Introduction to ludics

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Introduction

Ludics = a basis for an

- *interactive* (behaviours in context)
- resource-sensitive (both time and *space*)

account of logic.

I see three key ingredients:

- Paraproofs (in Curry-Howard correspondence with *errors*)
- Locations ("Only the location matters")
- Focalisation (abstracted from linear logic)

Two steps:

- A collection of designs (compare with untyped λ-terms)
- Behaviours (and connectives) as suitable sets of designs (compare with types as collections of untyped terms)

Errors – paraproofs

If err is admitted as a semantic value, if F is a procedure with two arguments and if

$$F(err, \perp) = err$$

then this test, or *interaction*

$$\text{between } F \text{ and } (err, \perp)$$

tells us that A needs its first argument (error propagation).

On the other side of Curry-Howard, if a "proof" of A is to interact with a "proof" of $A^\perp$ (negation of A), then one of them is not a proof...

Proofs provide a too scarcy filling of the space

$\rightarrow$ *paraproofs*, which are proofs where you can "give up".

Focalisation 1/6

Consider the multiplicative + additive rules of linear logic:

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \quad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma} \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}$$

- The rules for $\otimes$ and $\oplus$ are *irreversible*: context splitting ($\otimes$), choice of the subformula ($\oplus$).
- The rules for $\parr$ et $\&$ are *reversible*: shared context ($\&$), both subformulas kept ($\parr$).

$$\left.\begin{array}{l} \otimes \text{ and } \oplus \\ \& \text{ and} \parr \end{array}\right\} \text{ are called } \left\{\begin{array}{l} \textbf{positive} \\ \textbf{negative} \end{array}\right.$$

The only rule that augments the number of formulas in a sequent is the one for $\parr$.

$\to$ sequents with at most one negative formula.

Focalisation 2/6

Further evidence of the negative / positive grouping:

- $\otimes$ and $\oplus$ enjoy distributivity:

$$A \otimes (B \oplus C) \Leftrightarrow (A \otimes B) \oplus (A \otimes C)$$

- In contrast, one has

$$A \otimes (B \parr C) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad (A \otimes B) \parr C$$

$$\exists\, x \;\; \forall\, y \;\; P(x, y) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \forall\, y \;\; \exists\, x \;\; P(x, y)$$

No commutation between negative and positive!

Summarizing:

	Pos.	Neg.
Additive	$\oplus$	$\&$
Multiplicative	$\otimes$	$\parr$
Quantifiers	$\exists$	$\forall$

Focalisation 3/6

Grouping positive (resp. negative) connectives. This gives the following *synthetic connectives* (in prenex form):

$$P ::= \cdots \oplus (N_i^1 \otimes \cdots \otimes N_i^{j_i}) \oplus \cdots$$
$$N ::= \cdots \& (P_i^1 \parr \cdots \parr P_i^{j_i}) \& \cdots$$

and rules ($I_m = \{\langle m, 1\rangle, \ldots, \langle m, j_m\rangle\}$):

$$(+, \xi, I_m) \quad \frac{(N_m^1)^\perp \vdash \Lambda_1 \quad \cdots \quad (N_m^{j_m})^\perp \vdash \Lambda_{j_m}}{\vdash P, \Lambda}$$

(the Λ_i's form a partition of Λ)

$$(-, \xi, \{I_1, \ldots, I_n\}) \quad \frac{\vdash P_1^1, \ldots, P_1^{j_1}, \Lambda \quad \cdots \quad \vdash P_n^1, \ldots, P_n^{j_n}, \Lambda}{N^\perp \vdash \Lambda}$$

More on irreversibility / reversibility: choice of P / uniqueness of N.

Focalisation 4/6

Andreoli proved that the focalised rules above are complete:

1. a reversible rule can be applied early;
2. an irreversible one may not be applied too early,
3. but once a positive formula is decomposed, it can be decomposed all the way down.

Illustration of 2. :

$$\dfrac{\dfrac{\vdash A, C \qquad \vdash B, D}{\vdash A \otimes B, C, D}}{\vdash A \otimes B, C \parr D}$$

Focalisation 5/6

Illustration of 1. (negative can be early):

$$\dfrac{\dfrac{\vdash A, C \qquad \vdash A, D}{\vdash A, C \& D} \qquad \vdash B}{A \otimes B, C \& D}$$

can be replaced by:

$$\dfrac{\dfrac{\vdash A, C \qquad \vdash B}{\vdash A \otimes B, C} \qquad \dfrac{\vdash A, D \qquad \vdash B}{\vdash A \otimes B, D}}{\vdash A \otimes B, C \& D}$$

Focalisation 6/6

Illustration of 3 (permutation of positive rules):

$$\dfrac{\dfrac{\vdash A, C, \Gamma_{11} \qquad \vdash B, \Gamma_{12}}{\vdash A \otimes B, C, \Gamma_1 = \Gamma_{11}, \Gamma_{12}} \qquad \vdash D, \Gamma_2}{\vdash A \otimes B, C \otimes D, \Gamma = \Gamma_1, \Gamma_2}$$

can be replaced by:

$$\dfrac{\dfrac{\vdash A, C, \Gamma_{11} \qquad \vdash D, \Gamma_2}{\vdash A, C \otimes D, \Gamma_{11}, \Gamma_2} \qquad \vdash B, \Gamma_{12}}{\vdash A \otimes B, C \otimes D, \Gamma = \Gamma_{11}, \Gamma_{12}, \Gamma_2}$$

Designs 1/12

- *bias* = natural number (relative address)
- *ramification* = finite set I of biases
- *locus* = finite sequence ξ of loci (occurrence, or absolute address)
- *basis* = $\vdash \Lambda$ (positive), $\xi \vdash \Lambda$ (negative) (Λ finite set of loci)

logical information $\rightarrow$ spatial information:

$$\begin{array}{lcl} \text{formulas} & \rightarrow & \text{loci} \\ \text{sequents} & \rightarrow & \text{bases} \end{array}$$

Designs 2/12

Designs-**dessins** are trees built as follows:

- Positive (a premiss for each $i \in I$, Λ_i's disjoint and $\subseteq \Lambda$):

$$(+, \xi, I) \quad \frac{\cdots \quad \xi i \vdash \Lambda_i \quad \cdots}{\vdash \xi, \Lambda}$$

- Negative (a premiss for each $J \in \mathcal{N} \subseteq \mathcal{P}_f(\mathbf{N})$, $\Lambda_J \subseteq \Lambda$):

$$(-, \xi, \mathcal{N}) \quad \frac{\cdots \quad \vdash \xi \star J, \Lambda_J \quad \cdots}{\xi \vdash \Lambda}$$

- *Demon*:

$$\maltese \quad \frac{}{\vdash \Lambda}$$

Notice the $\subseteq$ above: shift to an *affine* framework, i.e., with weakening.

Designs 3/12

Designs as strategies.

$$(+,\xi,I)\quad \cfrac{\overbrace{\cdots\quad \overbrace{\cdots\quad \vdash \xi i\star J,\Lambda_{i,J}\quad \cdots}^{J\in\mathcal{P}_f(\mathbf{N})}\quad \cdots}^{i\in I}}{\vdash \xi,\Lambda}$$

(where Λ_{i_1,J_1}, Λ_{i_2,J_2} disjoint if $i_1 \neq i_2$).

- Player chooses ξ and I.
- Opponent picks an $i \in I$ and an *arbitrary* J.

Note that the premisses are organised as a finite set (i) of denumerable sets (J).

To recover $\mathcal{N} \neq \mathcal{P}_f(\mathbf{N})$, one adds:

$$\Omega\quad \cfrac{}{\vdash \Lambda}$$

Designs 4/12

Designs-**desseins**. Get rid of bases, and name Opponent's moves after a Player's move as $(-, \zeta, J)$ (choice of i such that $\zeta = \xi i$ and $i \in I$, and of J).

Syntax of *pseudo-designs*:

$$\begin{array}{ll} \text{Positive} & \phi ::= \Omega \parallel \maltese \parallel (+, \xi, I) \cdot \{\psi_{\xi i} \mid i \in I\} \\ \text{Negative} & \psi_\zeta ::= \{(-, \zeta, J)\phi_J \mid J \in \mathcal{P}_f(\mathbf{N})\} \end{array}$$

The moves $(+, \xi, I)$ et $(-, \zeta, J)$ are called positive and negative *actions*, respectively.

Designs 5/12

When is a pseudo-design a design, i.e. when does it come from a design-dessin? When a "type" can be infered:

$$\frac{}{\Omega : (\vdash \Lambda)} \qquad \frac{}{\maltese : (\vdash \Lambda)}$$

$$\frac{\cdots\ \psi_{\xi i} : (\xi i \vdash \Lambda_i)\ \cdots \left(\begin{array}{l} i \in I \\ \forall\, i\ \ \Lambda_i \subseteq \Lambda \\ \forall\, i_1, i_2\ \ (i_1 \neq i_2 \Rightarrow \Lambda_{i_1} \cap \Lambda_{i_2} = \emptyset) \end{array}\right)}{(+, \xi, I) \cdot \{\psi_{\xi i} \mid i \in I\} : (\vdash \xi, \Lambda)}$$

$$\frac{\cdots\ \phi_J : (\vdash \zeta \star J, \Lambda_J)\ \cdots \quad \left(\begin{array}{l} J \in \mathcal{P}_f(\mathbf{N}) \\ \forall\, J\ \ \Lambda_J \subseteq \Lambda \end{array}\right)}{\{(-, \zeta, J)\phi_J \mid J \in \mathcal{P}_f(\mathbf{N})\} : (\zeta \vdash \Lambda)}$$

Designs 6/12

Summary:

- Pseudo-designs = raw terms
- Designs-desseins = “typable” terms
- Designs-dessins = proofs of “typing”

Cf. Natural deduction versus its formulation with sequents, proof-nets w.r.t. sequents in linear logic. Here, by focalisation, the viewpoints are very close (isomorphic up to weakening).

Designs 7/12

Designs-desseins as trees with pointers:

- An Opponent's action $(-, \zeta, J)$ points to the immediately preceding Player's action $(+, \xi, I)$: $\zeta = \xi i$ (with $i \in I$).
- A Player's action $(+, \xi, I)$ may point to a non-local Opponent's action.

Cf. Hyland-Ong games. Cf. λ-calculus (coming up).

Designs 8/12

An example: the fax $Fax_{\xi,\xi'}$

- As a dessin:

$$
\dfrac{(-,\xi,\mathcal{P}_f(\mathbf{N})) \qquad \cdots \qquad \dfrac{(+,\xi',J_1) \qquad \dfrac{\cdots \quad Fax_{\xi' j_1, \xi j_1} \quad \cdots}{\vdash \xi \star J_1, \xi'}}{} \qquad \cdots}{\xi \vdash \xi'}
$$

- As a dessein:

$$
\begin{cases}
\vdots \\
(i,J_1)[(i',J_1),\overset{-}{\hookleftarrow}] \begin{cases} \vdots \\ (j_1,J_2)[(j_1,J_2,\overset{1}{\hookleftarrow}] \begin{cases} \vdots \\ (j_2,J_3)[(j_2,J_3,\overset{1}{\hookleftarrow}] \left\{ \vdots \right. \\ \vdots \end{cases} \\ \vdots \end{cases} \\
\vdots
\end{cases}
$$

Designs 9/12

chronicles = sequences of actions

Designs as sets of chronicles:

$$\Omega^\bullet = \emptyset$$

$$✠^\bullet = \{✠\}$$

$$((+, \xi, I) \cdot \{\psi_{\xi i} \mid i \in I\})^\bullet \\ = \bigcup_{i \in I} \{(+, \xi, I) r \mid r \in \psi^\bullet_{\xi i}\}$$

$$\{(-, \zeta, J)\phi_J \mid J \in \mathcal{P}_f(\mathbf{N})\}^\bullet \\ = \bigcup_{J \in \mathcal{P}_f(\mathbf{N})} \{(-, \zeta, J) r \mid r \in \phi^\bullet_J\}$$

$^\bullet$ is an injection.

Designs 10/12

Restricting

• to *slices* = designs-dessins where $\mathcal{N}$ is a singleton at each negative rule:

$$\frac{\phi_J : (\vdash \zeta \star J, \Lambda_J) \quad \begin{pmatrix} J \in \mathcal{P}_f(\mathbf{N}) \\ \forall\, J \;\; \Lambda_J \subseteq \Lambda \end{pmatrix}}{\{(-, \zeta, J)\phi_J\} \cup \{(-, \zeta, K)\Omega \mid K \neq J\} : (\zeta \vdash \Lambda)}$$

• and to affine λ-terms in normal form (with a record-like application, and a corresponding abstraction w.r.t. a set of variables,

one gets a bijection.

Designs 11/12

$$\frac{P : (\vdash \vec{x} : \zeta \star J, \Lambda') \quad (\Lambda' \subseteq \Lambda)}{\lambda\{x_j \mid j \in J\}.P : (\zeta \vdash \Lambda)}$$

$$\frac{\cdots \;\; M_i : (\xi i \vdash \Lambda_i) \;\; \cdots \left(\begin{array}{l} i \in I \\ \forall\, i \;\; \Lambda_i \subseteq \Lambda \\ \forall\, i_1, i_2 \;\; (i_1 \neq i_2 \Rightarrow \Lambda_{i_1} \cap \Lambda_{i_2} = \emptyset) \end{array} \right)}{x\{M_i \mid i \in I\} : (\vdash x : \xi, \Lambda)}$$

Note that:

- The cardinal of J (I) gives the number of head λ's (of arguments).
- If $\xi = \zeta \star j$, then ζ and j identify
  - the binding $\lambda\{x_j \mid j \in J\}$ and
  - the choice of x_j in the window $\{x_j \mid j \in J\}$

  (Cf. Huet, Danos-Regnier).

Designs 12/12

Extending to all designs, we get the following concrete syntax for designs:

$$
\begin{array}{l}
M ::= x \parallel \{J = \lambda\{x_j \mid j \in J\}.P_J \mid J \in \mathcal{P}_f(\omega)\} \\
P ::= (M \cdot I)\{M_i \mid i \in I\} \parallel \Omega \parallel \maltese
\end{array}
$$

Normalisation 1/7

The normalisation of $\phi : (\vdash \xi, \Lambda_1)$ against, or in the context $\psi' : (\xi \vdash \Lambda_2)$ (Λ_1 and Λ_2 disjoint) produces a design of basis $\vdash \Lambda_1, \Lambda_2$.

Notation:

- $\Psi' = \{\psi'_{\xi_1}, \ldots, \psi'_{\xi_n}\}$ finite set of negative designs.
- $(\xi \in \{\xi_1, \ldots, \xi_n\}) = (\Psi'$ *accepts* $\xi)$.
- $\Psi' \setminus \xi = \Psi' \setminus \psi'_\xi$.

Machine states = $\langle \phi | \Psi' \rangle$ (“$\langle$program$|$context$\rangle$”)

Normalisation 2/7

If Ψ' accepts ξ:

$$\text{(R)}\ \ \langle (+,\xi,I)\cdot\Psi \mid \Psi'\rangle \longrightarrow \langle {\Psi'}_{\xi,I} \mid \Psi \cup (\Psi' \setminus \xi)\rangle$$

Initially, $\Psi' = \{\psi'\}$.

Example:

$$\dfrac{\cdots\ \dfrac{\dfrac{\cdots}{(+,\zeta,J)}}{(-,\xi i_1,I_1)}\cdots\dfrac{\cdots}{(-,\xi i_2,I_2)}\cdots}{(+,\xi,I)} \qquad \cdots \qquad \dfrac{\dfrac{\cdots\ \dfrac{\dfrac{\cdots}{(+,\xi i_2,I_2)}}{(-,\zeta,J)}\ \cdots}{(+,\xi i_1,I_1)}}{(-,\xi,I)} \qquad \cdots$$

.

$$\begin{aligned}
\langle \phi \mid \psi' \rangle &\longrightarrow \langle \phi_1 \mid \Psi \cup (\psi' \setminus \xi)\rangle = \langle \phi_1 \mid \Psi \rangle \\
&\longrightarrow \langle \phi_2 \mid \Psi_1 \cup (\Psi \setminus \xi i_1)\rangle \\
&\longrightarrow \langle \phi_3 \mid \Psi_2 \cup (\Psi_1 \setminus \xi) \cup (\Psi \setminus \xi i_1)\rangle
\end{aligned}$$

Normalisation 3/7

Final states (in *head reduction*):

- $\langle \Omega \mid \Psi' \rangle$ (waiting for more information)
- $\langle \maltese | \Psi' \rangle$ (convergence: the two players have agreed – one has given up)
- $\langle (+, \xi_n, I_n) \cdot \Psi_n \mid \Psi'_n \rangle$ where Ψ'_n does not accept ξ_n (cf. λ-calculus' head normal forms).

In all three cases, the root of the normal form of $\langle \phi \mid \psi' \rangle$ has been obtained.

Obviously terminating in linear time (not a surprise = affine).

Normalisation 4/7

Strong reduction: recursive calls of the head machine, in call-by-need (*stream-like* computation)

Each invocation is indexed by the chronicle of the normal form under exploration (initially, index $= \epsilon$).

If Ψ' does not accept ξ and if $i \in I$:

$$\langle (+, \xi, I) \cdot \Psi \mid \Psi' \rangle_q \;\longrightarrow\; \langle \Psi_{\xi i, J} \mid \Psi' \rangle_{q(+,\xi,I)(-,\xi i,J)}$$

$$\langle \Omega \mid \Psi \rangle_q \;\longrightarrow\, !\, q\Omega \qquad \langle \maltese \mid \Psi \rangle_q \;\longrightarrow\, !\, q\maltese$$

The normal form $[\![\phi, \psi']\!]$ is:

$$\begin{array}{l} \{q(+,\xi,I) \mid \langle \phi \mid \psi' \rangle_\epsilon \;\longrightarrow^\star\; \langle \phi_n \mid \Psi_n \rangle_{q(+,\xi,I)(-,\xi i,J)} \cdots\} \\ \cup\ \{q\Omega \mid \langle \phi \mid \psi' \rangle_\epsilon \;\longrightarrow^\star\, !\, q\Omega\} \\ \cup\ \{q\maltese \mid \langle \phi \mid \psi' \rangle_\epsilon \;\longrightarrow^\star\, !\, q\maltese\} \\ \cup\ \{q\Omega \mid \text{computation loops} + q \text{ maximal}\} \end{array}$$

Normalisation 5/7

If $\phi : (\vdash \xi, \Lambda_1)$ and $\psi' : (\xi \vdash \Lambda_2)$, then

$$[\![\phi, \psi']\!] : (\vdash \Lambda_1, \Lambda_2)$$

In particular, when Λ_1 and Λ_2 are empty, then

$$[\![\phi, \psi']\!] : (\vdash)$$

There are only two designs over $\vdash$: Ω and ✠.

Notation:

- $\phi \perp \psi$ if $[\![\phi, \psi]\!] =$ ✠
- $\phi^\perp = \{\psi \mid \phi \perp \psi\}$

Behaviour = set $\mathbf{G}$ of designs such that $\mathbf{G}^{\perp\perp} = \mathbf{G}$

with all designs on the same base. $\mathbf{G}$ is called positive (negative) if the base is positive (negative).

Normalisation 6/7

In concrete syntax, the (affine) machine is a variant of (a stack-free form of) Krivine abstract machine:

$$\langle (x \cdot I)\{M_i \mid i \in I\} \mid \rho \cup (x \leftarrow \{\ldots, I = \lambda\{x_i \mid i \in I\}.P_I, \ldots$$

$$\downarrow$$

$$\langle P_I \mid \rho \cup (\textstyle\bigcup_{i \in I}(x_i \leftarrow M_i)\rangle$$

If one frees oneself from the requirement of affinity (disjoint contexts in the positive rules), then you get the following machine:

$$\langle (x \cdot I)\{M_i \mid i \in I\} \mid \rho\rangle$$

$$\downarrow$$

$$\langle P_I \mid \rho_I \cup (\textstyle\bigcup_{i \in I}(x_i \leftarrow \langle M_i \mid \rho\rangle))\rangle$$

where $\rho(x) = \{\ldots, I = \langle \lambda\{x_i \mid i \in I\}.P_I|\rho_I\rangle, \ldots\}$.

Normalisation 7/7

Fax as *renamer*: the normalization of

$$xM_1(\lambda y_1 y_2.y_1)M_3$$

against (a finite version of the fax)

$$\lambda z_1 z_2 z_3.x' z_1(\lambda u_1 u_2.z_2 u_1 u_2)z_3$$

yields:

$$x' M_1'(\lambda u_1 u_2.u_1)M_3'$$

Analytical theorems 1/2

A partial order on designs:

$$\frac{}{\Omega \sqsubseteq \phi} \qquad \frac{}{\phi \sqsubseteq \maltese}$$

$$\frac{\psi_{\xi i} \sqsubseteq \psi'_{\xi i}}{(+, \xi, I) \cdot \{\psi_{\xi i} \mid i \in I\} \sqsubseteq (+, \xi, I) \cdot \{\psi'_{\xi i} \mid i \in I\}}$$

$$\frac{\cdots \quad \phi_J \sqsubseteq \phi'_J \quad \cdots}{\{\ldots, (-, \zeta, J)\phi_J, \ldots\} \sqsubseteq \{\ldots, (-, \zeta, J)\phi'_J, \ldots\}}$$

Write

- $\phi_1 \sqsubseteq^R \phi_2$ (or $\phi_1 \leq \phi_2$) if $\phi_1 \sqsubseteq \phi_2$ is proved without using $\phi \sqsubseteq \maltese$ (stable ordering),
- $\phi_1 \sqsubseteq^L \phi_2$ if $\phi_1 \sqsubseteq \phi_2$ is proved without using $\Omega \sqsubseteq \phi$ ($\rightarrow$ extensional ordering).

Remark: there is a unique decomposition property:

$$\phi_1 \sqsubseteq \phi_2 \Rightarrow \exists ! \, \phi \;\; \phi_1 \sqsubseteq^L \phi \sqsubseteq^R \phi_2$$

(remove before adding ...).

Analytical theorems 2/2

- *Separation*:

$$\phi_1 \sqsubseteq \phi_2 \quad \Leftrightarrow \quad \phi_1^{\perp} \subseteq \phi_2^{\perp}$$

- *Associativity*:

$$[\![[\![\phi, \Psi_1]\!], \Psi_2]\!] = [\![\phi, \Psi_1 \cup \Psi_2]\!]$$

- *Monotonicity*:

$$[\![\phi_1, \psi]\!] \sqsubseteq [\![\phi_2, \psi]\!]$$

- *Stability*: $r \in [\![\phi, \psi]\!] \Rightarrow$

$\exists\, \phi_0 \leq \phi, \psi_0 \leq \psi$ minimum such that $r \in [\![\phi_0, \psi_0]\!]$

Behaviours 1/2

Alternative definition of behaviour = any set $O^\perp$
(O arbitrary set of designs = set of *tests*)

Example. A record

($\texttt{coord} = (p,q)$, $\texttt{colour} =$ green , $\texttt{shape} =$ circle)

can be encoded:

$$\frac{(+,2,\{n\})}{(-,\epsilon,\{2\})} \qquad \frac{(+,3,\{6\})}{(-,\epsilon,\{3\})} \qquad \frac{(+,8,\{2\})}{(-,\epsilon,\{8\})}$$

(attributing loci 2, 3, and 8 to `coord`, `colour`, and `shape`, and then biases n, 6, and 2 to (p,q), "green", and "circle").

Then the set of all records which have fields `coord`, `colour`, and `shape` is

$$\begin{array}{c}\{(+,\epsilon,\{2\})\cdot\{\{(-,2,J)\maltese \mid J\in\mathcal{P}_f(\mathbf{N})\}\},\\ (+,\epsilon,\{3\})\cdot\{\{(-,3,J)\maltese \mid J\in\mathcal{P}_f(\mathbf{N})\}\},\\ (+,\epsilon,\{8\})\cdot\{\{(-,8,J)\maltese \mid J\in\mathcal{P}_f(\mathbf{N})\}\}\}^\perp\end{array}$$

Behaviours 2/2

If $\mathbf{G}$ is a behaviour and if $\phi \in \mathbf{G}$, the **incarnation** $|\phi|_{\mathbf{G}}$ of ϕ in $\mathbf{G}$ is the smallest design $\leq \phi$ of $\mathbf{G}$.

Example: The incarnation of

$$(\texttt{coord} = (p, q)\ ,\ \texttt{colour} = \text{green}\ ,\ \texttt{shape} = \text{circle})$$

in

$$\begin{array}{c}\{\ (+, \epsilon, \{3\}) \cdot \{\{(-, 3, J)\maltese \mid J \in \mathcal{P}_f(\mathbf{N})\}\}, \\ (+, \epsilon, \{8\}) \cdot \{\{(-, 8, J)\maltese \mid J \in \mathcal{P}_f(\mathbf{N})\}\}\}^{\perp}\end{array}$$

is

$$(\texttt{coord} = (p, q)\ ,\ \texttt{shape} = \text{circle})$$

Additives 1/3

The set-theoretic intersection of two behaviours is a behaviour (cf. intersection types):

$$O_1^\perp \cap O_2^\perp = (O_1 \cup O_2)^\perp$$

For union, one needs to take the biorthogonal:

$$\mathbf{G}_1 \bigsqcup \mathbf{G}_2 = (\mathbf{G}_1 \cup \mathbf{G}_2)^{\perp\perp}$$

Ordinary additive connectives correspond to the special case where the above connectives are applied to *disjoint behaviours*.

Additives 2/3

- The *directory* $Dir(\mathbf{G})$ of a positive behaviour $\mathbf{G}$ is the set of those I's such that $\mathbf{G}$ contains a design starting with $(+, \epsilon, I)$.
- The directory $Dir(\mathbf{G})$ of a negative behaviour is defined by

$$Dir(\mathbf{G}) = Dir(\mathbf{G})^{\perp}$$

- Two behaviours $\mathbf{G}$ and $\mathbf{G'}$ on the same basis are called disjoint if their directories are disjoint.

If $\mathbf{G}$ et $\mathbf{H}$ are negative and disjoint, we write

$$\mathbf{G} \& \mathbf{H} = \mathbf{G} \cap \mathbf{H}$$

If $\mathbf{G}$ et $\mathbf{H}$ are positive and disjoint, we write

$$\mathbf{G} \oplus \mathbf{H} = \mathbf{G} \cup \mathbf{H}$$

(notice that there is no $^{\perp\perp}$ anymore!).

Additives 3/3

If $\mathbf{G}$ and $\mathbf{H}$ are negative and disjoint, then (*mystery of incarnation*):

$$|\mathbf{G} \& \mathbf{H}| = |\mathbf{G}| \times |\mathbf{H}|$$

If $\mathbf{G}$ and $\mathbf{H}$ are negative and disjoint, then

$$\mathbf{G} \cup \mathbf{H} \text{ is a behaviour}$$

- *Disjunction property*: a design in $\mathbf{G} \oplus \mathbf{H}$ is a design in $\mathbf{G}$ or a design in $\mathbf{H}$.
- Viewing $H^{\perp\perp}$ as the set of all designs "valid in all models of H", the property $H = H^{\perp\perp}$ can be read as the *completeness* of H.

Towards Multiplicatives 1/2

Competition for space

$\rightarrow$ different different notions of tensor product.

The (*non-commutative*) right tensor product $\otimes_r$ is defined as follows:

- For two designs ϕ, ϕ' of basis $\vdash \epsilon$

$$\frac{\phi = \maltese}{\phi \otimes_r \phi' = \maltese} \qquad \frac{\phi' = \maltese}{\phi \otimes_r \phi' = \maltese}$$

$$\frac{\phi = (+, \epsilon, I) \cdot \{\psi_i \mid i \in I\} \quad \phi' = (+, \epsilon, J) \cdot \{\psi'_j \mid j \in J\}}{\phi \otimes_r \phi' = (+, \epsilon, I \cup J) \cdot (\{\psi'_j \mid j \in J\} \cup \{\psi_i \mid i \in I \setminus J\})}$$

- If $\mathbf{G}$ and $\mathbf{G}'$ are two positive behaviours, set

$$\mathbf{G} \otimes_r \mathbf{G}' = \{\phi \otimes_r \phi' \mid \phi \in \mathbf{G} \text{ and } \phi' \in \mathbf{G}'\}^{\perp\perp}$$

Towards Multiplicatives 2/2

Notions of (partial) *application* for designs:

- If $\psi : (\epsilon \vdash)$ and $\phi(\vdash \epsilon)$, then there exists a unique negative design $\psi[\phi]$ such that for all $\phi' : (\vdash \epsilon)$:

$$[\![\phi \otimes_r \phi', \psi]\!] = [\![\phi', \psi[\phi]]\!]$$

- If $\psi : (\epsilon \vdash)$ and $\phi' : (\vdash \epsilon)$, then there exists a unique negative design $[\phi']\psi$ such that for all $\phi : (\vdash \epsilon)$:

$$[\![\phi \otimes_r \phi', \psi]\!] = [\![\phi, [\phi']\psi]\!] \ .$$