Uniform and Bernoulli measures
on the boundary of trace monoids

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Abstract

Trace monoids and heaps of pieces appear in various contexts in combinatorics. They also constitute a model used in computer science to describe the executions of asynchronous systems. The design of a natural probabilistic layer on top of the model has been a long standing challenge. The difficulty comes from the presence of commuting pieces and from the absence of a global clock. In this paper, we introduce and study the class of *Bernoulli* probability measures that we claim to be the simplest adequate probability measures on infinite traces. For this, we strongly rely on the theory of trace combinatorics with the Möbius polynomial in the key role. These new measures provide a theoretical foundation for the probabilistic study of concurrent systems.

1—Introduction

Trace monoids are finitely presented monoids with commutation relations between some generators, that is to say, relations of the form $a \cdot b = b \cdot a$.

Trace monoids have first been studied in combinatorics under the name of partially commutative monoids [7]. It was noticed by Viennot that trace monoids are ubiquitous both in combinatorics and in statistical physics [27]. Trace monoids have also attracted a lasting interest in the computer science community, since it was realized that they constitute a model of concurrent systems which are computational systems featuring parallel actions; typically, parallel access to distributed databases or parallel events in networked systems [11, 12]. In a nutshell, the co-occurrence of parallel actions corresponds to the commutation between generators in the trace monoid. The relationship with other models of concurrency has been extensively studied [23, 28]. In particular, in most concurrency models, the executions can be described as regular trace languages, that is to say, regular subsets of trace monoids. Hence trace monoids are among the most fundamental objects of concurrency theory.

There are several motivations for adding a probabilistic layer on top of trace monoids. In the concurrent systems context, it is relevant for network dimensioning and performance evaluation [24, 18]. It is also a question that has been considered for general combinatorial structures since the 80's, and which is crucial for the design of random sampling algorithms [16]. Consider for instance the model checking of asynchronous systems. Such systems are known to suffer from the "state-space explosion" problem. So it is in practice impossible to check for all the trajectories. The key idea in *statistical model checking* is to design testing procedures, relying on random sampling, that provide quantitative guarantees for the fair exploration of trajectories. In this paper, we design

a relevant probabilistic layer at the level of the full trace monoid. This is a first and necessary step, which has to be thoroughly understood, before pushing the analysis further towards the regular trace languages describing the trajectories of concurrent systems.

The elements of a trace monoid are called traces. Traces can be seen as an extended notion of words, where some letters are allowed to commute with each other. Traces carry several notions which are transposed from words. In particular, traces have a natural notion of length, the number of letters in any representative word, and they are partially ordered by the prefix relation inherited from words. A trace monoid can be embedded into a compact metric space where the boundary elements are *infinite traces*, which play the same role with respect to traces than infinite words play with respect to words.

One of our goals is to design a natural and effective notion of "uniform" probability measure on *infinite traces*. Let us illustrate the difficulties that have to be overcome.

An elementary challenge with no elementary solution. Consider the basic trace monoid $\mathcal{M} = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$. For $u \in \mathcal{M}$, denote by $\uparrow u$ the set of all infinite traces for which u is a possible prefix. Does there exist a probability $\mathbb{P}$ on infinite traces which is *uniform*, that is, which satisfies: $\mathbb{P}(\uparrow u) = \mathbb{P}(\uparrow v)$ if u and v have the same length ?

A first attempt consists in performing a random walk on the trace monoid (see [26]). Draw a random sequence of letters in $\{a, b, c\}$, the letters being chosen independently and with probabilities p_a, p_b, p_c. Then consider the infinite trace obtained by concatenating the letters. We invite the reader to check by hand that the probability measure thus induced on infinite traces does not have the uniform property, whatever the choice of p_a, p_b, p_c.

A second attempt consists in considering the well-defined sequence $(\nu_n)_{n \geq 0}$, where ν_n is the uniform measure on the finite set $\mathcal{M}_n = \{u \in \mathcal{M} : |u| = n\}$ of traces of length n. Observe that there are three traces of length 1, which are a, b and c, and eight traces of length 2, obtained from the collection of nine words of length 2 on $\{a, b, c\}$ by identifying the two words ab and ba. In particular, we have:

$$1/3 = \nu_1(a) \; \neq \; \nu_2(aa) + \nu_2(ab) + \nu_2(ac) = 3/8 \,.$$

Hence the pair (ν_1, ν_2) is *not consistent*; and neither is the pair (ν_n, ν_{n+1}) for all $n > 0$. There is therefore no probability measure on infinite traces that induces the family $(\nu_n)_{n \geq 0}$.

We seemingly face the Cornelian dilemma of choosing between consistent but non-uniform probabilities (first attempt), or uniform but non consistent probabilities (second attempt).

Results. In this paper, we prove the existence and uniqueness of the uniform probability measure $\mathbb{P}$ for any irreducible trace monoid—irreducibility corresponds to a connectedness property. The above dilemma is solved by playing on a variable which was thought to be fixed: the total mass of finite marginals. Indeed, the uniform *probability* on infinite traces induces a uniform *measure* on traces of a given length *whose mass exceeds* 1. As for the consistency conditions, they do not hold and they are replaced by compatibility conditions based on the inclusion-exclusion principle.

The uniform measure has the remarkable property of satisfying $\mathbb{P}\big(\uparrow(u{\cdot}v)\big) = \mathbb{P}(\uparrow u)\mathbb{P}(\uparrow v)$ for any traces $u, v \in \mathcal{M}$. More generally, we call Bernoulli measure any probability measure satisfying this identity, which corresponds to a memoryless property on traces. We exhibit infinitely many Bernoulli measures and characterize all of them by means of a finite family of intrinsic parameters obeying polynomial equations. Furthermore, we establish a realization theorem by proving that Bernoulli measures correspond to some particular Markov chains on the Cartier-Foata decomposition of traces. This realization result is a basis for further work on algorithmic sampling, a task that has not been tackled in the literature so far.

The Möbius polynomial associated with the trace monoid appears in all the results. For instance, we establish that the uniform measure satisfies $\mathbb{P}(\uparrow u) = p_0^{|u|}$, for all $u \in \mathcal{M}$, where p_0 is the unique root of smallest modulus of the Möbius polynomial. Also, in the realization result for Bernoulli measures, the relationship between the intrinsic parameters and the transition matrix of the Markov chain is based on a general Möbius transform in the sense of Rota [22]. This highlights the deep combinatorial structure of the probabilistic objects that we construct.

Related work. The uniform measure that we construct is closely related to two classical objects: the Parry measure and the Patterson-Sullivan measure. The *Parry measure* is the measure of maximal entropy on a sofic subshift, that is, roughly, the "uniform" measure on the infinite paths in a finite automaton [19]. Traces can be represented by their Cartier-Foata decompositions which are recognized by a finite automaton having an associated Parry measure. The limitation in this approach is that the link with the combinatorics of the trace monoid remains hidden in the construction. In a sense, our results reveal the inherent combinatorial structure of the Parry measure. The *Patterson-Sullivan measure* is also a uniform measure, which is classically constructed on the boundary at infinity of some geometric groups [17]. The proof of its existence is non-constructive and is based on the Poincaré series of the group, which, in the context of the trace monoid, is simply $\sum_{u \in \mathcal{M}} z^{|u|}$. Using that the Poincaré series of $\mathcal{M}$ is the inverse of the Möbius polynomial, we get an explicit and combinatorial identification of the Patterson-Sullivan measure for trace monoids. Hence our results provide the first discrete framework, outside trees [10], where the Patterson-Sullivan measure is explicitly identified.

Our approach radically differs from the probabilistic techniques found in the computer science literature and related to concurrent systems, such as Rabin's probabilistic automata [21] and their variants, probabilistic process algebra [15], or stochastic Petri nets [14]. All these approaches rely first on a transposition of the asynchronous system into a sequential one, after which a Markov chain structure is typically added. In contrast, we consider the randomization of the elements involving parallelism, and not sequentializations of those elements.

Organization of the paper. The paper is organized in four sections. Section 2 exposes the framework and contains the statements of the results, with no proofs. Section 3 illustrates the results through a study of two concrete examples. It also provides a first immediate application of our constructions to the computation of the "speedup" of trace monoids. Section 4 introduces auxiliary tools. Section 5 is devoted to the proofs of the results stated in Section 2. Last, a concluding section provides perspectives for future work.

3

2—Framework and results

2.1—Trace monoids and their boundary

An *independence pair* is an ordered pair (Σ, I), where Σ is a finite set, referred to as the *alphabet* and whose elements are called *letters*, and $I \subset \Sigma \times \Sigma$ is an irreflexive and symmetric binary relation on Σ. To each independence pair is attached another ordered pair (Σ, D), called the associated *dependence pair*, where D is defined by $D = (\Sigma \times \Sigma) \setminus I$, which is a symmetric and reflexive relation on Σ. Two letters $\alpha, \beta \in \Sigma$ such that $(\alpha, \beta) \in I$ are said to be *parallel*, denoted by $\alpha \parallel \beta$.

To each independence pair (Σ, I) is associated the finitely presented monoid

$$\mathcal{M}(\Sigma, I) = \langle \Sigma \,|\, \alpha \cdot \beta = \beta \cdot \alpha \text{ for } (\alpha, \beta) \in I \rangle.$$

Denoting by Σ^* the free monoid generated by Σ, the monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$ is thus the quotient monoid $\Sigma^*/\mathscr{R}$, where $\mathscr{R}$ is the congruence relation on Σ^* generated by $(\alpha\beta, \beta\alpha)$, for (α, β) ranging over I. Such a monoid $\mathcal{M}$ is called a *trace monoid*, and its elements are called *traces*. The concatenation in $\mathcal{M}$ is denoted by the dot "$\cdot$", the unit element in $\mathcal{M}$, the *empty* trace, is denoted 0. The trace monoid $\mathcal{M}$ is said to be *non-trivial* if $\Sigma \neq \emptyset$. *By convention, we only consider non-trivial trace monoids throughout the paper, even if not specified.* Throughout the paper, we consider a generic trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$.

Viennot's *heap of pieces* interpretation is an enlightening visualization of traces [27]. In this interpretation, a trace is identified with the *heap* obtained from any representative word as follows: each letter corresponds to a piece that falls vertically until it is blocked; a letter is blocked by all other letters but the ones which are parallel to it. We illustrate this in Figure 1 for the example monoid $\mathcal{M}_1 = \langle a, b, c \,|\, a \cdot b = b \cdot a \rangle$.
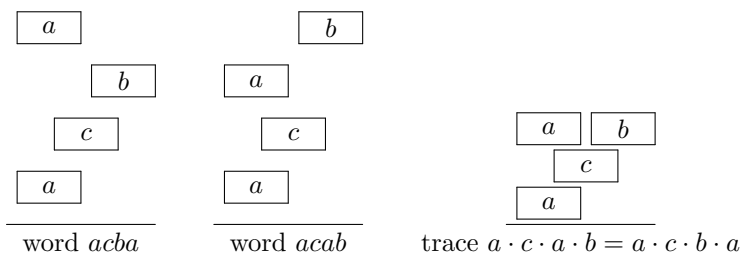

$$\text{word } acba \qquad \text{word } acab \qquad \text{trace } a \cdot c \cdot a \cdot b = a \cdot c \cdot b \cdot a$$

Figure 1: *Two congruent words and the resulting heap (trace) for $\mathcal{M}_1$*

The trace monoid $\mathcal{M}$ is said to be *irreducible* whenever the associated dependence pair (Σ, D), as an undirected graph, is connected. Note that a given trace monoid $\mathcal{M}(\Sigma, I)$ determines the independence pair (Σ, I) up to isomorphism, and hence the dependence pair (Σ, D) as well, which makes the definition of an irreducible trace monoid intrinsic to the monoid.

The notion of independence clique is central in the combinatorics of trace monoids. An element $c \in \mathcal{M}$ is said to be a *clique* if it is of the form $c = \alpha_1 \cdot \ldots \cdot \alpha_n$ for some integer $n \geq 0$ and for some letters $\alpha_1, \ldots, \alpha_n \in \Sigma$ such that $i \neq j \implies \alpha_i \parallel \alpha_j$. The set of cliques is denoted $\mathscr{C}_{\mathcal{M}}$, or simply $\mathscr{C}$. The

set $\mathscr{C} \setminus \{0\}$ of non-empty cliques is denoted $\mathfrak{C}_{\mathcal{M}}$, or $\mathfrak{C}$. Noting that a letter may occur at most once in any representative word of a clique, we identify a clique with the set of letters occurring in any of its representative words. In the heap representation, each layer of a heap is a clique.

Two cliques $c, c' \in \mathscr{C}$ are said to be *parallel* whenever $c \times c' \subseteq I$, which is denoted $c \parallel c'$. This relation extends to cliques the parallelism relation defined on letters. Observe that, since I is supposed to be irreflexive, two parallel cliques are necessarily disjoint.

For each clique $c \in \mathscr{C}$, we consider the sub-monoid $\mathcal{M}_c \subseteq \mathcal{M}$ defined as follows:

$$\Sigma_c = \{\alpha \in \Sigma \,:\, \alpha \parallel c\}, \quad I_c = I \cap (\Sigma_c \times \Sigma_c), \quad \mathcal{M}_c = \mathcal{M}(\Sigma_c, I_c). \quad (1)$$

So for instance, $\Sigma_0 = \Sigma$ and $\mathcal{M}_0 = \mathcal{M}$; while $\Sigma_c = \emptyset$ if c is a maximal clique, and then $\mathcal{M}_c = \{0\}$.

The *length* of a trace $u \in \mathcal{M}$ is defined as the length of any of its representative words in the free monoid Σ^*, and is denoted by $|u|$. Obviously, length is additive on traces, and 0 is the unique trace of length 0. The length of a trace corresponds to the number of pieces in the associated heap.

We consider the left divisibility relation of $\mathcal{M}$, denoted $\leq$, and defined by:

$$\forall u, v \in \mathcal{M} \quad u \leq v \iff \exists w \in \mathcal{M} \quad v = u \cdot w.$$

Trace monoids are cancellative [7]. This justifies the notation $v - u$ to denote the unique trace $w \in \mathcal{M}$ such that $v = u \cdot w$ whenever $u \leq v$ holds. The two properties mentioned above for the length of traces imply that $(\mathcal{M}, \leq)$ is a partial order.

Informally, infinite traces correspond to heaps with a countably infinite number of pieces. Following [1], a formal way to define infinite traces associated to $\mathcal{M}$ is to consider the completion of $\mathcal{M}$ with respect to least upper bound (*l.u.b.*) of non-decreasing sequences in $(\mathcal{M}, \leq)$. Say that a sequence $(u_k)_{k \geq 0}$ is *non-decreasing* in $\mathcal{M}$ if $u_k \leq u_{k+1}$ holds for all integers $k \geq 0$. Let $(\mathcal{H}, \preccurlyeq)$ be the pre-ordered set of all non-decreasing sequences in $\mathcal{M}$ equipped with the Egli-Milner pre-ordering relation, defined as follows:

$$(u_k)_{k \geq 0} \preccurlyeq (u'_k)_{k \geq 0} \iff \forall k \geq 0 \quad \exists k' \geq 0 \quad u_k \leq u'_{k'}.$$

Finally, let $(\mathcal{W}, \leq)$ be the collapse partial order associated with $(\mathcal{H}, \preccurlyeq)$. The elements of $\mathcal{W} = \mathcal{W}(\Sigma, I)$ are called *generalized traces*. Intuitively, any non-decreasing sequence in $\mathcal{M}$ defines a generalized trace, and two such sequences are identified whenever they share the same *l.u.b.* in a universal *l.u.b.*-completion of $\mathcal{M}$. In particular, there is a natural embedding of partial orders $\iota : \mathcal{M} \to \mathcal{W}$ which associates to each trace $u \in \mathcal{M}$ the generalized trace represented by the constant sequence, equal to u. In the heap model, generalized traces correspond to heaps with countably many pieces, either finitely or infinitely many.

By construction, any generalized trace is the *l.u.b.* in $\mathcal{W}$ of a non-decreasing sequence of traces in $\mathcal{M}$. Furthermore, $(\mathcal{W}, \leq)$ is shown to be closed with respect to *l.u.b.* of non-decreasing sequences, and also to enjoy the following compactness property: for every trace $u \in \mathcal{M}$ and for every non-decreasing sequence $(u_k)_{k \geq 0}$ in $\mathcal{M}$ such that $\bigvee \{u_k \,:\, k \geq 0\} \geq u$ holds in $\mathcal{W}$, there exists

an integer $k \geq 0$ such that $u_k \geq u$ holds in $\mathcal{M}$. This property is used to reduce problems concerning generalized traces to problems concerning traces.

The *boundary* of $\mathcal{M}$ is defined as a measurable space $(\partial \mathcal{M}, \mathfrak{F})$. The set $\partial \mathcal{M}$ is defined by $\partial \mathcal{M} = \mathcal{W} \setminus \mathcal{M}$, the set of *infinite traces*. For any trace $u \in \mathcal{M}$, the *elementary cylinder of base* u is the non-empty subset of $\partial \mathcal{M}$ defined by

$$\uparrow u = \{\xi \in \partial \mathcal{M} \ : \ u \leq \xi\} \, ;$$

and $\mathfrak{F}$ is the σ-algebra on $\partial \mathcal{M}$ generated by the countable collection of all elementary cylinders.

2.2—Finite measures on the boundary

In this section, we point out two basic facts which are valid for any finite measure on the boundary of a trace monoid $\mathcal{M}$.

First, it is known, see [4, p. 150], that any two traces $u, v \in \mathcal{M}$ have a *l.u.b.* $u \vee v$ in $\mathcal{M}$ if and only if there exists a trace $w \in \mathcal{M}$ such that $u \leq w$ and $v \leq w$, in which case u and v are said to be *compatible*. Using the compactness property mentioned above, we deduce:

$$\forall u, v \in \mathcal{M} \quad \uparrow u \cap \uparrow v = \begin{cases} \uparrow(u \vee v), & \text{if } u \text{ and } v \text{ are compatible,} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2}$$

In particular, two different elementary cylinders $\uparrow u$ and $\uparrow v$ may have a non-empty intersection, even if u and v have the same length. For instance, for the monoid $\mathcal{M}_1 = \langle a, b, c \, | \, a \cdot b = b \cdot a \rangle$, we have the identity $\uparrow a \cap \uparrow b = \uparrow(a \cdot b)$.

It follows from (2) that the collection of elementary cylinders, to which is added the empty set, is closed under finite intersections: this collection forms thus a π-system, which generates $\mathfrak{F}$. This implies that a *finite measure on* $(\partial \mathcal{M}, \mathfrak{F})$ *is entirely determined by its values on elementary cylinders.*

Second, we highlight a relation satisfied by any finite measure on the boundary (proof postponed to § 5.1).

• **Proposition 2.1**—*Let λ be a finite measure defined on the boundary $(\partial \mathcal{M}, \mathfrak{F})$ of a trace monoid $\mathcal{M}$. Then:*

$$\forall u \in \mathcal{M} \qquad \sum_{c \in \mathscr{C}_{\mathcal{M}}} (-1)^{|c|} \lambda\big(\uparrow(u \cdot c)\big) = 0 \, . \tag{3}$$

If $\mathcal{M} = \Sigma^*$ is a free monoid, corresponding to the trivial independence relation $I = \emptyset$, then the only non-empty cliques are the letters of the alphabet Σ. In this case, if λ_k is the marginal distribution of λ on the words of length $k \geq 0$, the relation (3) is equivalent to $\lambda_k(u) = \sum_{\alpha \in \Sigma} \lambda_{k+1}(u \cdot \alpha)$, the usual consistency relation between marginals. For general trace monoids however, the sum in (3) contains terms for cliques of length ≥ 2. This relates with (2).

2.3—Valuations and Bernoulli measures

Our central object of study is introduced in the following definition. Throughout the paper, $\mathbb{R}_+^*$ denotes the set of positive reals.

• **Definition 2.2**—*Let $\mathcal{M}$ be a trace monoid. We say that a probability measure $\mathbb{P}$ on $(\partial\mathcal{M}, \mathfrak{F})$ is a* Bernoulli measure *if it satisfies:*

$$\forall u \in \mathcal{M} \qquad \mathbb{P}(\uparrow u) > 0 \,, \tag{4}$$

$$\forall u, v \in \mathcal{M} \qquad \mathbb{P}\big(\uparrow (u \cdot v)\big) = \mathbb{P}(\uparrow u)\mathbb{P}(\uparrow v) \,. \tag{5}$$

The characteristic numbers *of $\mathbb{P}$ are defined by $p_\alpha = \mathbb{P}(\uparrow\alpha)$ for $\alpha \in \Sigma$.*

A Bernoulli measure $\mathbb{P}$ is entirely characterized by its characteristic numbers since, by (5), the value of $\mathbb{P}$ on all elementary cylinders is determined by the characteristic numbers. The characteristic numbers appear thus as the natural family of parameters of a Bernoulli measure.

The main property of Bernoulli measures, condition (5), corresponds to a *memoryless* property on traces. Note that, if $\mathcal{M} = \Sigma^*$ is a free monoid, then $(\partial\mathcal{M}, \mathfrak{F})$ is the standard sample space of infinite sequences with values in Σ, and measures satisfying (5) are indeed the standard Bernoulli measures corresponding to *i.i.d.* processes.

Condition (4) is there for convenience and does not involve any loss of generality. Indeed, it will be satisfied when restricting ourselves to the sub-monoid generated by those $\alpha \in \Sigma$ such that $p_\alpha > 0$.

Say that a function $f : \mathcal{M} \to \mathbb{R}_+^*$ which satisfies:

$$\forall u, v \in \mathcal{M} \quad f(u \cdot v) = f(u)f(v) \,, \tag{6}$$

is a *valuation*, and we insist that f only takes positive values. The numbers defined by $p_\alpha = f(\alpha)$ are called the *characteristic numbers* of the valuation, and it is readily seen that for any family of positive numbers $(q_\alpha)_{\alpha\in\Sigma}$, there exists a unique valuation with $(q_\alpha)_{\alpha\in\Sigma}$ as characteristic numbers. By definition, if $\mathbb{P}$ is a Bernoulli measure on $(\partial\mathcal{M}, \mathfrak{F})$, then the function $u \in \mathcal{M} \mapsto \mathbb{P}(\uparrow u)$ is a valuation, that is said to be *induced by* $\mathbb{P}$.

We recall next the notion of Möbius polynomial and the notion of Möbius transform of functions. The general notion of Möbius transform for partial orders has been introduced by Rota [22], and we particularize it to trace monoids.

Considering a trace monoid $\mathcal{M}$ and any real-valued function $f : \mathscr{C} \to \mathbb{R}$, the *Möbius transform* of f is the function $h : \mathscr{C} \to \mathbb{R}$ defined by:

$$\forall c \in \mathscr{C} \qquad h(c) = \sum_{c' \in \mathscr{C} \,:\, c \leq c'} (-1)^{|c'|-|c|} f(c') \,. \tag{7}$$

By convention, if $f : \mathcal{M} \to \mathbb{R}_+^*$ is a valuation, the Möbius transform of f is defined as the Möbius transform of its restriction $f\big|_{\mathscr{C}}$.

For each letter $\alpha \in \Sigma$, let X_α be a formal indeterminate, and let $\mathbb{Z}[\Sigma]$ be the ring of polynomials over $(X_\alpha)_{\alpha\in\Sigma}$. The *multi-variate Möbius polynomial* associated to (Σ, I) is $\mu_{\mathcal{M}} \in \mathbb{Z}[\Sigma]$ defined by:

$$\mu_{\mathcal{M}} = \sum_{c \in \mathscr{C}_{\mathcal{M}}} (-1)^{|c|} \prod_{\alpha \in c} X_\alpha \,. \tag{8}$$

The *evaluation* of the polynomial $\mu_{\mathcal{M}}$ over a family of real numbers $(p_\alpha)_{\alpha\in\Sigma}$ is obtained by substituting the real numbers p_α to the indeterminates X_α in the above expression. The result is denoted $\mu_{\mathcal{M}}\big((p_\alpha)_{\alpha\in\Sigma}\big)$.

When considering a valuation $f : \mathcal{M} \to \mathbb{R}_+^*$, with characteristic numbers $(p_\alpha)_{\alpha \in \Sigma}$, the Möbius transform h of f has the following simple expression involving the evaluation of Möbius polynomials:

$$\forall c \in \mathscr{C} \quad h(c) = f(c)\, \mu_{\mathcal{M}_c}\big((p_\alpha)_{\alpha \in \Sigma_c}\big), \tag{9}$$

where $\mathcal{M}_c$ is the sub-monoid defined in (1). The expression (9) derives immediately from the change of variable $c' = c \cdot \delta$, for δ ranging over $\mathscr{C}_{\mathcal{M}_c}$, in the defining sum (7) for $h(c)$, and using the multiplicative property (6). Two particular instances of (9) shall be noted: for $c = 0$, we obtain $h(0) = \mu_{\mathcal{M}}\big((p_\alpha)_{\alpha \in \Sigma}\big)$ since $f(0) = 1$, and if $c \in \mathscr{C}_{\mathcal{M}}$ is maximal then $h(c) = f(c)$ since $\mathscr{C}_{\mathcal{M}_c} = \{0\}$.

• **Definition 2.3**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid. A valuation $f : \mathcal{M} \to \mathbb{R}_+^*$ is a* Möbius valuation *if its Möbius transform $h : \mathscr{C}_{\mathcal{M}} \to \mathbb{R}$ satisfies the following two conditions:*

$$(a) \quad h(0) = 0\,, \qquad\qquad (b) \quad \forall c \in \mathfrak{C}_{\mathcal{M}} \quad h(c) > 0\,. \tag{10}$$

Equivalently, if $(p_\alpha)_{\alpha \in \Sigma}$ are the characteristic numbers of f, then f is a Möbius valuation if and only if:

$$(a) \quad \mu_{\mathcal{M}}\big((p_\alpha)_{\alpha \in \Sigma}\big) = 0\,, \qquad (b) \quad \forall c \in \mathfrak{C}_{\mathcal{M}} \quad \mu_{\mathcal{M}_c}\big((p_\alpha)_{\alpha \in \Sigma_c}\big) > 0\,. \tag{11}$$

Our first result transfers the initial problem of determining Bernoulli measures to the new problem of determining Möbius valuations.

• **Theorem 2.4**—*Let $\mathcal{M}(\Sigma, I)$ be an irreducible trace monoid. Then:*

1. *The valuation induced by any Bernoulli measure on the boundary of $\mathcal{M}$ is a Möbius valuation.*

2. *If $f : \mathcal{M} \to \mathbb{R}_+^*$ is a Möbius valuation, there exists a unique Bernoulli measure $\mathbb{P}$ on $(\partial\mathcal{M}, \mathfrak{F})$ such that $\mathbb{P}(\uparrow u) = f(u)$ for all $u \in \mathcal{M}$.*

Although Theorem 2.4 provides valuable information, it does not state the existence of Möbius valuations—and thus of Bernoulli measures. We will give a positive result on this point in § 2.5.

The basic relations (3), valid for any finite measure, when applied to a Bernoulli measure $\mathbb{P}$, reduce to the following:

$$\forall u \in \mathcal{M} \quad \sum_{c \in \mathscr{C}} (-1)^{|c|} \mathbb{P}\big(\uparrow(u \cdot c)\big) = 0\,, \text{ hence } \sum_{c \in \mathscr{C}} (-1)^{|c|} \mathbb{P}(\uparrow c) = 0\,.$$

Developing each $\mathbb{P}(\uparrow c)$ as a product of characteristic numbers p_α for α ranging over c, yields the relation $\mu_{\mathcal{M}}\big((p_\alpha)_{\alpha \in \mathcal{M}}\big) = 0$, proving that point (a) in (11) is a necessary condition for $\mathbb{P}$ to be a Bernoulli measure. This is the only elementary part in the proof of Theorem 2.4; the rest of the proof is postponed to § 5.1 for point 1 and to § 5.2 for point 2.

2.4—Cartier-Foata subshift and acceptor graph

We introduce now a subshift of finite type based on the Cartier-Foata decomposition of traces (only elementary notions related to subshifts will be used, and proper definitions will be recalled when needed). This is the starting point for a *realization result* in which Bernoulli measures are described as the law of the trajectories of a Markov chain.

Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid, and let $D \subseteq \Sigma \times \Sigma$ be the associated dependence relation. A pair $(c, c') \in \mathscr{C}_\mathcal{M} \times \mathscr{C}_\mathcal{M}$ of cliques is said to be *Cartier-Foata admissible*, denoted $c \to c'$, if:

$$\forall \beta \in c' \quad \exists \alpha \in c \quad (\alpha, \beta) \in D \,.$$

For every non-empty trace $u \in \mathcal{M}$, there exists a unique integer $n \geq 1$ and a unique sequence of non-empty cliques $(c_1, \ldots, c_n)$ such that $u = c_1 \cdot \ldots \cdot c_n$ and $c_i \to c_{i+1}$ holds for all i in $\{1, \ldots, n-1\}$. This sequence of cliques, denoted $c_1 \to \ldots \to c_n$, is called the *Cartier-Foata decomposition* or *Cartier-Foata normal form* of u [7, 27]. The integer n is called the *height* of the trace x, denoted by $n = \tau(x)$.

In the heap interpretation, this sequence of cliques corresponds to the successive layers of pieces in the heap.

The *Cartier-Foata acceptor graph* is the graph $(\mathfrak{C}, \to)$. The *Cartier-Foata subshift* of $\mathcal{M}$ is the set of right-infinite paths in the graph $(\mathfrak{C}, \to)$. (It is a "subshift of finite type" in the terminology of symbolic dynamics.) See for instance Figure 2 in § 3.1 for a concrete example of a Cartier-Foata acceptor graph. Denote by $(\Omega, \mathfrak{G})$ the measurable space corresponding to the right-infinite paths in the graph $(\mathfrak{C}, \to)$. Hence elements ω of Ω are given by infinite sequences $(c_k)_{k \geq 1}$ of non-empty cliques such that $c_k \to c_{k+1}$ holds for all $k \geq 1$, and $\mathfrak{G}$ is the σ-algebra of Ω induced by the product σ-algebra, where $\mathfrak{C}$ is equipped with the discrete σ-algebra.

The Cartier-Foata decomposition result can be rephrased as the fact that the finite paths in the graph $(\mathfrak{C}, \to)$ are in bijection with the traces of the monoid. More precisely, for each integer $k \geq 0$, traces of height k are in bijection with paths of length k in the graph.

In the same way, infinite paths of $(\mathfrak{C}, \to)$ correspond naturally to the points of the boundary of the monoid. We postpone the proof of this result to § 4.1 and admit for the time being that there exists a bi-measurable bijection $\Psi : \partial\mathcal{M} \to \Omega$, which associates to each point of the boundary $\xi \in \partial\mathcal{M}$ an infinite sequence $(c_k)_{k \geq 1}$ with values in $\mathfrak{C}_\mathcal{M}$, and entirely characterized by the following two properties:

$$\forall k \geq 1 \quad c_k \to c_{k+1} \,, \qquad \bigvee_{k \geq 1} (c_1 \cdot \ldots \cdot c_k) = \xi \,. \qquad (12)$$

The bijection $\Psi : \partial\mathcal{M} \to \Omega$ induces a bijection $\Psi_* : \mathbf{M}_1(\partial\mathcal{M}, \mathfrak{F}) \to \mathbf{M}_1(\Omega, \mathfrak{G})$ between the associated sets of probability measures. We shall always use this identification.

Both spaces $\partial\mathcal{M}$ and Ω come equipped with their own elementary cylinders: $\uparrow u$ with $u \in \mathcal{M}$ for $\partial\mathcal{M}$, and $\{\omega \in \Omega \;:\; C_1(\omega) = c_1, \ldots, C_n(\omega) = c_n\}$ with

$c_1 \to \ldots \to c_n$ for Ω. If $\mathbb{P}$ and $\mathbb{Q}$ are probability measures on $\partial\mathcal{M}$ and Ω related by $\mathbb{Q} = \Psi_*\mathbb{P}$, the effective correspondence between the values of $\mathbb{P}$ and $\mathbb{Q}$ on their respective elementary cylinders is non-trivial. For instance $\mathbb{P}(\uparrow u)$ differs in general from $\mathbb{Q}(C_1 = c_1, \ldots, C_n = c_n)$ where $c_1 \to \ldots \to c_n$ is the Cartier-Foata decomposition of u. The correspondence between cylinders is established in details in § 4.1.

- **Theorem 2.5**—*Let $\mathcal{M}$ be an irreducible trace monoid.*

  1. *Assume that $\mathbb{P}$ is a Bernoulli measure on $(\partial\mathcal{M}, \mathfrak{F})$, with $f_{\mathbb{P}}(\cdot) = \mathbb{P}(\uparrow\cdot)$ the induced valuation. Then, under probability $\mathbb{P}$, the sequence $(C_k)_{k\geq 1}$ of Cartier-Foata cliques is an irreducible and aperiodic Markov chain with values in $\mathfrak{C}$. The law of C_1 is the restriction to $\mathfrak{C}$ of the Möbius transform $h : \mathscr{C} \to \mathbb{R}$ of $f_{\mathbb{P}}$, and $h > 0$ on $\mathfrak{C}$. The transition matrix of the chain is $P = (P_{c,c'})_{(c,c')\in\mathfrak{C}\times\mathfrak{C}}$ given by:*

  $$P_{c,c'} = \begin{cases} h(c')/g(c), & \text{if } c \to c' \\ 0, & \text{if } \neg(c \to c') \end{cases} \quad \text{with } g(c) = \sum_{c'\in\mathfrak{C}\,:\,c\to c'} h(c')\,. \quad (13)$$

  Furthermore, for any integer $n \geq 1$, if $c_1, \ldots, c_n$ are n non-empty cliques such that $c_1 \to \ldots \to c_n$ holds, then:

  $$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f_{\mathbb{P}}(c_1) \cdots f_{\mathbb{P}}(c_{n-1})h(c_n)\,. \quad (14)$$

  2. *Conversely, let $f : \mathcal{M} \to \mathbb{R}_+^*$ be a Möbius valuation, and let $h : \mathscr{C} \to \mathbb{R}$ be the Möbius transform of f. Then the restriction $h\big|_{\mathfrak{C}}$ is a probability distribution on $\mathfrak{C}$. The Markov chain on $\mathfrak{C}$ with $h\big|_{\mathfrak{C}}$ as initial law, and with transition matrix P given as in (13) above, induces a Bernoulli measure $\mathbb{P}$ on $\partial\mathcal{M}$ which satisfies $\mathbb{P}(\uparrow u) = f(u)$ for all traces u in $\mathcal{M}$.*

It is worth observing that $h\big|_{\mathfrak{C}}$ is *not* the stationary distribution of P, implying that $(C_n)_{n\geq 1}$ is not stationary with respect to n under $\mathbb{P}$. Markovian measures with the property (14) also appear in the context of random walks on some infinite groups where they are called "Markovian multiplicative" [20].

The proof of Theorem 2.5 is postponed to § 5.1 for point 1 and to § 5.2 for point 2.

2.5—Uniform measures

So far we have obtained polynomial normalization conditions for the characteristic numbers of Bernoulli measures (§ 2.3) and we have identified Bernoulli measures with certain Markov measures on a combinatorial subshift (§ 2.4). The reader might have noticed that the actual existence of Bernoulli measures has not yet been stated.

In this subsection we state the existence of *uniform Bernoulli* measures, those having all their characteristic numbers identical. We also introduce the weaker notion of uniform measure. An equivalence between uniform measures and uniform Bernoulli measures is stated—a non-trivial result. Then we show how small deformations of the characteristic numbers around the particular value for the uniform measure lead to a continuum of distinct Bernoulli measures.

Let $\mathcal{M}$ be an irreducible trace monoid, and assume there exists a Bernoulli measure for which all characteristic numbers are equal, say to some real $p > 0$. Then, according to Theorem 2.4 point 1, and using the formulation stated in (11)–(a), the number p must be a root of the *single-variable Möbius polynomial* $\mu_{\mathcal{M}}(X) \in \mathbb{Z}[X]$ defined by:

$$\mu_{\mathcal{M}}(X) = \sum_{c \in \mathscr{C}} (-1)^{|c|} X^{|c|}. \qquad (15)$$

We therefore face two questions. First, among the roots of $\mu_{\mathcal{M}}(X)$, which ones correspond indeed to a Bernoulli measure? Such measures, and we shall prove their existence, we call *uniform Bernoulli measures*.

Obviously, any uniform Bernoulli measure satisfies the following property:

$$\forall u, v \in \mathcal{M} \qquad |u| = |v| \implies \mathbb{P}(\uparrow u) = \mathbb{P}(\uparrow v). \qquad (16)$$

We emphasize that the above property is purely metric. Say that a probability measure on $\partial \mathcal{M}$ satisfying (16) is *uniform*. The second question is: does any uniform measure belong to the class of Bernoulli measures? In other words, does (16) imply the memoryless property $\mathbb{P}\big(\uparrow(u \cdot v)\big) = \mathbb{P}(\uparrow u)\mathbb{P}(\uparrow v)$? Note that the answer is clearly affirmative in the case of a free monoid, but much less trivial for a trace monoid.

Next theorem brings answers to the two above questions. The statement requires the knowledge of the following fact, which will be given in an even more precise form below in Theorem 4.7: *the Möbius polynomial of an independence pair (Σ, I) has a unique root of smallest modulus. This root is real and lies in the open interval $(0, 1)$.*

• **Theorem 2.6**—*Let $\mathcal{M}$ be an irreducible trace monoid, and let p_0 be the unique root of smallest modulus of the Möbius polynomial $\mu_{\mathcal{M}}(X)$. Then:*

1. *There exists a unique uniform Bernoulli measure $\mathbb{P}_0$ on $(\partial \mathcal{M}, \mathfrak{F})$. It is entirely characterized by $\mathbb{P}_0(\uparrow u) = p_0^{|u|}$.*

2. *Any uniform measure is Bernoulli uniform. Hence $\mathbb{P}_0$ is also the unique uniform measure on $(\partial \mathcal{M}, \mathfrak{F})$.*

Point 2 in Theorem 2.6 appears as a confirmation of the central role of Bernoulli measures.

Having identified at least one Bernoulli measure for each irreducible trace monoid, we are able to construct many others by considering small variations around the value $(p_0, \ldots, p_0)$ for the family of characteristic numbers.

• **Proposition 2.7**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be irreducible with $|\Sigma| > 1$. There exists a continuous family of different Bernoulli measures on $\partial \mathcal{M}$.*

The proofs of Theorem 2.6 and of Proposition 2.7 are postponed to § 5.3.

3—Examples and Applications

3.1—Two illustrative examples

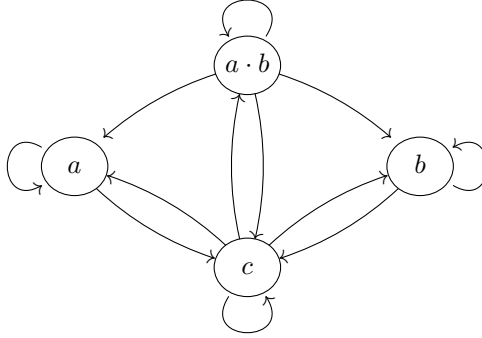We illustrate the above results on two specific examples.

Figure 2: *Cartier-Foata acceptor graph of* $\mathcal{M}_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$

We first concentrate on $\mathcal{M}_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$, and provide a complete description of associated Bernoulli measures. Non-empty cliques of $\mathcal{M}_1$ are given by $\{a, b, c, a \cdot b\}$, and the Cartier-Foata acceptor graph is depicted in Figure 2. The Möbius polynomials of $\mathcal{M}_1$ is $\mu_{\mathcal{M}_1}(X) = 1 - 3X + X^2$, with roots $(3 \pm \sqrt{5})/2$. Keeping the same symbols a, b, c to denote the characteristic numbers of a generic valuation $f : \mathcal{M}_1 \to \mathbb{R}_+^*$, the Möbius transform h of f is given as follows:

clique γ	0	a	b	c	$a \cdot b$
valuation $f(\gamma)$	1	a	b	c	ab
Möbius transform $h(\gamma)$	$1 - a - b - c + ab$	$a - ab$	$b - ab$	c	ab

Möbius valuations are thus determined, according to Definition 2.3, by the following conditions on parameters:

$$1 - a - b - c + ab = 0, \quad a(1 - b) > 0, \quad b(1 - a) > 0, \quad c > 0, \quad ab > 0.$$
$$(17)$$

It follows from Theorem 2.4 that Bernoulli measures on $\mathcal{M}_1$ are in bijective correspondence with the set of triples $(a, b, c) \in (\mathbb{R}_+^*)^3$ solutions of (17). The set of admissible triples forms a surface, a plot of which is given in Figure 3.

We can easily compute the transition matrix P associated with a generic Bernoulli measure with parameters (a, b, c) solution of (17). The normalization factor g from Theorem 2.5 is equal to 1 on all maximal cliques, which are $a \cdot b$ and c. We observe that $g(a) = a - ab + c = 1 - b$, taking into account that $1 - a - b - c + ab = 0$. Similarly, $g(b) = 1 - a$. According to formula (13) stated in Theorem 2.5, and indexing the rows and columns of the transition matrix according to the cliques $(a, b, c, a \cdot b)$ in this order, we get:

$$P = \begin{pmatrix} a & 0 & 1 - a & 0 \\ 0 & b & 1 - b & 0 \\ a - ab & b - ab & c & ab \\ a - ab & b - ab & c & ab \end{pmatrix}$$

It is readily checked by hand on this example, and this is true in general, that conditions (17) ensure that the above matrix has all its entries non negative and that all lines sum up to 1.

According to Theorem 2.6, the only uniform measure $\mathbb{P}_0$ on $\mathcal{M}_1$ is determined by the root $p_0 = (3 - \sqrt{5})/2 = 0.382 \cdots$ of $\mu_{\mathcal{M}_1}$, and satisfies

$\mathbb{P}_0(\uparrow u) = p_0^{|u|}$ for all u in $\mathcal{M}$. Note that, for this example, the other root of $\mu_{\mathcal{M}_1}$ is outside $(0,1)$, so it is immediate that only p_0 can correspond to a probability. But the Möbius polynomial might have several roots within $(0,1)$, as the next example reveals.

Consider the trace monoid $\mathcal{M}_2 = \mathcal{M}(\Sigma_2, I_2)$ with $\Sigma_2 = \{a_1, \ldots, a_5\}$, and which associated dependence relation D_2 is depicted in Figure 4. Then the Möbius polynomial is $\mu_{\mathcal{M}_2}(X) = 1 - 5X + 5X^2$, with roots $q_0 = 1/2 - \sqrt{5}/10$ and $q_1 = 1/2 + \sqrt{5}/10$. Hence, $\mu_{\mathcal{M}_2}$ has its two roots within $(0,1)$, so we need to use the full statement of Theorem 2.6 point 1 in order to rule out q_1 and retain only q_0 as defining a uniform measure.

This can be double-checked by hand on this example. Consider the valuation $f(u) = q_1^{|u|}$. Let h be the Möbius transform of f. We have, for $i \in \{1, \ldots, 5\}$: $h(a_i) = q_1(1 - 2q_1) = -q_1/\sqrt{5} < 0$. Therefore, the valuation f is not Möbius and q_1 does not qualify to define a uniform measure.

3.2—Computing the speedup

In a trace monoid, what is the "average parallelism"? Or what is the average speedup of the parallel execution time compared to the sequential one? Or what is the average density of a heap? The questions are natural and have been extensively studied [24, 8, 18, 3]. Obviously the probability assumptions need to be specified for the questions to make sense.

In the remaining of § 3.2, we consider an irreducible trace monoid $\mathcal{M}$.

Let $\tau(u)$ denote the *height* of a trace $u \in \mathcal{M}$, which is equivalently defined (see also § 2.4) either as the number of cliques in the Cartier-Foata decomposition of u, or as the height of the heap of pieces associated with u, and can also be interpreted as the "parallel execution time" of u.
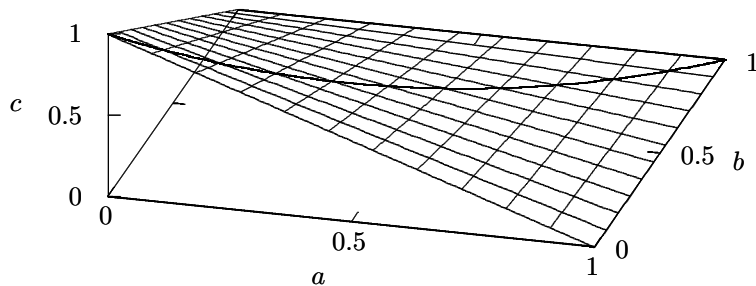

Figure 3: *Plot of the possible values for the characteristic numbers of a Bernoulli measure on $\mathcal{M}_1$*
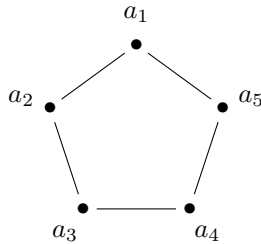
Figure 4: *Dependence relation D_2 for $\mathcal{M}_2$. Self-loops are omitted.*

One standard approach is to define the average parallelism as the quantity $\lambda_{\mathcal{M}} = \lim_{n \to \infty} n/\tau(x_1 \cdot x_2 \cdot \ldots \cdot x_n)$ where $(x_n)_n$ is an independent and uniformly distributed sequence of Σ-valued random variables. The limit exists indeed and is constant with probability one, by a sub-additivity argument [24]. The quantity $\lambda_{\mathcal{M}}$ is non-algebraic except for small trace monoids [18], and is NP-hard even to approximate [5].

Another approach, more closely related to our probabilistic model, is as follows. For all integer $n \geq 0$, set $\mathcal{M}_n = \{u \in \mathcal{M} \ : \ |u| = n\}$ and let ν_n be the uniform probability distribution on the finite set $\mathcal{M}_n$. Let $\varphi : \mathcal{M} \to \mathbb{R}_+$ be the function defined by:

$$\varphi(0) = 1, \qquad\qquad x \neq 0 \quad \varphi(x) = \frac{|x|}{\tau(x)}.$$

Clearly, $1 \leq \varphi(\cdot) \leq K$ holds for K the maximal size of cliques. Define, assuming existence, the two limits:

$$\gamma_{\mathcal{M}} = \lim_{n \to \infty} \mathbb{E}_{\nu_n} 1/\varphi(\cdot) = \lim_{n \to \infty} \frac{1}{\#\mathcal{M}_n} \sum_{u \in \mathcal{M}_n} \frac{\tau(u)}{n} \tag{18}$$

$$\rho_{\mathcal{M}} = \lim_{n \to \infty} \mathbb{E}_{\nu_n} \varphi(\cdot) = \lim_{n \to \infty} \frac{1}{\#\mathcal{M}_n} \sum_{u \in \mathcal{M}_n} \frac{n}{\tau(u)} \tag{19}$$

The quantities $\rho_{\mathcal{M}}$ and $\gamma_{\mathcal{M}}$ have been studied under the respective names of *speedup* and *average parallelism*. In [18], the limit in (18) is shown to exist and to be an algebraic number (the notation is "$\lambda_{\mathbb{M}}$" in [18]). The method is based on manipulations of the bi-variate generating series $L(x, y) = \sum_{u \in \mathcal{M}} x^{|u|} y^{\tau(u)}$. The expression obtained for $\gamma_{\mathcal{M}}$ is a ratio:

$$\gamma_{\mathcal{M}} = \frac{\left[(\partial L/\partial y)(x, 1) \cdot (p_0 - x)^2 \right]_{|x=p_0}}{p_0 \cdot \left[L(x, 1) \cdot (p_0 - x) \right]_{|x=p_0}}, \tag{20}$$

where p_0 is the root of smallest modulus of the Möbius polynomial of $\mathcal{M}$. The above expression is tractable to a certain extent since the series $L(x, y)$ is shown to be rational [18, Prop. 4.1].

Using the results of the present paper, we obtain much more. Let $\mathbb{P}$ be the uniform measure on $\partial \mathcal{M}$. Recall that the sequence $(C_n)_{n \geq 1}$ of cliques is then an irreducible and aperiodic Markov chain on $\mathfrak{C}$. Let π be the stationary measure of this chain, that is to say, π is the unique probability vector on $\mathfrak{C}$ satisfying: $\pi P = \pi$, where P is the transition matrix given in Theorem 2.5.

The distribution ν_n on $\mathcal{M}_n$ induces a distribution of the ratios $\varphi(x)$ on $[1, K]$, as well as a distribution of the inverses $1/\varphi(x)$ on $[1/K, 1]$. We denote respectively by $\varphi_* \nu_n$, and by $(1/\varphi)_* \nu_n$, the induced distributions.

• **Proposition 3.1**—*The limits in* (18)–(19) *exist. They are algebraic numbers, and satisfy the following formulas:*

$$\rho_{\mathcal{M}} = \sum_{c \in \mathfrak{C}} \pi(c)|c|, \qquad\qquad \gamma_{\mathcal{M}} = \frac{1}{\rho_{\mathcal{M}}}. \qquad (21)$$

Furthermore, the sequences of probability distributions $(\varphi_* \nu_n)_{n \geq 1}$ *and* $((1/\varphi)_* \nu_n)_{n \geq 1}$ *converge weakly respectively towards the Dirac measure* $\delta_{\rho_{\mathcal{M}}}$, *and toward the Dirac measure* $\delta_{\gamma_{\mathcal{M}}}$.

The formula in (24) provides a more tractable expression for $\gamma_{\mathcal{M}}$ than the one in (20). The concentration results are new. In a forthcoming work based on spectral methods, the first author and S. Gouëzel show the weak convergence of $\sqrt{n}(1/\varphi(\cdot) - \gamma_{\mathcal{M}})$ towards a normal law $\mathcal{N}(0, \sigma^2)$, non degenerated unless $\mathcal{M}$ is a free monoid.

Proof. We briefly sketch the proof. Set $\rho = \sum_{c \in \mathfrak{C}} \pi(c)|c|$. Clearly, ρ is algebraic since p_0 is algebraic, and since the coefficients $\pi(c)$ are solutions of a linear system involving only p_0 and its powers.

For every integer $n \geq 1$, let $Y_n = C_1 \cdot \ldots \cdot C_n$, of height n by construction, and let:

$$A_n = Y_{P_n}, \qquad\qquad P_n = \max\{q \in \mathbb{N} : |Y_q| \leq n\}.$$

Clearly, $P_n \geq n/K$ holds and thus $\lim_{n \to \infty} P_n = \infty$. By the Ergodic theorem for Markov chains (see [9, Part. I, § 15, Theorem 2], or [6, Theorem 4.1 p. 111]), the ratios $\varphi(Y_n) = |Y_n|/\tau(Y_n) = (|C_1| + \ldots + |C_n|)/n$ converge $\mathbb{P}$-a.s. towards ρ, and so do the ratios $\varphi(A_n) = \varphi(Y_{P_n})$.

Using the expression (14) of Theorem 2.5 on the one hand, and Proposition 4.12 below on the other hand, we see that there is a constant $C_1 > 0$ such that $\mathbb{P}(A_n = y) \geq C_1 p_0^n$ for all traces y belonging to the range of values of A_n. Hence, by the asymptotics $p_0^{-n} \sim C(\#\mathcal{M}_n)$ recalled in Theorem 4.7 below, there is a constant $C_2 > 0$ such that $\mathbb{P}(A_n = y) \geq C_2/(\#\mathcal{M}_n)$, for all such traces y.

Then, let $\varepsilon > 0$. We have:

$$\nu_n\big(\varphi(x) \notin [\rho - \varepsilon, \rho + \varepsilon]\big) = \sum_{y \in \mathcal{M}_n} \frac{1}{\#\mathcal{M}_n} \mathbf{1}_{\{\varphi(y) \notin [\rho - \varepsilon, \rho + \varepsilon]\}}$$

$$\leq \frac{1}{C_2} \sum_{y \in \mathcal{M}_n} \mathbf{1}_{\{\varphi(y) \notin [\rho - \varepsilon, \rho + \varepsilon]\}} \mathbb{P}(A_n = y)$$

$$\leq \frac{1}{C_2} \mathbb{P}\big(\varphi(A_n) \notin [\rho - \varepsilon, \rho + \varepsilon]\big) \to_{n \to \infty} 0$$

since the convergence $\varphi(A_n) \to \rho$ holds $\mathbb{P}$-almost surely, and thus in distribution. This proves the convergence in distribution of $\varphi(x)$ towards δ_ρ. The convergence in distribution of $1/\varphi(x)$ towards $\delta_{1/\rho}$ follows. And this implies of course the convergence of $\mathbb{E}_{\nu_n}\varphi(\cdot)$ towards ρ, and of $\mathbb{E}_{\nu_n}1/\varphi(\cdot)$ towards $1/\rho$. It follows that $\rho_{\mathcal{M}} = \rho$ and $\gamma_{\mathcal{M}} = 1/\rho$. $\qquad\square$

For instance, for the trace monoids $\mathcal{M}_1$ and $\mathcal{M}_2$ analyzed in § 3.1, we obtain, after computation of the associated stationary measure π:

$$\rho_{\mathcal{M}_1} = 5(7 - \sqrt{5})/22 = 1.0827 \cdots \qquad \rho_{\mathcal{M}_2} = (29 - \sqrt{5})/22 = 1.2165 \cdots$$
$$\gamma_{\mathcal{M}_1} = (7 + \sqrt{5})/10 = 0.924 \cdots \qquad \gamma_{\mathcal{M}_2} = (29 + \sqrt{5})/38 = 0.822 \cdots$$

consistently with the results of [18, Appendix B], where the case of $\mathcal{M}_1$ is included in the table. So parallelism increases the speed of execution by about 8% in the monoid $\mathcal{M}_1$ and by about 22% in the monoid $\mathcal{M}_2$.

4—Auxiliary tools

This section introduces auxiliary tools needed for the proofs of the previously stated results.

4.1—Elementary cylinders and sequences of cliques

Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid. In this subsection, we establish the correspondence between points of the boundary $\partial\mathcal{M}$, and infinite sequences of non-empty cliques satisfying the Cartier-Foata condition. We also describe how the order on traces transposes to their Cartier-Foata normal form.

To each non-empty trace $u \in \mathcal{M}$ of Cartier-Foata normal form $d_1 \to \ldots \to d_n$, we associate an infinite sequence $(c_k)_{k \geq 1}$ of cliques, defined as follows: $c_k = d_k$ if $1 \leq k \leq n$, and $c_k = 0$ for $k > n$. The sequence $(c_k)_{k \geq 1}$ is called the *extended Cartier-Foata decomposition* of u, abbreviated xCF.

Recall from § 2.4 that $\tau(u)$, the height of the trace u, is defined as the number of cliques in the Cartier-Foata normal form of u. Equivalently, $\tau(u)$ is the number of non empty cliques in the xCF decomposition of u.

The Cartier-Foata decomposition establishes, for each integer $n \geq 1$, a bijection between the set of traces of height n and a subset of the product $\mathscr{C}^n$. How the ordering between traces is read on their Cartier-Foata decompositions is the topic of next results. In particular, the obtained order on Cartier-Foata sequences is strictly coarser in general than the order induced by the product order on $\mathscr{C}^n$.

In the following result and later on, we make use of the notion of parallel cliques introduced in § 2.1, by writing $c \parallel c_1, \ldots, c_n$ if $c \parallel c_i$ for all integers $i \in \{1, \ldots, n\}$.

• **Lemma 4.1**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid and let $u, v \in \mathcal{M}$ be two non-empty traces. Let $c_1 \to \ldots \to c_n$ and $d_1 \to \ldots \to d_p$ be the Cartier-Foata decompositions of u and of v. Then $u \leq v$ if and only $n \leq p$ and there are n cliques $\gamma_1, \ldots, \gamma_n$ such that, for all $i \in \{1, \ldots, n\}$:*

1. *$\gamma_i \parallel c_i, \ldots, c_n$; and*

2. *$d_i = c_i \cdot \gamma_i$.*

Proof. If the Cartier-Foata normal forms of u and v satisfy the properties stated in points 1–2, then an easy induction argument shows that:

$$v = c_1 \cdot \ldots \cdot c_n \cdot (\gamma_1 \cdot \ldots \cdot \gamma_n) \cdot d_{n+1} \cdot \ldots \cdot d_p,$$

which implies that $u \leq v$.

Conversely, the proof is simple using the heap of pieces intuition. Here is the main argument. Assume that $u \leq v$, and let w be such that $v = u \cdot w$. If $w = 0$, the result is trivial. Otherwise, let $\delta_1 \to \ldots \to \delta_r$ be the Cartier-Foata normal form of w. Apply the following recursive construction: pick a letter $\alpha \in \delta_1$, and move α from δ_1 to the clique c_i where i is the smallest index such that $\alpha \parallel c_i, \ldots, c_n$. If there is no such index i, the letter α stays in δ_1. Then repeat the operation, until all letters of δ_1 have been dispatched. Once this is done, recursively apply the same procedure to δ_2 up to δ_r. Some cliques among the δ_i might entirely vanish during the procedure. The whole procedure yields the Cartier-Foata normal form of $u \cdot w$, under the requested form. $\square$

For each integer $p \geq 0$, we define the p-cut operation as the mapping $\kappa_p : \mathcal{M} \to \mathcal{M}$, $u \mapsto \kappa_p(u) = c_1 \cdot \ldots \cdot c_p$, where $(c_k)_{k \geq 1}$ is the xCF decomposition of u.

● **Corollary 4.2**—*Let $\mathcal{M}$ be a trace monoid, and let $u, v \in \mathcal{M}$ be two traces. Then $u \leq v$ if and only if $u \leq \kappa_{\tau(u)}(v)$.*

Proof. If $u \leq \kappa_{\tau(v)}(v)$, then $u \leq v$ since $\kappa_n(v) \leq v$ holds trivially for any integer $n \geq 0$. Conversely, assume that $u \leq v$. Then the Cartier-Foata normal forms $c_1 \to \ldots \to c_n$ and $d_1 \to \ldots \to d_p$ of u and v satisfy properties 1 and 2 in Lemma 4.1. We have $\tau(u) = n$ and

$$\kappa_{\tau(u)}(v) = (c_1 \gamma_1) \cdot \ldots \cdot (c_n \gamma_n) = (c_1 \cdot \ldots \cdot c_n) \cdot \gamma_1 \cdot \ldots \cdot \gamma_n \geq u.$$

The proof is complete. $\square$

● **Corollary 4.3**—*Let $\mathcal{M}$ be a trace monoid. Let $(c_k)_{k \geq 1}$ and $(d_k)_{k \geq 1}$ be the xCF decompositions of two traces u and v of $\mathcal{M}$. If $u \leq v$, then $c_k \leq d_k$ for all integers $k \geq 1$.*

Proof. The inequality $c_k \leq d_k$ is trivial if $c_k = 0$. And for $c_k \neq 0$, then c_k belongs to the Cartier-Foata normal form of u, and $c_k \leq d_k$ follows from Lemma 4.1 point 2. $\square$

We lift the xCF decomposition to generalized traces as follows.

● **Lemma 4.4**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid. Then, for every generalized trace $u \in \mathcal{W}(\Sigma, I)$, there exists a unique infinite sequence of cliques $(c_k)_{k \geq 1}$ such that $u = \bigvee_{k \geq 1}(c_1 \cdot \ldots \cdot c_k)$ and $c_k \to c_{k+1}$ holds for all integers $k \geq 0$.*

Proof. The result is clear if $u \notin \partial \mathcal{M}$. And if $u \in \partial \mathcal{M}$, we consider a non-decreasing sequence $(u_n)_{n \geq 0}$ in $\mathcal{M}$ such that $u = \bigvee_{n \geq 0} u_n$. Then, if $(c_{n,k})_{k \geq 1}$ is the xCF decomposition of u_n, it follows from Corollary 4.3 that $(c_{n,k})_{n \geq 0}$ is non-decreasing in $\mathscr{C}$ for each integer $k \geq 1$, and thus eventually constant, say equal to c_k. Routine verifications using the compactness property stated in § 2.1 show that the sequence $(c_k)_{k \geq 1}$ thus defined is the only adequate sequence. $\square$

Say that the sequence $(c_k)_{k \geq 1}$ associated to a generalized trace $u \in \mathcal{W}(\Sigma, I)$ as in Lemma 4.4 is the xCF *decomposition* of u. For each $\xi \in \partial \mathcal{M}$, the sequence $(c_k)_{k \geq 1}$ is the unique sequence of non-empty cliques announced in (12).

Recall that we have defined Ω as the set of infinite paths in the graph $(\mathfrak{C}, \rightarrow)$ of non-empty cliques. Mapping each point $\xi \in \partial \mathcal{M}$ to its xCF decomposition defines a well-defined application $\Psi : \partial \mathcal{M} \rightarrow \Omega$, which is the mapping announced in § 2.4. It is bijective; its inverse is given, if $\omega \in \Omega$ has the form $\omega = (c_k)_{k \geq 1}$, by:

$$\Psi^{-1}(\omega) = \bigvee_{k \geq 1} (c_1 \cdot \ldots \cdot c_k).$$

The following result, which is based on Lemma 4.1, explores how elementary cylinders transpose through the xCF decomposition, connecting the two different points of view on the boundary elements: the *intrinsic* point of view through elementary cylinders, and the *effective* point of view through sequences of cliques.

• **Proposition 4.5**—*Let $\mathcal{M}$ be a trace monoid. For any element $\xi \in \partial \mathcal{M}$, denote by $\big(C_k(\xi) \big)_{k \geq 1}$ the xCF decomposition of ξ. Let $n \geq 1$ be an integer, and let $c_1, \ldots, c_n \in \mathscr{C}$ be n cliques such that $c_1 \rightarrow \ldots \rightarrow c_n$ holds. Put $v = c_1 \cdot \ldots \cdot c_{n-1}$ and $u = v \cdot c_n$. Then the following equalities of subsets of $\partial \mathcal{M}$ hold:*

$$\uparrow u = \uparrow (c_1 \cdot \ldots \cdot c_n) = \big\{ \xi \in \partial \mathcal{M} \ : \ C_1(\xi) \cdot \ldots \cdot C_n(\xi) \geq u \big\}, \qquad (22)$$

$$\big\{ \xi \in \partial \mathcal{M} \ : \ C_1(\xi) = c_1, \ldots, C_n(\xi) = c_n \big\} = \uparrow u \setminus \Big(\bigcup_{\substack{c \in \mathscr{C} \ : \\ c_n < c}} \uparrow (v \cdot c) \Big), \qquad (23)$$

where $c_n < c$ means $c_n \leq c$ and $c_n \neq c$.

Proof. Put $\mathcal{W} = \mathcal{W}(\Sigma, I)$, and extend the cut operations $\kappa_p : \mathcal{M} \rightarrow \mathcal{M}$ defined above for all integers $p \geq 0$, to mappings $\kappa_p : \mathcal{W} \rightarrow \mathcal{M}$ in the obvious way. Combining the compactness property with Corollary 4.2 yields:

$$\forall u \in \mathcal{M} \quad \forall v \in \mathcal{W} \quad u \leq v \iff u \leq \kappa_{\tau(u)}(v). \qquad (24)$$

Applied to $u = c_1 \cdot \ldots \cdot c_n$ as in the statement and to $\xi \in \partial \mathcal{M}$ in place of v, this is (22).

We now prove (23). Set:

$$A = \big\{ \xi \in \partial \mathcal{M} \ : \ C_1(\xi) = c_1 \wedge \ldots \wedge C_n(\xi) = c_n \big\}, \quad B = \bigcup_{c \in \mathscr{C} \ : \ c_n < c} \uparrow (v \cdot c).$$

It is obvious that $A \subseteq \uparrow u$. We prove that $A \cap B = \emptyset$. For this, by contradiction, assume there exists $\xi \in A \cap B$, and let $(\delta_k)_{k \geq 1}$ be the xCF of ξ. Then $\delta_i = c_i$ for all $i \in \{1, \ldots, n\}$ since $\xi \in A$. Let $c \in \mathscr{C}$ be a clique such that $c_n < c$ and $\xi \in \uparrow (v \cdot c)$. Clearly, $\tau(v \cdot c) = n$. Applying (24) to $v \cdot c \leq \xi$ we get thus $c_1 \cdot \ldots \cdot c_{n-1} \cdot c \leq c_1 \cdot \ldots \cdot c_{n-1} \cdot c_n$, and then by left cancellativity of the monoid, $c \leq c_n$, a contradiction. This proves that $A \cap B = \emptyset$, and thus the $\subseteq$ inclusion of (23).

For the converse $\supseteq$ inclusion, let $\xi \in \uparrow u \setminus B$, keeping the notation $(\delta_k)_{k \geq 1}$ for its xCF decomposition. Since $\tau(u) = n$, it follows from (24) that $c_1 \cdot \ldots \cdot c_n \leq$

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$\delta_1 \cdot \ldots \cdot \delta_n$. Hence $\delta_i = c_i \cdot \gamma_i$ for some cliques $\gamma_1, \ldots, \gamma_n$ as in Lemma 4.1. Using the properties of the cliques γ_i's, we have $\xi \geq \delta_1 \cdot \ldots \cdot \delta_n = c_1 \cdot \ldots \cdot c_n \cdot \gamma_1 \cdot \ldots \cdot \gamma_n$. Since $\xi \notin B$ by assumption, this implies that $\gamma_i = 0$ for all $i \in \{1, \ldots, n\}$, and thus $\xi \in A$. □

• **Corollary 4.6**—*The bijection $\Psi : \partial\mathcal{M} \to \Omega$ which associates to each point $\xi \in \partial\mathcal{M}$ its* xCF *decomposition, is bi-measurable with respect to $(\partial\mathcal{M}, \mathfrak{F})$ and $(\Omega, \mathfrak{G})$.*

Proof. The fact that Ψ is measurable follows from (23). The fact that Ψ^{-1} is measurable follows from (22). □

4.2—Generating series and asymptotics

For $\mathcal{M}$ a trace monoid and $k \geq 0$ an integer, let $\lambda_\mathcal{M}(k)$ be the number of traces of length k:

$$\lambda_\mathcal{M}(k) = \#\{u \in \mathcal{M} \; : \; |u| = k\}\,.$$

Let $G_\mathcal{M}(X)$ be the generating series of $\mathcal{M}$, defined by:

$$G_\mathcal{M}(X) = \sum_{u \in \mathcal{M}} X^{|u|} = \sum_{k \geq 0} \lambda_\mathcal{M}(k) X^k\,.$$

The following result is standard, and is the basis of the combinatorial study of trace monoids [7, 27, 18, 13].

• **Theorem 4.7**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be a trace monoid, with $\mu_\mathcal{M}(X)$ the single-variable Möbius polynomial. We assume that $|\Sigma| > 1$. Then:*

1. *The following formal identity holds in $\mathbb{Z}[[X]]$:*

   $$G_\mathcal{M}(X) = 1/\mu_\mathcal{M}(X)\,.$$

   In particular, $G_\mathcal{M}(X)$ is a rational series.

2. *The polynomial $\mu_\mathcal{M}(X)$ has a unique root of smallest modulus, say p_0. This root is real and lies in $(0, 1)$. The radius of convergence of the power series $G_\mathcal{M}(z)$ is thus equal to p_0, and the series $G_\mathcal{M}(z)$ is divergent at p_0.*

3. *If N is the multiplicity of the root p_0 in $\mu_\mathcal{M}(X)$, then the following estimate holds for some constant $C > 0$:*

   $$\lambda_\mathcal{M}(k) \sim_{k \to \infty} C k^{N-1} \big(1/p_0\big)^k\,. \tag{25}$$

   Furthermore, the root p_0 has multiplicity 1 if $\mathcal{M}$ is irreducible.

• **Lemma 4.8**—*Let $\mathcal{M}$ be an irreducible trace monoid. Then for any non-empty clique $c \in \mathfrak{C}_\mathcal{M}$, we have:*

$$\lim_{k \to \infty} \lambda_{\mathcal{M}_c}(k)/\lambda_\mathcal{M}(k) = 0\,. \tag{26}$$

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Proof. The argument is rather standard. We sketch it and illustrate it for $\mathcal{M}_1$. Start by considering the Cartier-Foata automaton of $\mathcal{M}$ and transform it by expanding each node corresponding to a clique c of cardinality strictly larger than one, into $|c|$ nodes. The first of the expanded nodes is initial and the last of the expanded nodes is final. The non-expanded nodes are both initial and final. See [18, p.148] for the details of the construction and see Figure 5 for an illustration.

Let $\mathcal{A}$ be the resulting automaton. Let $\mathcal{A}_c$ be the automaton obtained from $\mathcal{A}$ by keeping the same nodes, the same initial and final nodes, but by keeping only the arcs entering into the nodes labeled by the letters of Σ_c. Admissible paths of length k in $\mathcal{A}$ are in bijection with traces of length k in $\mathcal{M}$. And admissible paths of length k in $\mathcal{A}_c$ are in bijection with traces of length k in $\mathcal{M}_c$ (recall that a path in an automaton is *admissible* if it starts with an initial state and ends up with a final state). Denote by A the incidence matrix of the automaton $\mathcal{A}$, and by A_c the one of $\mathcal{A}_c$. By construction, we have:

$$A_c \leq A, \qquad\qquad A_c \neq A. \qquad\qquad (27)$$

According to Lemma 4.16 below, since $\mathcal{M}$ is irreducible, the matrix A is primitive. So we are in the domain of applicability of [25, Theorem 1.2 p.9 and Theorem 1.1 point (e) p.4], the strong version of Perron-Frobenius Theorem for non-negative matrices. According to it, the strict inequality (27) yields that the spectral radius of A_c is strictly smaller than the spectral radius of A. The limit (26) follows. $\qquad\square$

In Figure 5, we illustrate the construction of the proof for the trace monoid $\mathcal{M}_1 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ by showing the automaton $\mathcal{A}$, to be compared with the original automaton depicted in Figure 2. In $\mathcal{A}$, the initial nodes are $\{a, b, c, (a \cdot b)_1\}$ and the final nodes are $\{a, b, c, (a \cdot b)_2\}$. For the clique $\{a\}$, for instance, the automaton $\mathcal{A}_{\{a\}}$ has the same nodes as $\mathcal{A}$ but a single arc: the self-loop around the node b.
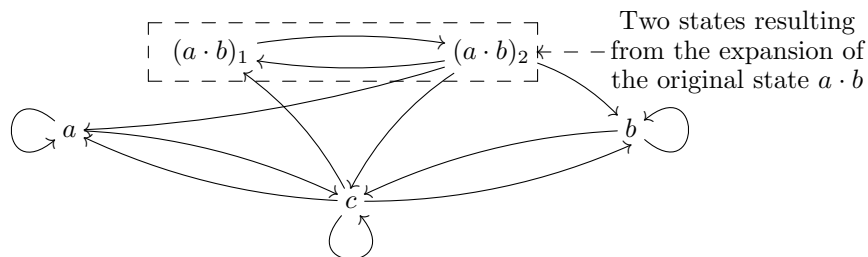

Figure 5: *The expanded automaton $\mathcal{A}$.*

● **Proposition 4.9**—*Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be an irreducible trace monoid, and let p_0 be the root of smallest modulus of the Möbius polynomial $\mu_{\mathcal{M}}(X)$. Then $\mu_{\mathcal{M}_c}(p_0) > 0$ for all non-empty cliques $c \in \mathfrak{C}_{\mathcal{M}}$.*

Proof. By point 3 of Theorem 4.7, and since $\mathcal{M}$ is assumed to be irreducible, we have the estimate $\lambda_{\mathcal{M}}(k) \sim C(1/p_0)^k$ for $k \to \infty$. Let $c \in \mathfrak{C}_{\mathcal{M}}$. If c is maximal, then $\mu_{\mathcal{M}_c}(X) = 1$ and thus $\mu_{\mathcal{M}_c}(p_0) > 0$ holds trivially. Otherwise, the monoid $\mathcal{M}(\Sigma_c, I_c)$ is non-trivial, let p_c be the root of smallest modulus

of $\mu_{\mathcal{M}_c}$. Then $\lambda_{\mathcal{M}_c}(k) \sim C' k^{N-1} (1/p_c)^k$ for some constant $C' > 0$ and where N is the multiplicity of p_c in $\mu_{\mathcal{M}_c}(X)$, by point 3 of Theorem 4.7. In view of Lemma 4.8, it follows that $p_c > p_0$.

In particular, and by point 2 of Theorem 4.7, p_0 is in the open disc of convergence of the series $G_{\mathcal{M}_c}(z) = \sum_{k \geq 0} \lambda_{\mathcal{M}_c}(k) z^k$. Applying point 1 of Theorem 4.7 to the trace monoid $\mathcal{M}_c$, and converting the formal equality into an equality between reals, we get $\mu_{\mathcal{M}_c}(p_0) = 1/G_{\mathcal{M}_c}(p_0) > 0$, completing the proof. $\qquad\square$

4.3—Möbius inversion formula and consequences

The notion of Möbius function of a partial order is due to Rota [22]. Due to the formal equality $G_{\mathcal{M}}(X)\mu_{\mathcal{M}}(X) = 1$, recalled in Theorem 4.7 point 1, the Möbius function, element of the incidence algebra in the sense of Rota, is easily found to be $\nu_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{Z}$ given by $\nu_{\mathcal{M}}(x, y) = (-1)^{|y|-|x|}$ if $x \leq y$ and if $y - x$ is a clique, and 0 otherwise. From this, we derive the following form of the Möbius inversion formula [22, Prop.2] for trace monoids.

• **Proposition 4.10**—*Let $\mathscr{C}$ be the set of cliques associated with an independence pair (Σ, I). If $f, h : \mathscr{C} \to \mathbb{R}$ are two functions, then h is the Möbius transform of f, that is, satisfies:*

$$\forall c \in \mathscr{C} \qquad h(c) = \sum_{c' \in \mathscr{C} \,:\, c \leq c'} (-1)^{|c'|-|c|} f(c') \,, \qquad (28)$$

if and only if the following holds:

$$\forall c \in \mathscr{C} \qquad f(c) = \sum_{c' \in \mathscr{C} \,:\, c' \geq c} h(c') \,. \qquad (29)$$

The formula (29) is called the *Möbius inversion formula*, since it allows to recover any function $f : \mathscr{C} \to \mathbb{R}$ from its Möbius transform. We give an enhanced version in Proposition 4.13 below which applies outside the mere set $\mathscr{C}$.

• **Corollary 4.11**—*Let $\mathscr{C}$ be the set of cliques associated with an independence pair (Σ, I). Let $h : \mathscr{C} \to \mathbb{R}$ be the Möbius transform of a function $f : \mathscr{C} \to \mathbb{R}$ such that $f(0) = 1$. Then $h(0) = 0$ if and only if $\sum_{c \in \mathfrak{C}} h(c) = 1$.*

Proof. The Möbius inversion formula (29) applied to $c = 0$ writes as follows: $1 = h(0) + \sum_{c \in \mathfrak{C}} h(c)$, whence the result. $\qquad\square$

We recall that, by convention, the Möbius transform of a valuation $f : \mathcal{M} \to \mathbb{R}_+^*$ is defined as the Möbius transform of its restriction to $\mathscr{C}_{\mathcal{M}}$.

• **Proposition 4.12**—*Let $h : \mathscr{C} \to \mathbb{R}$ be the Möbius transform of a valuation $f : \mathcal{M} \to \mathbb{R}_+^*$, where $\mathscr{C}$ is associated to an independence pair (Σ, I). Assume that $h(0) = 0$, and let $g : \mathscr{C} \to \mathbb{R}$ be the function defined by:*

$$\forall c \in \mathscr{C} \quad g(c) = \sum_{c' \in \mathscr{C} \,:\, c \to c'} h(c') \,.$$

Then the following formula holds:

$$\forall c \in \mathscr{C} \quad g(c)f(c) = h(c).$$

Proof. The identity $g(0)f(0) = h(0)$ is trivial since $0 \to c'$ if and only if $c' = 0$, and thus $g(0) = h(0)$, while $f(0) = 1$. For $c \in \mathfrak{C}$ a non-empty clique, and by definition of h and of g, one has:

$$g(c) = \sum_{c' \in \mathfrak{C}} (-1)^{|c'|} f(c') \sum_{\delta \in \mathfrak{C} \,:\, \delta \leq c' \wedge c \to \delta} \mathbf{1}_{\{c \to \delta\}} \mathbf{1}_{\{\delta \leq c'\}} (-1)^{|\delta|} \,.$$

For any $c' \in \mathfrak{C}$, the range of δ in the above sum is $\big\{ \delta \in \mathfrak{C} \,:\, \delta \leq c' \cap \{\alpha \in \Sigma \,:\, c \to \alpha\} \big\}$, and the binomial formula yields thus:

$$\sum_{\delta \in \mathfrak{C} \,:\, \delta \leq c' \wedge c \to \delta} (-1)^{|\delta|} = -\tau_c(c') \,, \quad \text{with } \tau_c(c') = \begin{cases} 0, & \text{if } c' \parallel c, \\ 1, & \text{if } \neg(c' \parallel c). \end{cases}$$

We obtain thus:

$$g(c) = -\sum_{c' \in \mathfrak{C}} (-1)^{|c'|} f(c') \tau_c(c') \,. \tag{30}$$

The assumption $h(0) = 0$ writes as:

$$1 + \sum_{c' \in \mathfrak{C} \,:\, c' \parallel c} (-1)^{|c'|} f(c') + \sum_{c' \in \mathfrak{C} \,:\, \tau_c(c') = 1} (-1)^{|c'|} f(c') = 0 \,. \tag{31}$$

Combining (30) and (31) yields:

$$g(c) = 1 + \sum_{c' \in \mathfrak{C} \,:\, c' \parallel c} (-1)^{|c'|} f(c') \,. \tag{32}$$

We multiply both sides of (32) by $f(c)$ and apply the change of variable $c'' = c \cdot c'$. Using that f is multiplicative, this yields:

$$f(c)g(c) = f(c) + \sum_{c'' \in \mathfrak{C} \,:\, c'' > c} (-1)^{|c''| - |c|} f(c'') = h(c) \,,$$

which was to be proved. $\qquad\square$

Next result is a generalization of the Möbius inversion formula (29). Whereas the original Möbius inversion formula is valid for any function $f : \mathscr{C}_\mathcal{M} \to \mathbb{R}$, the generalized version applies to valuations only.

Let $f : \mathcal{M} \to \mathbb{R}_+^*$ be a valuation. In (28), the Möbius transform of f was defined as a function $h : \mathscr{C}_\mathcal{M} \to \mathbb{R}$. Here, we extend the domain of definition of h to the whole monoid $\mathcal{M}$ as follows. If $u \in \mathcal{M}$ is a non-empty trace, we write $u = v \cdot c$, where $c \in \mathfrak{C}_\mathcal{M}$ is the *last* clique in the Cartier-Foata normal form of u, and v is the unique trace such that $u = v \cdot c$ holds. The *extended Möbius transform* $h : \mathcal{M} \to \mathbb{R}$ is then defined by:

$$\forall u \in \mathcal{M} \qquad h(u) = f(v)h(c) \,. \tag{33}$$

● **Proposition 4.13**—*Let $f : \mathcal{M} \to \mathbb{R}$ be a valuation defined on a trace monoid. Let $u \in \mathcal{M}$ be a non-empty trace, and let $\mathcal{M}(u)$ denote the set:*

$$\mathcal{M}(u) = \{ u' \in \mathcal{M} \,:\, \tau(u') = \tau(u), \; u \leq u' \} \,, \tag{34}$$

where $\tau(\cdot)$ is the height function defined in § 4.1. Then we have the identity:

$$\sum_{u' \in \mathcal{M}(u)} h(u') = f(u), \tag{35}$$

where $h : \mathcal{M} \to \mathbb{R}$ is the extended Möbius transform of f defined in (33).

Proof. We fix a non-empty trace $u \in \mathcal{M}$, and we set

$$S_0 = \sum_{u' \in \mathcal{M}(u)} h(u').$$

Let $c_1 \to \ldots \to c_n$ be the Cartier-Foata normal form of u, and set $c = c_n$ and $v = c_1 \cdot \ldots \cdot c_{n-1}$. We apply Lemma 4.1 to derive:

$$S_0 = \sum_{(\gamma_1, \ldots, \gamma_n) \in J(c_1, \ldots, c_n)} h(c_1 \cdot \gamma_1 \cdot \ldots \cdot c_n \cdot \gamma_n), \tag{36}$$

where we have set:

$$J(c_1, \ldots, c_n) = \big\{ (\gamma_1, \ldots, \gamma_n) \in \mathscr{C}^n \ : \ \gamma_i \parallel c_i, \ldots, c_n \text{ for } 1 \le i \le n,$$
$$c_1 \cdot \gamma_1 \to \ldots \to c_n \cdot \gamma_n \big\}.$$

By definition of h, this yields:

$$S_0 = f(v) S_1, \quad \text{with} \quad S_1 = \sum_{(\gamma_1, \ldots, \gamma_n) \in J(c_1, \ldots, c_n)} f(\gamma_1) \ldots f(\gamma_{n-1}) h(c_n \cdot \gamma_n).$$
$$\tag{37}$$

We define, for $x, y \in \mathscr{C}$:

$$\lambda(x, y) = \sum_{\delta \in \mathscr{C} \ : \ (x \to \delta) \wedge (\delta \ge y)} h(\delta). \tag{38}$$

Rewriting S_1 using the above notation, we get:

$$S_1 = \sum_{(\gamma_1, \ldots, \gamma_{n-1}) \in K(c_1, \ldots, c_n)} f(\gamma_1) \cdots f(\gamma_{n-1}) \lambda(c_{n-1} \cdot \gamma_{n-1}, c_n),$$

where we have set:

$$K(c_1, \ldots, c_n) = \big\{ (\gamma_1, \ldots, \gamma_{n-1}) \in \mathscr{C}^{n-1} \ : \ \gamma_i \parallel c_i, \ldots, c_n \text{ for } 1 \le i \le n-1,$$
$$c_1 \cdot \gamma_1 \to \ldots \to c_{n-1} \cdot \gamma_{n-1} \big\}.$$

Applying Lemma 4.14 below yields, for any γ_{n-1} in the scope of the sum defining S_1:

$$\lambda(c_{n-1} \cdot \gamma_{n-1}, c_n) = f(c_n) \sum_{\delta \in \mathscr{C} \ : \ \delta \parallel c_{n-1} \cdot \gamma_{n-1}, \, c_n} (-1)^{|\delta|} f(\delta).$$

Therefore $S_1 = f(c_n) S_2$ where S_2 is defined by:

$$S_2 = \sum_{\substack{(\gamma_1, \ldots, \gamma_{n-1}) \in K(c_1, \ldots, c_n) \\ \delta \in \mathscr{C} \ : \ \delta \parallel c_{n-1} \cdot \gamma_{n-1}, \, c_n}} f(\gamma_1) \cdots f(\gamma_{n-1}) (-1)^{|\delta|} f(\delta). \tag{39}$$

Since $S_2 = 1$ according to Lemma 4.15 below, we conclude that $S_1 = f(c_n)$ and finally $S_0 = f(v) f(c_n) = f(u)$, which was to be proved. $\qquad\square$

In the course of the above proof, we have used the following two lemmas.

• **Lemma 4.14**—*If $x, y \in \mathscr{C}$ are two cliques such that $x \to y$ holds, then the quantity $\lambda(x, y)$ defined in (38) satisfies:*

$$\lambda(x, y) = f(y) \sum_{\delta \in \mathscr{C} \,:\, \delta \| x, y} (-1)^{|\delta|} f(\delta) \,.$$

Proof. By definition of the Möbius transform h, one has:

$$\lambda(x, y) \;=\; \sum_{\delta \in \mathscr{C} \,:\, (x \to \delta) \wedge (y \leq \delta)} h(\delta) \;=\; \sum_{z \in \mathscr{C} \,:\, y \leq z} (-1)^{|z|} f(z) H(x, y, z) \qquad (40)$$

$$\text{where} \quad H(x, y, z) = \sum_{\delta \in \mathscr{C} \,:\, (x \to \delta) \wedge (y \leq \delta \leq z)} (-1)^{|\delta|} \,.$$

Consider δ as in the sum defining $H(x, y, z)$. Since $x \to y$ holds by assumption, the following equivalence holds: $x \to \delta \iff x \to (\delta - y)$. The binomial formula yields thus:

$$H(x, y, z) = \begin{cases} (-1)^{|y|}, & \text{if } (z - y) \| x \,, \\ 0, & \text{otherwise.} \end{cases}$$

Reporting the latter value of $H(x, y, z)$ in (40) and considering the change of variable $z = y \cdot \delta$ yields the expected expression for $\lambda(x, y)$. $\qquad \square$

• **Lemma 4.15**—*For any integer $n \geq 1$ and for any cliques $c_1, \ldots, c_n$ such that $c_1 \to \ldots \to c_n$ holds, the quantity S_2 defined in (39) satisfies $S_2 = 1$.*

Proof. We substitute the variable $\delta' = \delta \cdot \gamma_{n-1}$ to δ in the defining sum for S_2. For each γ_{n-1} in the scope of the sum, one has $\gamma_{n-1} \| c_{n-1}, c_n$, as specified by the definition of $K(c_1, \ldots, c_n)$. Hence the set $\{\delta \in \mathscr{C} \,:\, \delta \| c_{n-1} \cdot \gamma_{n-1}, c_n\}$ corresponds to the set $\{\delta' \in \mathscr{C} \,:\, (\delta' \geq \gamma_{n-1}) \wedge (\delta' \| c_{n-1}, c_n)\}$, and the change of variable yields:

$$S_2 = \sum_{\delta \in \mathscr{C} \,:\, \delta \| c_{n-1}, c_n} (-1)^{|\delta|} f(\delta) \sum_{(\gamma_1, \ldots, \gamma_{n-2}) \in L(c_1, \ldots, c_n)} f(\gamma_1) \cdots f(\gamma_{n-2}) R(\gamma_{n-2}) \,,$$

$$(41)$$

$$\text{with} \quad L(c_1, \ldots, c_n) = \big\{ (\gamma_1, \ldots, \gamma_{n-2}) \in \mathscr{C}^{n-2} \,:\, \gamma_i \| c_i, \ldots, c_n \text{ for } 1 \leq i \leq n-2, $$
$$c_1 \cdot \gamma_1 \to \ldots \to c_{n-2} \cdot \gamma_{n-2} \big\}$$

$$\text{and} \quad R(\gamma_{n-2}) = \sum_{\substack{\gamma_{n-1} \in \mathscr{C} \,: \\ \gamma_{n-1} \| c_{n-1}, c_n \\ \gamma_{n-1} \leq \delta \\ c_{n-2} \cdot \gamma_{n-2} \to \gamma_{n-1}}} (-1)^{|\gamma_{n-1}|} \,.$$

In the scope of the sum defining $R(\gamma_{n-2})$, the condition "$c_{n-2} \cdot \gamma_{n-2} \to c_{n-1} \cdot \gamma_{n-1}$" has been replaced by "$c_{n-2} \cdot \gamma_{n-2} \to \gamma_{n-1}$", which is equivalent since $c_{n-2} \to c_{n-1}$ already holds by assumption.

Since $\delta \| c_{n-1}, c_n$, and by the binomial formula, the sum defining $R(\gamma_{n-2})$ evaluates as follows:

$$R(\gamma_{n-2}) = \mathbf{1}_{\{\delta \| c_{n-2} \cdot \gamma_{n-2}\}} \,. \qquad (42)$$

Substituting the right side of (42) into (41), we obtain:

$$S_2 = \sum_{\substack{(\gamma_1,\ldots,\gamma_{n-2})\in L(c_1,\ldots,c_n) \\ \delta\in\mathscr{C} \,:\, \delta\|c_{n-2}\cdot\gamma_{n-2},c_{n-1},c_n}} f(\gamma_1)\cdots f(\gamma_{n-2})(-1)^{|\delta|}f(\delta)\,.$$

Applying recursively the same transformation eventually yields:

$$S_2 = \sum_{\substack{\gamma,\delta\in\mathscr{C} \,: \\ \gamma\|c_1,\ldots,c_n \\ \delta\|c_1\cdot\gamma,c_2,\ldots,c_n}} f(\gamma)(-1)^{|\delta|}f(\delta)\,,$$

and after yet the same transformation:

$$S_2 = \sum_{\delta\in\mathscr{C} \,:\, \delta\|c_1,\ldots,c_n} (-1)^{|\delta|}f(\delta)\sum_{\gamma\leq\delta}(-1)^{|\gamma|} = f(0) = 1\,,$$

completing the proof. $\qquad\square$

4.4—Combinatorial lemmas

The following result is known, see for instance [18, Lemma 3.2]. We provide an alternative proof in [2].

• **Lemma 4.16**—*If $\mathcal{M}$ is an irreducible trace monoid, then $(\mathfrak{C}_{\mathcal{M}},\to)$ is a strongly connected graph.*

Next lemma will be a key in proving the uniqueness of uniform measures. We recall first that for any trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$, the *mirror* mapping $\mathsf{rev} : \mathcal{M} \to \mathcal{M}$ is defined as the quotient mapping of the mapping $\Sigma^* \to \Sigma^*$ defined on words by $\mathsf{rev}(\alpha_1\cdots\alpha_n) = \alpha_n\cdots\alpha_1$. Given $u \in \mathcal{M}$, the heap of $\mathsf{rev}(u)$ is obtained from the heap of u by considering it upside-down. If $s_1 \to \ldots \to s_k$ and $r_1 \to \ldots \to r_\ell$ are the respective Cartier-Foata decompositions of u and $\mathsf{rev}(u)$, then:

$$k = \ell\,, \qquad \begin{cases} r_k \leq s_1 \\ r_{k-1} \cdot r_k \leq s_2 \cdot s_1 \\ \ldots \\ r_1 \cdot r_2 \cdot \ldots \cdot r_k \leq s_k \cdot s_{k-1} \cdot \ldots \cdot s_1 \end{cases} \tag{43}$$

The properties in (43) are easy to visualize using the heap interpretation.

• **Lemma 4.17**—*(Hat lemma) Let $\mathcal{M}$ be an irreducible trace monoid. Then there exists a trace $w \in \mathcal{M}$ with the following property:*

$$\forall u, v \in \mathcal{M} \qquad (|u| = |v|) \wedge (u \neq v) \implies \uparrow(u \cdot w) \cap \uparrow(v \cdot w) = \emptyset\,. \tag{44}$$

Proof. Let $\mathcal{M} = \mathcal{M}(\Sigma, I)$, and let D be the associated dependence relation. Since $\mathcal{M}$ is assumed to be irreducible, we consider a sequence $(\alpha_i)_{1\leq i\leq q}$ with $\alpha_i \in \Sigma$ such that: 1) every $\alpha \in \Sigma$ occurs at least once in the sequence; and 2) $(\alpha_i, \alpha_{i+1}) \in D$ for all $i \in \{1,\ldots,q-1\}$. We introduce the trace

$$w = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{q-1} \cdot \alpha_q \cdot \alpha_{q-1} \cdot \alpha_{q-2} \cdot \ldots \cdot \alpha_1\,,$$

and we aim at showing that w satisfies (44).

Claim (∗) For all $u \in \mathcal{M}$, the *first* q cliques in the Cartier-Foata normal form of $w \cdot u$ are $\alpha_1 \to \alpha_2 \to \ldots \to \alpha_q$, and the *last* q cliques in the Cartier-Foata normal form of $u \cdot w$ are $\alpha_q \to \alpha_{q-1} \to \ldots \to \alpha_1$.

We prove the claim (∗). By construction, the Cartier-Foata decomposition of w is

$$\alpha_1 \to \alpha_2 \to \ldots \to \alpha_{q-1} \to \alpha_q \to \alpha_{q-1} \to \ldots \to \alpha_2 \to \alpha_1 \, .$$

Consider the trace $w \cdot u$ for some $u \in \mathcal{M}$. Let $d_1 \to \ldots \to d_p$ be the Cartier-Foata decomposition of $w \cdot u$. Applying Lemma 4.1 to traces w and $w \cdot u$, we conclude in particular that $2q - 1 \leq p$, and for all $i \in \{1, \ldots, q\}$, we have $d_i = \alpha_i \cdot \gamma_i$ for some clique $\gamma_i \in \mathscr{C}$ such that $\gamma_i \parallel \alpha_i, \ldots, \alpha_q, \alpha_{q-1}, \ldots, \alpha_2, \alpha_1$. Since $\alpha_q, \ldots, \alpha_1$ range over all letters of Σ, it follows that $\gamma_i = 0$. So we have proved that $d_1 \to \ldots \to d_q = \alpha_1 \to \ldots \to \alpha_q$.

Now for the second part of the claim (∗), consider the trace $u \cdot w$ for some $u \in \mathcal{M}$. We have $\mathsf{rev}(u \cdot w) = \mathsf{rev}(w) \cdot \mathsf{rev}(u) = w \cdot \mathsf{rev}(u)$. According to the above, the Cartier-Foata decomposition of $w \cdot \mathsf{rev}(u)$ starts with $\alpha_1 \to \ldots \to \alpha_q$. According to (43), the q last cliques $d_1 \to \ldots \to d_q$ of $u \cdot w = \mathsf{rev}\big(w \cdot \mathsf{rev}(u)\big)$ satisfy:

$$d_q \leq \alpha_1 \, , \qquad d_{q-1} \cdot d_q \leq \alpha_2 \cdot \alpha_1 \, , \qquad \cdots \qquad d_1 \cdot \ldots \cdot d_q \leq \alpha_q \cdot \ldots \cdot \alpha_1 \, .$$

Since the α_i are minimal in $\mathfrak{C}$, it follows that $d_q = \alpha_1 \, , \ d_{q-1} = \alpha_2 \, , \ldots, d_1 = \alpha_q \, ,$ which completes the proof of the claim (∗).

We now come to the proof of (44) for w. Let $u, v \in \mathcal{M}$ such that $|u| = |v|$ and $\uparrow(u \cdot w) \cap \uparrow(v \cdot w) \neq \emptyset$. According to (2), it follows that $u \cdot w$ and $v \cdot w$ are compatible. Hence there are $u', v' \in \mathcal{M}$ such that $u \cdot w \cdot u' = v \cdot w \cdot v'$. Set $\widehat{w} = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{q-1}$, so that $w = \widehat{w} \cdot \alpha_q \cdot \mathsf{rev}(\widehat{w})$. It follows from the claim (∗) that the Cartier-Foata decomposition of $u \cdot \widehat{w} \cdot \alpha_q$ is of the form $c_1 \to \ldots \to c_k \to \alpha_q$, and the Cartier-Foata decomposition of $\alpha_q \cdot \mathsf{rev}(\widehat{w}) \cdot u'$ is of the form $\alpha_q \to d_1 \to \ldots \to d_\ell$. Hence the Cartier-Foata decomposition of $u \cdot w \cdot u'$ is obtained by concatenating the ones of $u \cdot w$ and of u'. By the same argument, the Cartier-Foata decomposition of $v \cdot w \cdot v'$ is obtained by concatenating the ones of $v \cdot w$ and of v'.

Hence, by uniqueness of the Cartier-Foata decomposition of $u \cdot w \cdot u' = v \cdot w \cdot v'$, between the decompositions of u' and of v', one is a suffix of the other. On the other hand, $u \cdot w \cdot u' = v \cdot w \cdot v'$ and $|u| = |v|$ imply $|u'| = |v'|$, and therefore $u' = v'$. Since $\mathcal{M}$ is cancellative, we conclude that $u = v$, which completes the proof. □

5—Proofs of the main results

5.1—From Bernoulli measures to Markov chains and Möbius valuations

In this section, we prove Proposition 2.1 and point 1 of Theorem 2.5 and point 1 of Theorem 2.4. The three results correspond to *necessary* conditions for a probability measure on the boundary of a trace monoid to be Bernoulli. We start with the two latter points.

The setting is the following: we consider an irreducible trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$, and we assume that $\mathbb{P}$ is a Bernoulli measure defined on $(\partial\mathcal{M}, \mathfrak{F})$. We consider the valuation $f : \mathcal{M} \to \mathbb{R}_+^*$ associated with $\mathbb{P}$, defined by $f(u) = \mathbb{P}(\uparrow u)$ for $u \in \mathcal{M}$, and we let $h : \mathscr{C} \to \mathbb{R}$ be the Möbius transform of f.

We start by proving formula (14) in Theorem 2.5, which implies most of the other affirmations. Hence, let $n \geq 1$ be an integer and let $c_1 \to \ldots \to c_n$ be n non-empty cliques. According to formula (23) in Proposition 4.5, one has:

$$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f(u) - \delta, \quad \text{with } \delta = \mathbb{P}\Big(\bigcup_{\substack{c \in \mathscr{C} : \\ c > c_n}} \uparrow(v \cdot c)\Big), \quad (45)$$

where $v = c_1 \cdot \ldots \cdot c_{n-1}$ and $u = c_1 \cdot \ldots \cdot c_n$. For any $\xi \in \partial\mathcal{M}$, one has $\xi \in \uparrow(v \cdot c)$ for some clique $c > c_n$ if and only if there is a letter $\alpha \parallel c_n$ such that $\xi \in \uparrow(v \cdot c_n \cdot \alpha)$. Let $\{\alpha_1, \ldots, \alpha_r\}$ be an enumeration of those letters $\alpha \parallel c_n$. Applying the inclusion-exclusion principle, we obtain:

$$\delta = \sum_{k=1}^r (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq r} \mathbb{P}\big(\uparrow(v \cdot c_n \cdot \alpha_{i_1}) \cap \cdots \cap \uparrow(v \cdot c_n \cdot \alpha_{i_k})\big).$$

For $i_1, \ldots, i_k$ indices as in the above sum, put $\gamma = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ and $\gamma' = c_n \cdot \gamma$. The related intersection is then either empty if γ is not a clique, or equal to $\uparrow(v \cdot c_n \cdot \gamma) = \uparrow(v \cdot \gamma')$. if γ is a clique, which is equivalent to $\gamma' = c_n \cdot \gamma$ being a clique. By construction, the cliques γ' range over the cliques $c' \in \mathscr{C}$ such that $c' > c_n$, and thus, taking into account that $\mathbb{P}(\uparrow \cdot) = f(\cdot)$ is multiplicative, we obtain:

$$\delta = \sum_{c' \in \mathscr{C} : c' > c_n} (-1)^{|c'|-|c|+1} \mathbb{P}\big(\uparrow(v \cdot c')\big) = f(v) \sum_{c' \in \mathscr{C} : c' > c_n} (-1)^{|c'|-|c|+1} f(c').$$

Injecting the above in (45), and writing $f(u) = f(v)f(c_n)$, we get:

$$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f(v) \sum_{c' \in \mathscr{C} : c' \geq c_n} (-1)^{|c'|-|c|} f(c') = f(v)h(c_n),$$

Since $f(v)h(c_n) = f(c_1) \cdots f(c_{n-1})h(c_n)$, we have the desired result.

As a particular case for $n = 1$, it follows at once that $h\big|_{\mathfrak{C}}$ coincides with the probability distribution of C_1. Therefore, by the total probability law, $\sum_{c \in \mathfrak{C}} h(c) = 1$, and by Corollary 4.11, $h(0) = 0$. It remains only to prove that $h > 0$ on $\mathfrak{C}$ to obtain that f is a Möbius valuation.

For this, let $c \in \mathfrak{C}$, and let c' be a maximal clique in $\mathfrak{C}$. Since $(\mathfrak{C}, \to)$ is connected according to Lemma 4.16, there exists a sequence $c_1, \ldots, c_n$ of cliques such that $c_1 = c$, $c_n = c'$, and $c_i \to c_{i+1}$ holds for all $i \in \{1, \ldots, n-1\}$. Since c' is maximal, the definition of the Möbius transform yields $h(c') = f(c')$, and thus $\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) = f(c_1) \cdots f(c_n) > 0$. This implies that $h(c) = \mathbb{P}(C_1 = c) > 0$. We have proved that f is a Möbius valuation, and completed the proof of point 1 in Theorem 2.4.

Finally, it remains only to show that $(C_n)_{n \geq 1}$ is an aperiodic and irreducible Markov chain with the specified transition matrix, since the law of C_1 has already been identified as $h\big|_{\mathfrak{C}}$.

From the general formula proved above, we derive, if $c_1 \to \ldots \to c_n$ holds:

$$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n | C_1 = c_1, \ldots, C_{n-1} = c_{n-1}) = \frac{1}{h(c_{n-1})} f(c_{n-1})h(c_n).$$

Since $h(0) = 0$, it follows from Proposition 4.12, and using the same notation g, that $h(c_{n-1}) = f(c_{n-1})g(c_{n-1})$. Therefore:

$$\mathbb{P}(C_1 = c_1, \ldots, C_n = c_n | C_1 = c_1, \ldots, C_{n-1} = c_{n-1}) = \frac{h(c_n)}{g(c_{n-1})}.$$

Since the latter quantity only depends on (c_{n-1}, c_n), it follows that $(C_n)_{n \geq 1}$ is a Markov chain with the transition matrix described in the statement of point 1 of Theorem 2.5.

The chain is irreducible since $(\mathfrak{C}, \rightarrow)$ is connected, as already observed. And it is aperiodic since $c \rightarrow c$ holds for any $c \in \mathfrak{C}$. The proof is complete.

Proof of Proposition 2.1. Consider any measure of finite mass on $\partial \mathcal{M}$. Let $u \in \mathcal{M}$ be any trace. The elementary cylinder $\uparrow u$ writes as the union: $\uparrow u = \bigcup_{\alpha \in \Sigma} \uparrow (u \cdot \alpha)$. Applying inclusion-exclusion principle as above yields the expected formula (3).

5.2—From Möbius valuations to Bernoulli measures, through Markov chains

In this section, we consider a trace monoid $\mathcal{M}$ equipped with a Möbius valuation $f : \mathcal{M} \rightarrow \mathbb{R}_+^*$, and we establish the existence and uniqueness of a probability measure on $(\partial \mathcal{M}, \mathfrak{F})$ such that $f(\cdot) = \mathbb{P}(\uparrow \cdot)$. This corresponds to the proof of point 2 of Theorem 2.4 and of point 2 of Theorem 2.5.

It must be noted that we do not use the irreducibility of $\mathcal{M}$ in this part of the proof.

The uniqueness of $\mathbb{P}$ follows from the remark made in § 2.2 that elementary cylinders form a π-system generating $\mathfrak{F}$.

For proving the existence of $\mathbb{P}$, we proceed by considering first the Markov chain on the Cartier-Foata subshift which is necessarily induced by $\mathbb{P}$, if it exists (even though it was only established for an irreducible trace monoid). Let $h : \mathscr{C} \rightarrow \mathbb{R}$ be the Möbius transform of the Möbius valuation f. By assumption, $h(0) = 0$, and therefore, thanks to Corollary 4.11, $\sum_{c \in \mathfrak{C}} h(c) = 1$. Since $h > 0$ on $\mathfrak{C}$ by assumption, it follows that $h\big|_{\mathfrak{C}}$ defines a probability distribution on $\mathfrak{C}$.

Furthermore, the normalization factor defined by

$$g(c) = \sum_{c' \in \mathfrak{C} \,:\, c \rightarrow c'} h(c')$$

is non-zero on $\mathfrak{C}$. Hence the stochastic matrix $P = (P_{c,c'})_{(c,c') \in \mathfrak{C} \times \mathfrak{C}}$ is well defined by

$$P_{c,c'} = \begin{cases} h(c')/g(c), & \text{if } c \rightarrow c' \\ 0, & \text{if } \neg(c \rightarrow c') \end{cases}.$$

Let $\mathbb{Q}$ be the probability measure on the space $(\Omega, \mathfrak{G})$, corresponding to the law of the Markov chain on $\mathfrak{C}$ with $h\big|_{\mathfrak{C}}$ as initial measure and with P as transition matrix. Let finally $\mathbb{P}$ be the probability measure on $(\partial \mathcal{M}, \mathfrak{F})$ associated with $\mathbb{Q}$. Then we claim that $\mathbb{P}(\uparrow u) = f(u)$ holds for all traces $u \in \mathcal{M}$.

First, we observe that, for any integer $n \geq 1$ and any sequence of cliques $\delta_1 \to \cdots \to \delta_n$, the following identity holds:

$$\mathbb{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = f(\delta_1) \cdots f(\delta_{n-1}) h(\delta_n). \qquad (46)$$

Indeed, $h(0) = 0$ by assumption, and this implies $h = fg$ on $\mathscr{C}$ according to Proposition 4.12. Using the form of the transition matrix P and the definition of the initial law of the chain $(C_n)_{n \geq 1}$, we have thus:

$$\mathbb{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = h(\delta_1) \frac{h(\delta_2)}{g(\delta_1)} \cdots \frac{h(\delta_n)}{g(\delta_n)} = f(\delta_1) \cdots f(\delta_{n-1}) h(\delta_n),$$

which proves (46). We recognize the generalized form of the Möbius transform introduced in (33) for the valuation f, and obtain thus:

$$\mathbb{P}(C_1 = \delta_1, \ldots, C_n = \delta_n) = h(\delta_1 \cdot \ldots \cdot \delta_n). \qquad (47)$$

We now prove $\mathbb{P}(\uparrow u) = f(u)$ for $u \in \mathcal{M}$. Trivially, $\mathbb{P}(\uparrow 0) = f(0) = 1$. Let u be a non-empty trace, and let $n = \tau(u)$ be the height of u. It follows from (22) stated in Proposition 4.5 that we have:

$$\mathbb{P}(\uparrow u) = \mathbb{P}(C_1 \cdot \ldots \cdot C_n \geq u). \qquad (48)$$

The random trace $C_1 \cdot \ldots \cdot C_n$ ranges over traces of height n. Combining (47) and (48) yields thus:

$$\mathbb{P}(\uparrow u) = \sum_{u' \in \mathcal{M} \,:\, \tau(u') = \tau(u), \, u' \geq u} h(u').$$

By Proposition 4.13, we deduce that $\mathbb{P}(\uparrow u) = f(u)$, as claimed. This completes the proofs of point 2 of Theorem 2.4 and of point 2 of Theorem 2.5.

5.3—Uniform measures: existence and uniqueness

This section is devoted to the proof of Theorem 2.6 and of Proposition 2.7.

We consider an irreducible trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$, and we let p_0 be the unique root of smallest modulus of $\mu_{\mathcal{M}}$, which is well defined according to Theorem 4.7. Let also $f_0(u) = p_0^{|u|}$ be the uniform valuation associated to p_0.

Existence of a uniform Bernoulli measure. We aim at applying Theorem 2.4 to obtain the existence of a probability measure $\mathbb{P}$ on $\partial \mathcal{M}$ such that $\mathbb{P}(\uparrow \cdot) = f_0(\cdot)$ on $\mathcal{M}$.

Accordingly, we only have to check that the uniform valuation f_0 is a Möbius valuation. As already noted in § 2.5, if $h : \mathscr{C} \to \mathbb{R}$ is the Möbius transform of f_0, the condition $h(0) = 0$ is equivalent to p_0 being a root of $\mu_{\mathcal{M}}$, which is fulfilled. According to the equivalence stated in Definition 2.3, the condition $h > 0$ on $\mathfrak{C}$ amounts to check that $\mu_{\mathcal{M}_c}(p_0) > 0$ for all $c \in \mathfrak{C}$, and this derives from Proposition 4.9, since $\mathcal{M}$ is assumed to be irreducible. Hence f_0 is indeed a Möbius valuation, which implies the existence of the desired probability measure.

Uniqueness of the uniform measure. The uniqueness of uniform probability measures entails the uniqueness of Bernoulli uniform measures, hence we restrict ourselves to proving the following: if $(\gamma_n)_{n\geq 0}$ is a sequence of real numbers such that

$$\forall n \geq 0 \quad \forall u \in \mathcal{M} \quad |u| = n \implies \mathbb{P}(\uparrow u) = \gamma_n, \qquad (49)$$

then $\gamma_n = p_0^n$ for all $n \geq 0$.

Let $\lambda_n = \lambda_{\mathcal{M}}(n)$ denote the number of traces of length n in $\mathcal{M}$ for all integer $n \geq 0$. Consider the following two generating series:

$$G(X) = \sum_{n \geq 0} \lambda_n X^n, \qquad\qquad S(X) = \sum_{n \geq 0} \gamma_n X^n. \qquad (50)$$

According to Theorem 4.7 point 1, we have $G(X) = 1/\mu_{\mathcal{M}}(X)$ where $\mu_{\mathcal{M}}(X)$ is the Möbius polynomial of $\mathcal{M}$. By developing $G(X)\mu_{\mathcal{M}}(X)$, we obtain in particular:

$$\forall n \geq \max_{c \in \mathscr{C}} |c| \qquad \sum_{c \in \mathscr{C}} (-1)^{|c|} \lambda_{n-|c|} = 0. \qquad (51)$$

Now let us turn our attention to $S(X)$. According to Proposition 2.1, we have: $\sum_{c \in \mathscr{C}} (-1)^{|c|} \mathbb{P}(\uparrow(u \cdot c)) = 0$ for all $u \in \mathcal{M}$. Using (49), it translates as:

$$\forall n \geq 0 \qquad \sum_{c \in \mathscr{C}} (-1)^{|c|} \gamma_{n+|c|} = 0. \qquad (52)$$

In view of (51) and (52), we are steered to consider $G(X)$ and $S(X)$ as being sort of dual. We are going to build upon this.

Equation (52) can be rewritten as $\gamma_n = \sum_{c \in \mathfrak{C}} (-1)^{|c|+1} \gamma_{n+|c|}$. By injecting this identity in $S(X)$, we get

$$
\begin{aligned}
S(X) &= \sum_{n \geq 0} \Big(\sum_{c \in \mathfrak{C}} (-1)^{|c|+1} \gamma_{n+|c|} \Big) X^n \\
&= \sum_{c \in \mathfrak{C}} (-1)^{|c|+1} X^{-|c|} \Big(S(X) - \sum_{i=0}^{|c|-1} \gamma_i X^i \Big).
\end{aligned}
$$

Collecting the different terms involving $S(X)$, we recognize the coefficients of the Möbius polynomial $\mu_{\mathcal{M}}(X)$ and obtain:

$$S(X)\mu_{\mathcal{M}}(1/X) = \sum_{c \in \mathfrak{C}} (-1)^{|c|} X^{-|c|} \Big(\sum_{i=0}^{|c|-1} \gamma_i X^i \Big).$$

Note that this proves already that $S(X)$ is rational.

Set $\ell = \max_{c \in \mathscr{C}} |c|$. Then $\mu_{\mathcal{M}}(X)$ is a polynomial of degree ℓ. Let $p_0, \ldots, p_{\ell-1}$ be the roots of $\mu_{\mathcal{M}}(X)$, with $p_0 < |p_1| \leq |p_2| \leq \ldots \leq |p_{\ell-1}|$. Denoting by "$\propto$" the proportionality relation, we have $\mu_{\mathcal{M}}(1/X) \propto X^{-\ell}(1-p_0 X) \cdots (1-p_{\ell-1} X)$, which yields:

$$S(X) \propto \frac{P(X)}{(1-p_0 X) \cdots (1-p_{\ell-1} X)} \propto \frac{P(X)}{(X - 1/p_0) \cdots (X - 1/p_{\ell-1})}, \qquad (53)$$

$$P(X) = \sum_{c \in \mathfrak{C}} (-1)^{|c|} X^{\ell-|c|} \sum_{i=0}^{|c|-1} \gamma_i X^i.$$

We observe that P is a polynomial of degree at most $\ell - 1$.

Let w be a trace as in the hat lemma 4.17, that is, satisfying (44). Set $|w| = q$. Define, for all integers $n \geq q$, the set $\mathcal{D}_n = \{u \cdot w \mid u \in \mathcal{M}_{n-q}\}$ where $\mathcal{M}_k = \{u \in \mathcal{M} \; : \; |u| = k\}$ for all integers $k \geq 0$. Observe that $\mathcal{D}_n \subseteq \mathcal{M}_n$ and, by cancellativity of the trace monoid $\mathcal{M}$, that $\mathcal{D}_n$ is in bijection with $\mathcal{M}_{n-q}$. Hence:

$$|\mathcal{D}_n| = |\mathcal{M}_{n-q}| \sim_{n \to \infty} C_1 \, (1/p_0)^n \,, \tag{54}$$

for some constant $C_1 > 0$, according to Theorem 4.7 point 3. The cylinders $\uparrow u$ for u ranging over $\mathcal{D}_n$ are disjoint by construction of w, we have thus $\sum_{u \in \mathcal{D}_n} \mathbb{P}(\uparrow u) \leq 1$. But, according to (49), we have $\sum_{u \in \mathcal{D}_n} \mathbb{P}(\uparrow u) = |\mathcal{D}_n| \gamma_n$. So we get $|\mathcal{D}_n| \cdot \gamma_n \leq 1$. Using (54), we obtain

$$\forall n \geq 0 \qquad \gamma_n \leq C_2 \, p_0^n \,, \tag{55}$$

for some constant $C_2 > 0$.

Returning to the expression (53) for S, the roots of the denominator are: $1/|p_{\ell-1}| \leq 1/|p_{\ell-2}| \leq \cdots \leq 1/|p_1| < 1/p_0$. Hence, would any of the roots $1/p_j$ with $j > 0$ not be a root of the numerator P, that would prevent (55) to hold. Since P is of degree at most $\ell - 1$, we deduce that $1/p_{\ell-1}, \ldots, 1/p_1$ are exactly all the roots of P, and (53) rewrites as $S(X) = K/(1 - p_0 X)$ for some constant $K \neq 0$. Evaluating both members at $X = 0$ yields $K = 1$ since $\gamma_0 = 1$, and thus $S(X) = 1/(1 - p_0 X)$. Since $S(X) = \sum_{n \geq 0} \gamma_n X^n$ by definition, we obtain that $\gamma_n = p_0^n$ for all $n \geq 0$, and the proof is complete.

Proof of Proposition 2.7. Let p_0 be the unique root of smallest modulus of the Möbius polynomial. Start with the valuation f defined by $f(\alpha) = p_0$ for all α in Σ. By Theorem 2.6, f is a Möbius valuation.

Now consider a collection of reals $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \Sigma}$ such that $p_0 + \varepsilon_\alpha \in (0, 1)$ for all $\alpha \in \Sigma$. Let f_ε be the valuation defined by: $f_\varepsilon(\alpha) = p_0 + \varepsilon_\alpha$ for all $\alpha \in \Sigma$. Let h_ε be the associated Möbius transform. The goal is to show that there exist a continuous family of values for $(\varepsilon_\alpha)_{\alpha \in \Sigma}$ such that f_ε is Möbius, that is:

$$(i) \; h_\varepsilon(0) = 0 \,, \qquad\qquad (ii) \; \forall c \in \mathfrak{C} \quad h_\varepsilon(c) > 0 \,.$$

First, observe that condition (ii) is an open condition and that it is satisfied for $\varepsilon = 0$. So it is still satisfied if $|\varepsilon_\alpha|$ is small enough, for all $\alpha \in \Sigma$.

Now let us concentrate on (i). Fix a letter $a \in \Sigma$. The equation $h_\varepsilon(0) = 0$ is an affine equation in ε_a if the values ε_α for $\alpha \neq a$ are fixed:

$$(p_0 + \varepsilon_a) A_\varepsilon + B_\varepsilon = 0 \,,$$

with

$$A_\varepsilon = \sum_{c \in \mathscr{C} \, : \, a \in c} (-1)^{|c|} \prod_{\alpha \in c, \alpha \neq a} (p_0 + \varepsilon_\alpha) \,, \quad B_\varepsilon = \sum_{c \in \mathscr{C} \, : \, a \notin c} (-1)^{|c|} \prod_{\alpha \in c} (p_0 + \varepsilon_\alpha) \,.$$

Observe that we have:

$$A_0 = \sum_{c \in \mathscr{C} \, : \, a \in c} (-1)^{|c|} p_0^{|c|-1} \,, \quad B_0 = \sum_{c \in \mathscr{C} \, : \, a \notin c} (-1)^{|c|} p_0^{|c|} \,.$$

We recognize in B_0 the Möbius polynomial of the independence pair (Σ', I'), with $\Sigma' = \Sigma \setminus \{a\}$ and $I' = I \cap (\Sigma' \times \Sigma')$, evaluated at p_0. But $\mathcal{M}$ is irreducible,

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and as already observed in the proof of Proposition 4.9, the comparison of growth rates entails that p_0 is strictly smaller in modulus than all the roots of the polynomial $\mu_{\mathcal{M}(\Sigma', I')}$. Hence $B_0 \neq 0$. Since $p_0 A_0 + B_0 = 0$, we conclude that $A_0 \neq 0$ and thus $A_\varepsilon \neq 0$ for ε small enough. Consequently, the equation $(p_0 + \varepsilon_a) A_\varepsilon + B_\varepsilon = 0$ has a unique solution in ε_a if all ε_α are small enough for $\alpha \neq a$. Since $|\Sigma| > 1$ by assumption, there is indeed an uncountable number of values for $(\varepsilon_\alpha)_{\alpha \neq a}$ arbitrarily close to 0.

We have proved the existence of a continuous family of distinct Möbius valuations, and we conclude by Theorem 2.4.

6—Conclusion and perspectives

The paper has introduced and characterized Bernoulli measures on irreducible trace monoids, interpreted as a probabilistic model of concurrent systems with a memoryless property. The combinatorics of trace monoids plays a central role in the characterization of Bernoulli measures, and in particular the notion of Möbius transform and of Möbius polynomial. The existence and uniqueness of uniform measures has been established, as well as the fact that they belong to the class of Bernoulli measures. A realization result allows for effective sampling of Bernoulli measures, paving the way for future applications.

The extension to non-irreducible trace monoid works out nicely and the complete description of Bernoulli measures is postponed to a future work.

Further developments of this work can be expected. First, it is natural to adapt our construction to trace groups. It would also be interesting to generalize our approach to other monoids or groups. Braid monoids and groups, and more generally Artin monoids and groups of finite Coxeter type, are natural candidates.

Another extension consists in studying Markovian measures on infinite traces instead of Bernoulli measures. Applications to the construction of Markovian measures for the executions of 1-bounded Petri nets are expected. In this case, the executions are described as a regular trace language.

References

[1]   S. Abbes. "On countable completions of quotient ordered semigroups". In: *Semigroup Forum* 3.77 (2008), pp. 482–499.

[2]   S. Abbes and J. Mairesse. *Uniform and Bernoulli measures on the boundary of trace monoids.* `arXiv 1407.5879`. http://arxiv.org/abs/1407.5879. 2014.

[3]   A. Bertoni and R. Radicioni. "Approximating the mean speedup in trace monoids". In: *Internat. J. Found. Comput. Sci.* 19.3 (2008), pp. 497–511.

[4]   A. Bertoni et al. "Counting techniques for inclusion, equivalence and membership problems". In: *The Book of Traces.* Ed. by V. Diekert and G. Rozenberg. World Scientific, 1994. Chap. 5, pp. 131–163.

[5]   V. Blondel, S. Gaubert, and J. Tsitsiklis. "Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard". In: *IEEE Trans. Automat. Control* 45.9 (2000), pp. 1762–1765.

[6] P. Brémaud. *Markov chains: Gibbs fields, Monte Carlo simulations, and queues.* Springer, 1999.

[7] P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements.* Vol. 85. Lecture Notes in Math. Springer, 1969.

[8] C. Cérin and A. Petit. "Speedup of recognizable trace languages". In: *Proc. Math. Found. Comput. Sci.* Lecture Notes in Comput. Sci. 711. Springer, 1993, pp. 332–341.

[9] K.L. Chung. *Markov chains with stationary transition probabilities.* Springer, 1960.

[10] M. Coornaert. "Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov". In: *Pacific J. Math.* 159.2 (1993), pp. 241–270.

[11] V. Diekert. *Combinatorics on traces.* Vol. 454. Lecture Notes in Comput. Sci. Springer, 1990.

[12] V. Diekert and G. Rozenberg, eds. *The Book of traces.* World Scientific, 1995.

[13] M. Goldwurm and M. Santini. "Clique polynomials have a unique root of smallest modulus". In: *Inform. Process. Lett.* 75.3 (2000), pp. 127–132.

[14] P. Haas. *Stochastic Petri Nets. Modelling, stability, simulation.* Springer, 2002.

[15] J. Hillston. *A compositional approach to performance modelling.* Cambridge Univ. Press, 1996.

[16] M.R. Jerrum, L.G. Valliant, and V.V. Vazirani. "Random generation of combinatorial structures from a uniform distribution". In: *Theoret. Comput. Sci.* 43 (1986), pp. 169–188.

[17] V. Kaimanovich. "Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds". In: *Ann. Inst Henri Poincaré (A)* 53.4 (1990), pp. 361–393.

[18] D. Krob, J. Mairesse, and I. Michos. "Computing the average parallelism in trace monoids". In: *Discrete Math.* 273 (2003), pp. 131–162.

[19] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding.* Cambridge Univ. Press, 1995.

[20] J. Mairesse. "Random walks on groups and monoids with a Markovian harmonic measure". In: *Electron. J. Probab.* 10 (2005), pp. 1417–1441.

[21] M.O. Rabin. "Probabilistic Automata". In: *Inform. Control* 6.3 (1963), pp. 230–245.

[22] G.-C. Rota. "On the foundations of combinatorial theory I. Theory of Möbius functions". In: *Z. Wahrscheinlichkeitstheorie* 2 (1964), pp. 340–368.

[23] B. Rozoy and P.S. Thiagarajan. "Event structures and trace monoids". In: *Theoret. Comput. Sci.* 91 (1991), pp. 285–313.

[24] N. Saheb. "Concurrency measure in commutation monoids". In: *Discrete Appl. Math.* 24 (1989), pp. 223–236.

[25] E. Seneta. *Non-negative matrices and Markov chains. Revised printing.* Springer, 1981.

[26] A. Vershik, S. Nechaev, and R. Bikbov. "Statistical properties of locally free groups with applications to braid groups and growth of random heaps". In: *Comm. Math. Phys.* 212.2 (2000), pp. 469–501.

[27] X. Viennot. "Heaps of pieces, I : basic definitions and combinatorial lemmas". In: *Combinatoire énumérative.* Vol. 1234. Lecture Notes in Math. Springer, 1986, pp. 321–350.

[28] G. Winskel and M. Nielsen. "Models for concurrency". In: *Handbook of Logic in Computer Science.* Ed. by S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum. Vol. 4. Oxford Univ. Press, 1995, pp. 1–148.