\mathcal{B})$

$$\mathcal{M}_{\mathcal{B}}(D) = \begin{array}{c} \begin{array}{ccccc} D(1) & D(x) & D(x^2) & \dots & D(x^n) \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & & 0 \\ | & 0 & 2 & & | \\ | & | & 0 & \ddots & | \\ | & | & | & \ddots & 0 \\ 0 & 0 & | & & n \\ & & 0 & \dots & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ x \\ | \\ x^{n-1} \\ x^n \end{array}$$

$\hookrightarrow \det D = 0$

On sait que $\forall P \in \mathbb{R}_n[X]$, $D^{n+1}(P) = 0$

donc $P = X^{n+1}$ annule D

(ii) $\mathcal{B} = \{\sin x, \cos x\}$ $\quad V = \mathrm{Vect}(\mathcal{B})$

$$\mathcal{M}_{\mathcal{B}}(D) = \begin{array}{cc} D(\sin x) & D(\cos x) \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{array}{l} \sin x \\ \cos x \end{array}$$

$\hookrightarrow \det D = 1$

Si $A = \mathcal{M}_{\mathcal{B}}(D)$, on note que $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

donc $P = X^2 + 1$ annule D

(iii) $\mathcal{B} = \{e^x, e^{2x}, e^{3x}\}$ $\quad V = \mathrm{Vect}(\mathcal{B})$

$$\mathcal{M}_{\mathcal{B}}(D) = \begin{array}{ccc} D(e^x) & D(e^{2x}) & D(e^{3x}) \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{array}{l} e^x \\ e^{2x} \\ e^{3x} \end{array}$$

$\hookrightarrow \det D = 6$

soit $P = X^3 + aX^2 + bX + c$ qui annule D,

on a $D^2 = \begin{pmatrix} 1 & & 0 \\ & 4 & \\ 0 & & 9 \end{pmatrix}$ et $D^3 = \begin{pmatrix} 1 & & 0 \\ & 8 & \\ 0 & & 27 \end{pmatrix}$

donc $\begin{cases} 1 + a + b + c = 0 \\ 8 + 4a + 2b + c = 0 \\ 27 + 9a + 3b + c = 0 \end{cases}$ $\Leftrightarrow \begin{cases} c = -1 - a - b \\ 7 + 3a + b = 0 \\ 26 + 8a + 2b = 0 \end{cases}$

$\Leftrightarrow \begin{cases} c = -1 - a - b \\ b = -7 - 3a \\ 12 + 2a = 0 \end{cases}$

$\Leftrightarrow \begin{cases} a = -6 \\ b = 11 \\ c = -6 \end{cases}$

donc $P = X^3 - 6X^2 + 11X - 6$ annule D