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A Cartesian Closed Category of Event Structures with Quotients

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We introduce a new class of morphisms for event structures. The category obtained is cartesian closed, and a natural notion of quotient event structure is defined within it. We study in particular the topological space of maximal configurations of quotient event structures. We introduce the compression of event structures as an example of quotient: the compression of an event structure E is a minimal event structure with the same space of maximal configurations as E.

Keywords: event structure, maximal elements, quotient semantics

1 Introduction

Prime event structures—we say *event structures* for short, always meaning prime event structures—have been introduced by Winskel as a model of concurrent computational processes [14]. Applications of event structures to concurrency theory are numerous, in particular to the theory of Petri nets and to trace theory [11, 13, 12]. An event structure is defined as a triple $(E, \leq, \#)$, where $(E, \leq)$ is a poset whose elements are called events, and $\#$ is a binary symmetric and irreflexive relation satisfying $e_1 \# e_2 \leq e_3 \Rightarrow e_1 \# e_3$. The order relation $\leq$ is called the *causality* relation, while the relation $\#$ is called the *conflict* relation. An event structure E—we identify event structures and sets of events—is called *finitary* if E is a countable set, and if for every $e \in E$, the subset $\downarrow e =_{[\text{def}]} \{e' \in E : e' \leq e\}$ is finite. For E an event structure, the poset $\widehat{E}$ of configurations of E is defined as the set of conflict-free and downward closed subsets of E, ordered by set-theoretic inclusion. Configurations represent the set of executions of the system modeled by the event structure.

Event structures were introduced together with a definition of morphisms between them [15]. A mapping $f : E \to F$ between event structures is a Winskel morphism if $f(x)$ is a configuration of F for every configuration x of E, and if the restriction $f|_x : x \to f(x)$ is injective for every configuration x of E. Remark that each morphism of event structures $f : E \to F$ is associated with a morphism of partial orders $\widehat{f} : \widehat{E} \to \widehat{F}$ between the associated posets of configurations, given by the set-theoretic action of f on subsets. This association is moreover functorial, i.e., $\widehat{\text{Id}_E} = \text{Id}_{\widehat{E}}$ and $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

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In the present paper, we study an alternative definition of morphisms. To motivate this change, consider the following situation: two communicating systems A and B represented by event structures are driven in a non-deterministic fashion by one another. That is to say, the occurrence, say, of a in A, allows the occurrences of b or b' in B, and so on. Assume moreover that the behaviors of B allowed by the occurrence of, say, two events a or a' in A, are the same. For example, a server B may treat clients in a fair manner, so that different requests of A lead to the same behavior of B. Hence, some events of A may be identified from the point of view of B. A natural model for such a situation is to consider an equivalence relation on events of A. This requires having at hand a theory for quotients of event structures.

Unfortunately, there is no hope that the original definition of morphisms allows the projection mapping, from an event structure to its quotient set—assuming we can equip this quotient as an event structure—, to be a morphism. Indeed, morphisms are supposed to be injective when restricted to configurations. Hence, it is not possible to identify two compatible events. We are thus brought to relax the condition that morphisms should map distinct compatible events to distinct events[(i)]. The topic of the following paper is a systematic study of such general morphisms. The definition we introduce for a mapping $f : E \to F$ to be a morphism is the following: f must be order-preserving with respect to the underlying posets, and moreover, $f(e)\#f(e') \Rightarrow e\#e'$. This requires that compatible events are mapped to compatible events, although the set-theoretic action $f(x)$ of f on a configuration x may not be a configuration itself. Therefore, we define $\widehat{f}(x)$, the action of f on a configuration x, as the smallest configuration that contains $f(x)$. The association $f \to \widehat{f}$ is then functorial, i.e., $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

In the sense explained above, our definition relaxes Winskel's definition. However all Winskel morphisms are not morphisms in our sense, since Winskel morphisms are not required to be order-preserving. Our definition is thus simply different. It is based on the idea that a morphism $f : E \to F$ should transport configurations of E into configurations of F, but not necessarily through the set-theoretic action of f.

It turns out that the mathematical framework that we define with our new class of morphisms presents some interesting properties. In particular, the category that we define is cartesian closed, i.e., it has all finite products and exponentiation, and moreover quotients of event structures can be naturally defined within it. The importance of cartesian closed categories in Computer Science has been pointed out by several authors [8, 6]. Indeed, the exponentiation is a natural computational operation, since it represents the operation of *evaluating* a mapping on its argument.

Moreover, we study a sub-category of event structures, where morphisms have the property of mapping maximal configurations to maximal configurations. The motivation for paying a particular attention to the space of maximal configurations of event structures comes from domain theory. It is indeed a classical domain theoretic approach to the computational study of some continuous space S to identify S with the space of maximal elements of a domain, equipped with the restriction of the Scott topology of the domain [9, 5]. For event structures, the space of maximal configurations of the domain of configurations plays moreover a fundamental role in the theory of probabilistic event structures, since it represents the space of complete histories of the system modeled by the event structure [1, 2]. This justifies that we study in detail how the space of maximal configurations of an event structure fits into the present framework.

We show in particular the following result: let $(E, \leq, \#)$ and $(F, \leq, \#)$ be two event structures, and let $f : E \to F$ be a mapping, order-preserving with respect to the underlying posets. Assume that f is onto, and satisfies: $\forall e, e' \in E,\ e\#e' \iff f(e)\#f(e')$. Then the spaces of maximal configurations of E and

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[(i)] Although morphisms like those proposed here have been recently considered by Winskel in [16] as "demand maps" obtained as Kleisli maps through the presence of "persistent" events, no systematic study of such morphisms has been carried out.

F are homeomorphic. The proof makes a fair use of the notion of adjunction pair between DCPO. As an application, we introduce the notion of compression of an event structure: the compression of an event structure E is a quotient of E, that shares the same space of maximal configurations as E.

The paper is organized in three sections. In Section 2, we define the category of event structures that will be our study topic. We review some classical categorical constructions within this category. A subsection analyzes the role of maximal configurations of events structures in this framework. We define quotients of event structure in Section 3, and we give their main properties. The example of compression of event structures is the topic of Section 4. Finally, Section 5 concludes the paper and proposes future work.

2 Event Structures and Morphisms

2.1 Posets and Prime Algebraic Coherent Posets

We rely on the classical notions for partially ordered sets—called *posets* for short. If $(D, \leq)$ is a poset, we do not mention the relation $\leq$ if it is clear from the context. The least upper bound (l.u.b.) of a subset $X \subseteq D$ is denoted $\sup X$ if it exists. A mapping $f : D \to F$ between two posets is said to be *order preserving* if $e \leq e' \Rightarrow f(e) \leq f(e')$ for all $e, e' \in D$. For any subset $A \subseteq D$, we denote by $\downarrow A$ the *downward closure* of A, defined by $\downarrow A = \{e \in D : \exists e' \in A,\, e \leq e'\}$. For singletons, we simply note $\downarrow e = \downarrow \{e\}$. We say that a set A is *downward closed* if $\downarrow A = A$.

A subset $A \subseteq D$ is said to be *upward closed* if for any $a \in A$, any element $b \in D$ such that $b \geq a$ belongs to A. A subset $A \subseteq D$ is said to be *directed* if any two elements $a, b \in A$ have an upper bound in A. An *ideal* is a downward closed directed set. A poset D is called a DCPO (Directed Complete POset) if every directed subset of D has a l.u.b. in D. A subset $A \subseteq D$ of a DCPO D is said to be *Scott-open* if it is upward closed, and if the following holds: for any directed subset X of D, $\sup X \in A \Rightarrow A \cap X \neq \emptyset$. Scott-open sets form a topology on D, called the Scott topology. A function $f : D \to F$ between two DCPO is said to be *Scott-continuous* if f is continuous with respect to the Scott topologies. Equivalently, f is Scott-continuous if and only if f is order-preserving, and if $f(\sup X) = \sup f(X)$ for every directed subset $X \subseteq D$ [6, Prop. II-2.1, p.157]; the later condition may only be verified on ideals instead of directed subsets.

Let $(D, \leq)$ be a DCPO. An element $x \in D$ is said to be *compact* if for every ideal $I \subseteq D$, if $x \leq \sup I$ then $x \in I$. Observe that $\uparrow x = \{y \in D : x \leq y\}$ is then a Scott-open set of D. The DCPO D is said to be *algebraic* when every element $x \in D$ is the l.u.b. of the set of compact elements $y \in D$ such that $y \leq x$.

Quite subtle are the differences between the above concepts, and the following (definitions found in [11]). A subset X of a poset D is said to be *pairwise consistent* if any two elements of X have an upper bound in D. A poset D is said to be *coherent* if every pairwise consistent subset has l.u.b. in D. An element p of a poset is a *complete prime* if for any subset X that has a l.u.b., if $p \leq \sup X$, then there exists $x \in X$ such that $p \leq x$. We denote by $\mathcal{P}(D)$ the subset of complete primes of a poset D. Then a poset D is said to be *prime algebraic* if for every $x \in D$, if we set $D_x = \{p \in \mathcal{P}(D) : p \leq x\}$, then D_x has a l.u.b. and $x = \sup D_x$. Obviously, coherent posets are DCPO and complete primes are compact, thus prime algebraic coherent posets are algebraic DCPO. It is also quite easy to check that any coherent poset is a complete semi-lattice, i.e., every subset has a greatest lower bound (indeed, for every subset X, the set of lower bounds of X, even if empty, is coherent, and has thus a l.u.b., which is the greatest lower bound of X).

We introduce the category $\mathbf{PrAlg}$, with prime algebraic coherent posets as objects, and with morphisms between two objects D and D' the mappings $f : D \to D'$ such that:

1. f preserves complete primes, i.e., $f\big(\mathcal{P}(D)\big) \subseteq \mathcal{P}(D')$;
2. $f(x) = \sup f(D_x)$ for every $x \in D$.

Note that the existence of $\sup f(D_x)$ follows from D' being coherent. Observe also that f is then order-preserving; for, if $x \leq y$, then $D_x \subseteq D_y$ and thus $f(x) \leq f(y)$ by property 2 above. A first result is the following:

Proposition 2.1 *Any arrow in* $\mathbf{PrAlg}$ *is Scott-continuous.*

Proof: Let $f : D \to D'$ be an arrow in $\mathbf{PrAlg}$, let I be an ideal of D, and let $d = \sup I$. We show that $f(d) = \sup f(I)$. Put $D_d = \{p \in \mathcal{P}(D) : p \leq d\}$. Then $D_d \subseteq I$ since I is an ideal and since complete primes are compact, and $\sup D_d = d$ holds. Therefore: $f(d) = \sup f(D_d) \leq \sup f(I)$. Since $f(d) \geq \sup f(I)$ is trivially true, the expected equality holds. Since f is known to be order-preserving, this shows that f is Scott-continuous. □

Observe that Scott-continuity is not enough *a priori* to guarantee that $f(x) = \sup f(D_x)$ holds, since the set D_x shall not be directed in general.

2.2 A Category of Event Structures

Event structures that we consider are defined as follows. Observe that the finitary assumption is considered separately.

Definition 2.1 (event structures and morphisms) *An* event structure *is a triple* $(E, \leq, \#)$*, where* $(E, \leq)$ *is a poset, called the* underlying poset *of the event structure, and* $\#$ *is a symmetric and irreflexive binary relation on* E *satisfying the following*

$$\text{inheritance axiom:} \qquad \forall e_1, e_2, e_3 \in E, \quad (e_1 \# e_2,\ e_2 \leq e_3) \Rightarrow e_1 \# e_3\,.$$

Elements of E *are called* events*. We identify the event structure* $(E, \leq, \#)$ *with the set of events* E *when no confusion occurs on the relations* $\leq$ *and* $\#$ *involved. The event structure* E *is said to be* finitary *if* E *is countable, and if* $\downarrow e$ *is a finite subset of* E *for every* $e \in E$*.*

A mapping $f : E \to F$ *between two event structures is said to be a* morphism of event structures*, or shortly a* morphism*, if* f *is order-preserving between the underlying posets and if moreover* f *reflects conflict, in the following sense:*

$$\forall e_1, e_2 \in E, \quad f(e_1) \# f(e_2) \Rightarrow e_1 \# e_2. \tag{1}$$

We denote by $\mathrm{Id}_E : E \to E$ *the identity morphism.*

A direct consequence of Definition 2.1 is the following:

Proposition 2.2 *Morphisms of event structures compose: if* $g : F \to G$ *and* $f : E \to F$ *are two morphisms of event structures, so is* $g \circ f : E \to G$*. Hence event structures form a category with the morphisms of Definition* 2.1*, denoted* $\mathbf{ES}$.

Note that finitary event structures form a full sub-category of $\mathbf{ES}$.

Configurations of an event structure are understood as the states of the event structure. They are defined as follows:

Definition 2.2 (compatible events, configurations, action of morphisms on configurations) *Two events e, e' of an event structure E are said to be* compatible *if $e\#e'$ does* not *hold. A subset $x \subseteq E$ is said to be a* configuration *if x is downward-closed and if any two events in x are compatible. By extension, an event e is said to be compatible with a configuration $x \subseteq E$ whenever $x \cup \downarrow e$ is a configuration of E.*

Configurations are ordered by set-theoretic inclusion. We denote by $\widehat{E}$ the poset of configurations of E.

If $f : E \to F$ is a morphism of event structures, we define $\widehat{f} : \widehat{E} \to \widehat{F}$ by:

$$\forall x \in \widehat{E}, \quad \widehat{f}(x) = \downarrow f(x), \tag{2}$$

where $f(x) = \{f(e),\ e \in x\}$ is the set-theoretic action of f on x.

Note that $\widehat{f} : \widehat{E} \to \widehat{F}$ is well defined thanks to property (1) of morphisms.

It is well known that, for each event structure E, $\widehat{E}$ is a coherent prime algebraic poset [11, Th.9, p.102], see also [4]. More precisely, the association $E \to \widehat{E}$ is one-to-one and onto, up to isomorphism, from event structures to prime algebraic coherent posets. In $\widehat{E}$, existing l.u.b. are given by set-theoretic union, and greatest lower bounds are given by set-theoretic intersection. Moreover, for any event structure E, the complete prime elements of $\widehat{E}$ are exactly the configurations of the form $\downarrow e$, with e ranging over E, and the compact elements are the bounded, finite unions of complete primes. With our definition of morphisms, this leads to the following:

Proposition 2.3 *There is a functor $\mathbf{F} : \mathbf{ES} \to \mathbf{PrAlg}$, defined by $\mathbf{F}(E) = \widehat{E}$ on objects, and by $\mathbf{F}(f) = \widehat{f}$ on morphisms. The functor $\mathbf{F}$ determines an equivalence of categories.*

Proof: $\mathbf{F} : \mathbf{ES} \to \mathbf{PrAlg}$ *is a functor.* The fact that $\mathbf{F}(E) = \widehat{E}$ is a prime algebraic coherent poset for every event structure E follows from the result already mentioned of [11]. We show that $\mathbf{F}(f)$ is a morphism of $\mathbf{PrAlg}$ for every morphism $f : E \to F$ of $\mathbf{ES}$. A direct inspection of (2) shows that $\widehat{f} : \widehat{E} \to \widehat{F}$ is order-preserving. Moreover $\widehat{f}$ maps complete prime elements to complete prime elements. Indeed, since f is order-preserving, we have for every $e \in E$:

$$\widehat{f}(\downarrow e) = \downarrow f(\downarrow e) = \downarrow f(e).$$

It remains to show that $\widehat{f}$ satisfies $\widehat{f}(x) = \sup \widehat{f}(D_x)$ for every $x \in \widehat{E}$, where $D_x = \{p \in \mathcal{P}(\widehat{E}) \ : \ p \subseteq x\}$. Thanks to (2), we have:

$$\widehat{f}(x) = \bigcup_{e \in x} \downarrow f(e) = \bigcup_{e \in x} \widehat{f}(\downarrow e) = \sup \widehat{f}(D_x).$$

Hence $\mathbf{F}(f) : \mathbf{F}(E) \to \mathbf{F}(F)$ is a morphism of $\mathbf{PrAlg}$ for every morphism $f : E \to F$ of $\mathbf{ES}$.

It is clear that $\mathbf{F}(\mathrm{Id}_E) = \mathrm{Id}_{\mathbf{F}(E)}$. To complete the proof that $\mathbf{F}$ is a functor, we show that $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$ for any pair of morphisms $f : E \to F$ and $g : F \to G$. Let $h = g \circ f$. By (2), we have for any $x \in \widehat{E}$:

$$\widehat{h}(x) = \bigcup_{e \in x} \downarrow h(e) = \bigcup_{e \in x} \downarrow g \circ f(e). \tag{3}$$

Let $y = \widehat{f}(x) = \bigcup_{e\in x} \downarrow f(e)$. We have:

$$\widehat{g}(y) = \bigcup_{e'\in y} \downarrow g(e') = \bigcup_{e\in x}\Bigg(\bigcup_{e'\in\downarrow f(e)} \downarrow g(e')\Bigg). \tag{4}$$

From (3) and (4) it is clear that $\widehat{h}(x) \subseteq \widehat{g}(y)$. We prove the converse inclusion. For every $e \in x$ and every $e' \in \downarrow f(e)$, we have $e' \le f(e)$ and thus $\downarrow g(e') \subseteq \downarrow g \circ f(e)$ since g is order-preserving. Hence $\widehat{g}(y) \subseteq \widehat{h}(x)$, which means $\widehat{g} \circ \widehat{f}(x) \subseteq \widehat{h}(x)$, as claimed. This shows that $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$, and thus $\mathbf{F} : \mathbf{ES} \to \mathbf{PrAlg}$ is a functor, what was to be shown.

Definition of a functor which will later prove to be the adjoint of $\mathbf{F}$ *in an equivalence of categories*. Let $\mathbf{G} : \mathbf{PrAlg} \to \mathbf{ES}$ be defined as follows: if D is an object of $\mathbf{PrAlg}$, $\mathbf{G}(D)$ is defined as the natural event structure with events the set of complete prime elements of D, see [11] for details. If $u : D \to D'$ is a morphism of $\mathbf{PrAlg}$, $\mathbf{G}(u) : \mathbf{G}(D) \to \mathbf{G}(D')$ is defined as follows: for any $e \in \mathbf{G}(D)$, $\mathbf{G}(u)(e)$ is defined as the unique event $e' \in \mathbf{G}(D')$ such that $u(\downarrow e) = \downarrow e'$, which is well defined since by definition of morphisms in $\mathbf{PrAlg}$, u maps complete prime elements to complete prime elements.

We show that $\mathbf{G}$ is a functor. Clearly, $\mathbf{G}(\mathrm{Id}_D) = \mathrm{Id}_{\mathbf{G}(D)}$ holds for any object D in $\mathbf{PrAlg}$. Let $u : D \to D'$ and $v : D' \to D''$ be arrows in $\mathbf{PrAlg}$, and let $f = \mathbf{G}(u)$, $g = \mathbf{G}(v)$, and $h = \mathbf{G}(v \circ u)$. We show that $h = g \circ f$. Let e be any event in E. By definition of $\mathbf{G}$ we have:

$$\downarrow f(e) = u(\downarrow e), \quad \downarrow g\big(f(e)\big) = v\big(\downarrow f(e)\big), \quad v \circ u(\downarrow e) = \downarrow h(e).$$

Therefore:

$$\downarrow h(e) = v\big(u(\downarrow e)\big) = v\big(\downarrow f(e)\big) = \downarrow g \circ f(e),$$

and thus $h(e) = g \circ f(e)$. This shows that $\mathbf{G}(v \circ u) = \mathbf{G}(v) \circ \mathbf{G}(u)$, hence $\mathbf{G}$ is a functor.

The pair $(\mathbf{F}, \mathbf{G})$ *defines an equivalence of categories*. We show that there are two natural isomorphisms

$$\phi : \mathbf{Id_{ES}} \to \mathbf{G} \circ \mathbf{F}, \quad \text{and} \quad \psi : \mathbf{F} \circ \mathbf{G} \to \mathbf{Id_{PrAlg}}.$$

For any event structure E, $\phi_E : E \to \mathbf{G} \circ \mathbf{F}(E)$ is the natural isomorphism that maps an event $e \in E$ to $\downarrow e$, seen as an event in $\mathbf{G}(\widehat{E})$.

For any object D in $\mathbf{PrAlg}$, $\psi_D : D \to \mathbf{F} \circ \mathbf{G}(D)$ is defined by $\psi_D(x) = \{p \in \mathcal{P}(D) : p \le x\}$, for any $x \in D$. Then ψ_D is obviously an isomorphism in $\mathbf{PrAlg}$. It remains only to show that ψ thus defined is natural, i.e., for any arrow $u : D \to D'$ in $\mathbf{PrAlg}$, the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "\psi_D"] \arrow[d, "u"'] & \mathbf{F} \circ \mathbf{G}(D) \arrow[d, "\mathbf{F}\circ\mathbf{G}(u)"] \\ D' \arrow[r, "\psi_{D'}"'] & \mathbf{F} \circ \mathbf{G}(D') \end{tikzcd}$$

In other words, we must prove $v \circ \psi_D = \psi_{D'} \circ u$, where $v = \mathbf{F} \circ \mathbf{G}(u)$. Since all four arrows belong to $\mathbf{PrAlg}$, it is enough to show:

$$\forall q \in \mathcal{P}(D), \quad v \circ \psi_D(q) = \psi_{D'} \circ u(q). \tag{5}$$

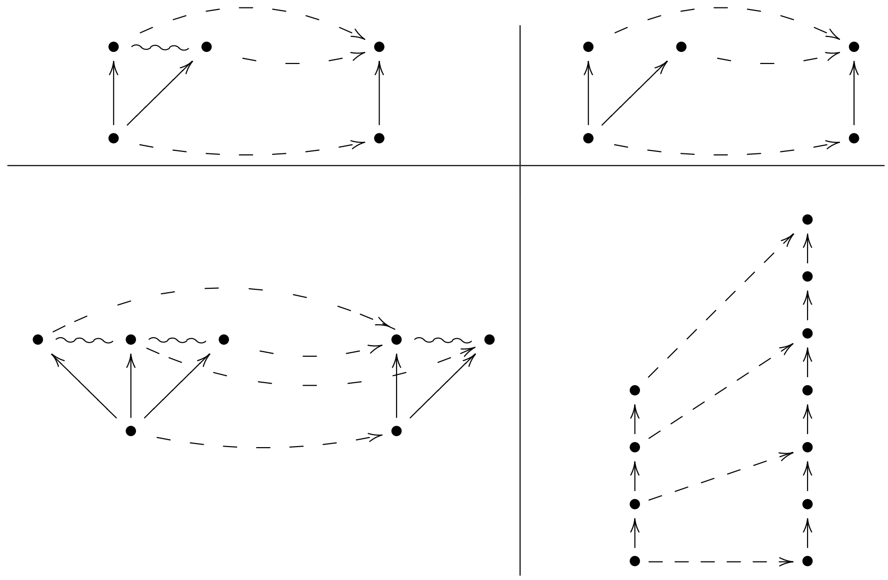

Fig. 1: *Examples of morphisms of event structures. Dotted arrows depict the mappings.*

Then, for any $q \in \mathcal{P}(D)$, we find that both members in (5) are equal to $\{p \in \mathcal{P}(D') \;:\; p \leq u(q)\}$, which completes the proof. $\square$

As a consequence of Proposition 2.3, everything that is stated below on the category $\mathbf{ES}$ can be interpreted as statements on the category $\mathbf{PrAlg}$. We believe however that the event structure representation is more intuitive. Proposition 2.3 can be seen as a further mathematical justification of our definition for morphisms of event structure.

2.3 Examples

Graphical conventions for drawing event structures. We depict event structures as follows (see Figure 1). Events are depicted by bullets. The causality relation is the reflexive transitive closure of the relation depicted by the arcs. Some conflicts are depicted by wave arcs. The whole conflict relation is obtained by inheritance from the conflicts depicted.

Figure 1 depicts 4 examples of morphisms of event structures.

Winskel morphisms are defined as follows [15]: $f : E \to F$ is a *Winskel morphism* if $f(x)$ is a configuration of F for any configuration x of E, and if the restriction $f|_x : x \to f(x)$ is injective for every configuration x.

A Winskel morphism is not necessarily a morphism in the sense of Definition 2.1, since it may not be order-preserving. Conversely, the morphisms that we study may not be Winskel morphisms. For example, the morphisms depicted in Figure 1 in the upper-right and lower-right corners are not Winskel. The two other morphisms are both Winskel morphisms and morphisms in the sense of Definition 2.1.

Winskel's *rigid morphisms* are defined as follows [15]: a Winskel morphism $f : E \to F$ is said to be *rigid* if:

$$\forall x \in \widehat{E}, \quad \forall y \in \widehat{F}, \quad y \subseteq f(x) \Rightarrow \exists z \in \widehat{E} \,:\, z \subseteq x,\, f(z) = y.$$

This implies in particular that a rigid morphism is order-preserving. Moreover any Winskel morphism satisfies $\forall e, e' \in E, \quad f(e)\#f(e') \Rightarrow e\#e'$. Indeed, if $e, e' \in E$ are not in conflict, there is a $x \in \widehat{E}$ such that $e, e' \in x$. Then $f(x)$ is a configuration of F that contains $f(e)$ and $f(e')$, hence $f(e)$ and $f(e')$ are not in conflict. We obtain thus: *Winskel's rigid morphisms are morphisms in the sense of Definition 2.1. They satisfy:* $\forall x \in \widehat{E},\ \widehat{f}(x) = f(x)$.

The later point illustrates the idea that we use to build our morphisms: a morphism f of event structures should act on configurations, but not necessarily through the set-theoretic action of f, as it occurs for Winskel's morphisms.

2.4 Categorical Constructions for Event Structures

In this subsection, we review some classical constructions from category theory, for the category $\mathbf{ES}$. As discussed in the Introduction, the most important fact is that the category $\mathbf{ES}$ is cartesian closed.

The category theoretic definitions involved are briefly recalled (initial and final objects, equalizers, products, coproducts, pullbacks and exponentiation). The reader is referred for example to [10] for more details. The proofs consist of direct verifications that the conditions stated in Definition 2.1 are satisfied. Most of them do not present difficulty and are left to the reader as an exercise, except for the exponentiation, treated in details. The categorical properties of $\mathbf{ES}$ that we obtain are summarized in the following theorem.

Theorem 2.1 *The category* $\mathbf{ES}$ *has initial and final objects, which are both finitary. It has all finite products and coproducts, it has equalizers and thus pullbacks, and is moreover cartesian closed. Products, coproducts, equalizers and thus pullbacks preserve the finitary property.*

The constructions are detailed below.

Initial and Final Event Structures. The category $\mathbf{ES}$ has both an initial and a final object. The initial object is the empty event structure, denoted $\mathbf{0}$, defined as the unique event structure with the empty set as set of events. For any event structure E, there is a unique morphism $\mathbf{0} \to E$.

A final object is any event structure with a singleton as set of events, and with the unique causality and conflict relations making an event structure from it. We denote it $\mathbf{1}$. For any event structure E, there is a unique morphism $E \to \mathbf{1}$.

Sub-structures and equalizers. Let F be an event structure, and let E be a subset of F. Equip E with the causality $\leq|_E$ and with the conflict $\#|_E$, defined by:

$$\leq|_E = \leq \cap (E \times E), \qquad \#|_F = \# \cap (E \times E).$$

Then $(E, \leq|_E, \#|_E)$ is an event structure, we call it a *sub-structure* of F. The injection mapping $i : E \to F$ is a morphism of event structures. If E is finitary, so is any sub-structure of E.

Sub-structures allow to define equalizers. Let E, F be two event structures, and let $f, g : E \to F$ two morphisms. Define the substructure of E:

$$L = \{e \in E \,:\, f(e) = g(e)\}.$$

Then L with the injection $i : L \to E$ is the equalizer of f, g. That is to say, for any event structure H, and any morphism $h : H \to E$ such that $f \circ h = g \circ h$, there is a unique morphism $k : H \to L$ such that

$i \circ k = h$, making the following diagram commutative:

$$
\begin{tikzcd}
L \arrow[dr, "i"] & & \\
& E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & F \\
H \arrow[uu, "k"] \arrow[ur, "h"'] & &
\end{tikzcd}
$$

If E is finitary, so is the equalizer of any pair $f, g : E \to F$.

Products, Coproducts, Pullbacks. We consider two event structures $(E_1, \leq_1, \#_1)$ and $(E_2, \leq_2, \#_2)$.
The *product* of E_1 and E_2 is defined as the event structure with $E =_{[\text{def}]} E_1 \times E_2$ as set of events, and which causality and conflict relations are defined by:

- *Causality:* $(e_1, e_2) \leq (e'_1, e'_2)$ if $e_1 \leq_1 e'_1$ and $e_2 \leq_2 e'_2$;
- *Conflict:* $(e_1, e_2)\#(e'_1, e'_2)$ if $e_1 \#_1 e'_1$ or $e_2 \#_2 e'_2$.

$E_1 \times E_2$ is an event structure, and the projections $p_1 : E \to E_1$ and $p_2 : E \to E_2$ are two morphisms. The product thus defined is indeed the categorical product in **ES**: for any event structure G, and pair of morphisms $f : G \to E_1$, $g : G \to E_2$, there is a unique morphism $h : G \to E$ making the following diagram commutative:

$$
\begin{tikzcd}
E_1 & E \arrow[l, "p_1"'] \arrow[r, "p_2"] & E_2 \\
& G \arrow[ul, "f"] \arrow[u, "h"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

The construction of products allows the standard construction, for any two morphisms $f : E \to F$ and $g : G \to H$, of the *product morphism* $f \times g : E \times G \to F \times H$, defined as the unique morphism making the following diagram commutative, where the vertical arrows are the projection morphisms:

$$
\begin{tikzcd}
E \arrow[r, "f"] & F \\
E \times G \arrow[u] \arrow[d] \arrow[r, "f \times g"] & F \times H \arrow[u] \arrow[d] \\
G \arrow[r, "g"] & H
\end{tikzcd}
$$

Since **ES** has products and equalizers, the pullback of any pair of morphisms $E \to A$ and $G \to A$ exists in **ES** [10, Th.4.5 p.42].

The *sum* or *coproduct* of E_1 and E_2 is the event structure $E =_{[\text{def}]} E_1 \sqcup E_2$ with set of events the disjoint union of E_1 and E_2, and with causality and conflict relations defined by:

- *Causality:* $e \leq e'$ if $e, e' \in E_1$ and $e \leq_1 e'$, or if $e, e' \in E_2$ and $e \leq_2 e'$;

- *Conflict:* $e\#e'$ if $e,e'\in E_1$ and $e\#_1e'$, or if $e,e'\in E_2$ and $e\#_2e'$.

E is an event structure, the two injections $i_1 : E_1 \to E$ and $i_2 : E_2 \to E$ are two morphisms, and (E,i_1,i_2) is a coproduct in the category **ES**: for any event structure H, and pair of morphisms $f : E_1 \to H$, $g : E_2 \to H$, there is a unique morphism $h : E \to H$ making the following diagram commutative:

$$\begin{tikzcd} E_1 \arrow[r, "i_1"] \arrow[dr, "f"'] & E \arrow[d, "h"] & E_2 \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & H & \end{tikzcd}$$

Finally, note that products, coproducts and pullbacks of finitary event structures are finitary.

Exponentiation. The *exponentiation* of an event structure F by an event structure E is the event structure F^E which events are the morphisms $f : E \to F$, and with causality and conflict relations defined by:

- *Causality:* $f \le f'$ if $\forall e \in E,\ f(e) \le f'(e)$;
- *Conflict:* $f\#f'$ if there exists two compatible events $e,e'\in E$ such that $f(e)\#f'(e')$.

Let $ev : F^E \times E \to F$ be the *evaluation* mapping, defined by $ev(f,e) = f(e)$ for any $f \in F^E$ and $e \in E$. Then we claim that F^E is an event structure, and that ev is a morphism of event structures with the following exponentiation property: for any event structure G and morphism $g : G\times E \to F$, there is a unique morphism $\overline{g} : G \to F^E$ making the following diagram commutative:

$$\begin{tikzcd} F^E & F^E \times E \arrow[r, "ev"] & F \\ G \arrow[u, "\overline{g}"'] & G\times E \arrow[u, "\overline{g}\times \mathrm{Id}_E"] \arrow[ur, "g"'] & \end{tikzcd}$$

We prove the claim in details: it is clear that $(F^E,\le)$ is a poset, and that $\#$ is symmetric. *The conflict is irreflexive:* Assume that $f\in F^E$ satisfies $f\#f$. Then there are two compatible events $e,e'\in E$ such that $f(e)\#f(e')$. Since f is a morphism of event structures, this implies that $e\#e'$, which contradicts that e and e' are compatible. Hence $\#$ is irreflexive. *The conflict is inherited:* Let $f_1,f_2,f_3\in F^E$ such that $f_1\#f_2$ and $f_2\le f_3$. Then there are two compatible events $e,e'\in E$ such that $f_1(e)\#f_2(e')$. Then $f_2(e')\le f_3(e')$, and this implies that $f_1(e)\#f_3(e')$, and thus $f_1\#f_3$, showing that $\#$ is inherited. Hence we have shown that F^E is an event structure.

We prove that $ev : F^E\times E \to F$ is a morphism. Let $(f,e),(f',e')\in F^E\times E$ such that $(f,e)\le(f',e')$. Then, using the definition of the order on products, we have $f\le f'$ and $e\le e'$. Thus $ev(f,e) = f(e) \le f'(e) \le f'(e')$, the later ordering since f' is order-preserving. Hence $ev(f,e)\le ev(f',e')$, and thus ev is order-preserving. Assume that $ev(f,e)\#ev(f',e')$. Then, if f and f' are not in conflict, this implies according to the definition of conflict in F^E that e and e' are in conflict. Hence we either have $f\#f'$ or $e\#e'$, i.e., according to the definition of conflict in products, $(f,e)\#(f',e')$. This shows that ev is a morphism of event structures.

We now prove the exponentiation property. Let G be an event structure, and let $g : G \times E \to F$ be a morphism. If $\overline{g} : G \to F^E$ exists as in the above diagram, it is necessarily given by $\overline{g}(e)(e') = g(e, e')$ for any $e \in G$ and $e' \in E$. Hence let $\overline{g} : G \to F^E$ be defined by this way. $\overline{g}$ *is a morphism:* First we prove that $\overline{g}$ is order-preserving. Let $e_1, e_2 \in G$ with $e_1 \leq e_2$. Then for any $e' \in E$, we have $(e_1, e') \leq (e_2, e')$ and thus $\overline{g}(e_1)(e') = g(e_1, e') \leq g(e_2, e') = \overline{g}(e_2)(e')$. This shows that $\overline{g}(e_1) \leq \overline{g}(e_2)$, and thus $\overline{g}$ is order-preserving. To complete the proof that $\overline{g}$ is a morphism, let $e_1, e_2 \in G$ such that $\overline{g}(e_1)\#\overline{g}(e_2)$, and we show that $e_1\#e_2$. Indeed, there are two compatible events $e', e'' \in E$ such that $g(e_1, e')\#g(e_2, e'')$. By the definition of conflict for products, and since g is a morphism, this implies that $e_1\#e_2$ or $e'\#e''$. Since e' and e'' are compatible, we have thus $e_1\#e_2$, what was to be shown. Now by construction, $\overline{g} \times \mathrm{Id}_E$ makes the above diagram commute. Since $\overline{g}$ is the unique morphism with this property, we have completed the proof of the exponentiation property. □

The exponentiation of finitary event structures is not finitary in general. However if E is *finite* and F is finitary, then F^E is finitary.

2.5 The Max-Synchronous Sub-Category

The motivations for studying the topological space of maximal configurations of an event structure have been discussed in the Introduction. This subsection analyzes how our framework can provide elements for this study. The main result is the identification of a class of morphisms that preserve the space of maximal configurations between event structures.

Maximal configurations. As a DCPO, the domain $\widehat{E}$ of configurations of an event structure contains maximal elements. More precisely, it follows from Zorn's lemma that every configuration $x \in \widehat{E}$ is subset of a maximal configuration v, i.e., a configuration v such that $\forall y \in \widehat{E},\ y \supseteq v \Rightarrow y = v$. We denote by Ω_E the set of maximal configurations of E.

Topology of Ω_E. Ω_E is equipped with the restriction of the Scott topology on $\widehat{E}$. That is to say, an open set of Ω_E has the form $U = \Omega_E \cap V$, where V is a Scott-open set of $\widehat{E}$. Although this belongs to domain theory folklore, we recall the proof of the following result:

Proposition 2.4 *For any object D in* $\mathbf{PrAlg}$*, the space Ω of maximal elements of D is Hausdorff.*

Proof: Let u, u' be two distinct elements of Ω. We claim that there are two compact elements $x \leq u$ and $x' \leq u'$, such that $\{x, x'\}$ has no upper bound. Assume there are no such x, x'. Then for any two compact elements x, x' with $x \leq u$ and $x' \leq u'$, the sup $x \vee x'$ is well defined since D is a coherent poset. For each compact element $x \leq u$, consider the set $K_x = \{x \vee x' : x' \text{ compact and } x' \leq u'\}$. Then K_x is directed and satisfies $\sup K_x \geq \sup\{x' : x' \text{ compact and } x' \leq u'\} = u'$. Since u' is maximal, this implies $\sup K_x = u'$. Hence $u' \geq x$ for any compact element $x \leq u$, and therefore $u = u'$, a contradiction.

Now for such an unbounded pair x, x', the two open sets $U = \{\omega \in \Omega : \omega \geq x\}$ and $U' = \{\omega \in \Omega : \omega \geq x'\}$ separate u and u'. □

Max-synchronous morphisms. We have seen that, if $f : E \to F$ is a morphism of event structures, $\widehat{f} : \widehat{E} \to \widehat{F}$ defined in Definition 2.2 is a morphism in $\mathbf{PrAlg}$, and is in particular Scott-continuous. However, nothing guarantees in general that $\widehat{f}$ maps maximal elements of $\widehat{E}$ to maximal elements of $\widehat{F}$.

We introduce therefore the following "pointwise" definition, to be interpreted later as a first step towards a global preservation of the space of maximal configurations (Theorem 2.2 and Corollary 3.1 below).

Definition 2.3 *We say that a morphism $f : E \to F$ in $\mathbf{ES}$ is* max-synchronous *if $\widehat{f}(v)$ is maximal in $\widehat{F}$ for every maximal configuration $v \in \widehat{E}$.*

Max-synchronous morphisms form the arrows of a sub-category of $\mathbf{ES}$. We denote by $\mathbf{ES_{max}}$ the category of event structures whose arrows are max-synchronous morphisms.

Proposition 2.5 *Let $\mathbf{TOP}$ be the category of Hausdorff topological spaces. There is a functor $\mathbf{G} : \mathbf{ES_{max}} \to \mathbf{TOP}$ defined by $\mathbf{G}(E) = \Omega_E$ on objects, and by $\mathbf{G}(f) = \widehat{f}|_{\Omega_E} : \Omega_E \to \Omega_F$ on arrows.*

Proof: It is clear that $\mathbf{G}(f \circ g) = \mathbf{G}(f) \circ \mathbf{G}(g)$ and that $\mathbf{G}(\mathrm{Id}_E) = \mathrm{Id}_{\Omega_E}$. It remains to check that $g = \mathbf{G}(f)$ is continuous for every max-synchronous morphism $f : E \to F$. This follows from the continuity of $\widehat{f}$ on $\widehat{E}$, which holds according to Proposition 2.1 since $\widehat{f}$ is an arrow in $\mathbf{PrAlg}$ (Proposition 2.3). $\square$

The following result gives a sufficient condition for a morphism $f : E \to F$ to be max-synchronous. Surprisingly, this sufficient condition implies that the spaces Ω_E to Ω_F are then homeomorphic. We first need a lemma, that may be well known from domain theory although we didn't find it explicitly in the literature.

Lemma 2.1 *Let f be the lower adjoint of an adjunction pair $D \underset{g}{\overset{f}{\rightleftarrows}} D'$ between two* DCPO *D and D', i.e., f and g are order-preserving and satisfy* [6, Def. O-3.1, p.22]*:*

$$\forall x \in D, \quad \forall y \in D', \quad f(x) \leq y \iff x \leq g(y).$$

Assume moreover that f is is onto. Then f maps the space Ω_D of maximal elements of D homeomorphically onto the space $\Omega_{D'}$ of maximal elements of D'.

Proof: Recall first that the properties of adjunction pairs imply that $g \circ f \geq \mathrm{Id}_D$, and that $f \circ g = \mathrm{Id}_{D'}$ since f is onto [6, Section O-3]. Moreover, f is Scott-continuous since lower adjoints preserve directed sups.

We claim first that f maps Ω_D *to* $\Omega_{D'}$: for any $x \in \Omega_D$ and $y \in D'$ such that $y \geq f(x)$, then $g(y) \geq g \circ f(x) \geq x$, hence $g(y) = x$ since x is maximal and thus $f \circ g(y) = f(x)$, i.e., $y = f(x)$. This shows that $f(x)$ is maximal in D', and proves the claim. Let $h = f|_{\Omega_D} : \Omega_D \to \Omega_{D'}$ be the restriction of f to Ω_D. It is easy to see that h is *onto*: for any $y \in \Omega_{D'}$, there is an element $x \in D$ such that $f(x) = y$ since f is onto, and any element $z \geq x$ and maximal in D will satisfy $h(z) = y$ since y is maximal. Now we show that h is injective. This follows from the observation that $g \circ f(x) = x$ for any $x \in \Omega_D$, the later since $g \circ f(x) \geq x$ but x is maximal.

So far we have shown that $h : \Omega_D \to \Omega_{D'}$ is a bijection. Also, h is continuous since f is. It remains only to show that h is open, i.e., maps any open set of Ω_D onto an open set of $\Omega_{D'}$. But, since f is the lower adjoint of an adjunction pair, this follows exactly from [6, Rem. IV-1.5 (2'), p.269]. The proof is complete. $\square$

Theorem 2.2 *Let $f : E \to F$ be a morphism of event structures. Assume that f is onto F, and that f satisfies moreover $f(\#) \subseteq \#$, i.e.: $\forall e, e' \in E$, $e\#e' \Rightarrow f(e)\#f(e')$. Then f satisfies the following properties:*

1. *$\widehat{f} : \widehat{E} \to \widehat{F}$ is the lower adjoint of an adjunction pair, the upper adjoint of which is $h : \widehat{F} \to \widehat{E}$ defined by $h(y) = f^{-1}(y)$ for all $y \in \widehat{F}$; i.e., $h(y) = \{e \in E \,:\, f(e) \in y\}$. Moreover, $\widehat{f}$ is onto.*

2. *f is max-synchronous, and the restriction $\widehat{f}|_{\Omega_E} : \Omega_E \to \Omega_F$ is a homeomorphism.*

Proof: 1. First we show that $h : \widehat{F} \to \widehat{E}$ is well defined. Let $y \in \widehat{F}$ and let $x = f^{-1}(y)$. Then x is downward closed since f is order-preserving. Moreover x only contains compatible events. Otherwise there would be two events $e, e' \in x$ with $e \# e'$. This would imply $f(e) \# f(e')$, contradicting that $f(e)$ and $f(e')$ belong to configuration y. Hence x is a configuration of E, and thus h is well defined. A straightforward inspection shows that the following adjunction identity holds:

$$\forall y \in \widehat{F}, \quad \forall x \in \widehat{E}, \quad \widehat{f}(x) \subseteq y \iff x \subseteq h(y),$$

and that $\mathrm{Id}_{\widehat{F}} = \widehat{f} \circ h$, hence $\widehat{f}$ is onto.

2. This follows from the above point, combined with Lemma 2.1. □

3 Quotients of Event Structures

This section describes the operation of quotient on event structures—the motivation for introducing quotients of event structures has been discussed in the Introduction. This operation extends the same operation for posets, taking furthermore into account the conflict relation of event structures. The quotient of a poset with respect to a general equivalence relation is a preorder, and is not necessarily a poset. A further identification is needed to obtain a poset; the same phenomenon arises for event structures.

3.1 The Lattice of Event Structures Built upon a Poset

Given a poset $(F, \leq)$, several binary relations $\#$ may define a conflict relation such that $(F, \leq, \#)$ is an event structure. Seen as subsets of $F \times F$, these relations $\#$ are ordered by set-theoretic inclusion. The construction of quotient of event structures rests on the following key lemma:

Lemma 3.1 *Let $(E, \leq, \#_E)$ be an event structure, let $(F, \leq)$ be a poset, and let $f : E \to F$ be an order-preserving mapping. Let $\mathcal{C}_f$ be the set of binary relations on F defined by:*

$$\mathcal{C}_f = \big\{ \# \subseteq F \times F \,:\, \begin{array}{l} (F, \leq, \#) \text{ is an event structure} \\ \text{and } f : E \to F \text{ is a morphism in } \mathbf{ES} \end{array} \big\}.$$

Then $\mathcal{C}_f$ is a nonempty complete lattice with $\emptyset$ as minimum.

Proof: Routine verifications show that, if $(\#_i)_{i \in I}$ is any family of conflict relations in $\mathcal{C}_f$, then the join and meet of $(\#_i)_{i \in I}$ in $\mathcal{C}_f$ are respectively given by:

$$\bigvee_{i \in I} \#_i = \bigcup_{i \in I} \#_i\,, \qquad \bigwedge_{i \in I} \#_i = \bigcap_{i \in I} \#_i\,.$$

Since $\emptyset \in \mathcal{C}_f$, $\mathcal{C}_f$ is nonempty and $\emptyset$ is indeed the minimum of $\mathcal{C}_f$. □

Remark. We open a short parenthesis that makes clearer the relationship between event structures and their underlying posets. A similar result to Lemma 3.1 is the following (proof left to the reader):

Lemma 3.2 *Let $(E, \leq)$ be a poset. The family of binary relations $\#$ on E such that $(E, \leq, \#)$ is an event structure is a complete lattice, whose minimal and maximal elements $\#_0$ and $\#_1$ are respectively given by:*

$$\begin{aligned} &\#_0 = \emptyset, \\ \forall e, e' \in E, \quad &e \#_1 e' \iff e \text{ and } e' \text{ are unbounded in } E. \end{aligned}$$

The lemma can be interpreted in terms of adjunctions of functors [10, p.89]. Let $\mathbf{PO}$ be the category of posets, and let $\mathbf{F} : \mathbf{ES} \to \mathbf{PO}$ be the forgetful functor defined by $\mathbf{F}(E, \leq, \#) = (E, \leq)$ on objects, and that leaves unchanged the morphisms. Let $\mathbf{F}_0, \mathbf{F}_1 : \mathbf{PO} \to \mathbf{ES}$ be the two functors defined as follows:

$$\mathbf{F}_0(E, \leq) = (E, \leq, \#_0), \quad \mathbf{F}_1(E, \leq) = (E, \leq, \#_1),$$

where $\#_0$ and $\#_1$ are the conflict relations defined in Lemma 3.2. Then we leave to the reader to prove:

Proposition 3.1 $(\mathbf{F}, \mathbf{F}_0)$ *and* $(\mathbf{F}_1, \mathbf{F})$ *are two adjunction pairs of functors.*

Since the forgetful functor $\mathbf{F}$ is both a left and a right adjoint, it preserves products, coproducts and exponentiation. This explains that the partial order structures on products, coproducts and exponentiation defined in §2.4 for event structures coincide with the analogous for posets. We close the parenthesis.

3.2 Quotients of Event Structures

We are now ready to define the quotient of event structures. Let $(E, \leq, \#)$ be an event structure, and let $\sim$ be an equivalence relation on E. We first follow the classical construction of quotient posets (see for instance [3]). Let $\overline{\leq}$ be the transitive closure of the relation $(\sim \cup \leq)$. Define also the equivalence relation $\mathcal{R}$ on E by:

$$\forall e, e' \in E, \quad e \mathcal{R} e' \iff e \overline{\leq} e' \text{ and } e' \overline{\leq} e.$$

We denote by $[e]_{\mathcal{R}}$ the equivalence class of e with respect to $\mathcal{R}$, by $F = E/_{\mathcal{R}}$ the quotient set, and by $\pi : E \to F$ the canonical projection. F is equipped with the relation $\preceq$ defined by:

$$\begin{aligned} \forall u, u' \in F, \quad u \preceq u' &\iff \exists e, e' \in E \,:\, [e]_{\mathcal{R}} = u,\ [e']_{\mathcal{R}} = u',\ e \overline{\leq} e' \\ &\iff \forall e, e' \in E, \quad (\,[e]_{\mathcal{R}} = u,\ [e']_{\mathcal{R}} = u') \Rightarrow e \overline{\leq} e'. \end{aligned}$$

By construction, $(F, \preceq)$ is a poset, called the *quotient poset*, and $\pi : E \to F$ is order preserving.

Definition 3.1 *With the above notations, we define the* quotient *of $(E, \leq, \#)$ with respect to the equivalence relation $\sim$ as the event structure $(F, \preceq, \#_F)$, with underlying poset $(F, \preceq)$ defined above, and with conflict relation $\#_F$ defined by:*

$$\#_F = \max(\mathcal{C}_\pi),$$

where $\pi : E \to F$ is the canonical projection.

Intuitively, we keep in the quotient as much conflicts as we can from the original event structure. The definition is motivated by the following property of quotients:

Theorem 3.1 *Let E an event structure, let F be the quotient of E with respect to an equivalence relation $\sim$, as defined in Definition 3.1, and let $\pi : E \to F$ be the canonical projection. Then π is a morphism of event structures, and the pair (F, π) has the following universal property: for any event structure G, and morphism $f : E \to G$ satisfying:*

$$\forall e, e' \in E, \quad e \sim e' \Rightarrow f(e) = f(e'),$$

there is a unique morphism $g : F \to G$ making the following diagram commutative:

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "f"'] & F \arrow[dl, "g"] \\ G & \end{tikzcd}$$

The pair (F, π) is unique, up to a unique isomorphism, with this property.

Proof: By construction of $\#_F$, F is an event structure and $\pi : E \to F$ is a morphism of event structures.

Now let G be an event structure and let $f : E \to G$ be a morphism satisfying $e \sim e' \implies f(e) = f(e')$. By the universal property of the quotient poset $(F, \preceq)$, there is a unique order preserving mapping $g : F \to G$ making the following diagram commutative:

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "f"'] & F \arrow[dl, "g"] \\ G & \end{tikzcd}$$

Therefore, if it exists, the morphism of event structures g in the statement of the theorem must be this one. We need thus only to check that g thus defined is a morphism of event structures.

Consider the binary relation $\#_g$ on F defined by:

$$\forall v, v' \in F, \quad v \#_g v' \iff g(v) \# g(v').$$

Then $\#_g$ is symmetric, and irreflexive. Moreover $\#_g$ satisfies the inheritance axiom: if $v, v', v'' \in F$ are such that $v \#_g v'$ and $v' \preceq v''$, then $g(v) \# g(v')$, and $g(v') \leq g(v'')$ since g is order-preserving, and therefore $g(v) \# g(v'')$. This shows that $v \#_g v''$, and thus $(F, \preceq, \#_g)$ is an event structure. Moreover, $\pi : E \to (F, \preceq, \#_g)$ is a morphism of event structures. Indeed, we have already observed that π is order-preserving. And if $e, e' \in E$ are two events such that $\pi(e) \#_g \pi(e')$, then by definition of $\#_g$, this implies $g\big(\pi(e)\big) \# g\big(\pi(e')\big)$, i.e., $f(e) \# f(e')$. Since f is a morphism of event structures, this implies that $e \# e'$. This shows that $\pi : E \to (F, \preceq, \#_g)$ is a morphism of event structures, and thus, $\#_g \in \mathcal{C}_\pi$. By definition of the conflict relation $\#_F$ in F, this implies:

$$\#_g \subseteq \#_F. \tag{6}$$

Therefore, for any two events $v, v' \in F$, we have:

$$g(v) \# g(v') \Rightarrow v \#_g v' \Rightarrow v \#_F v',$$

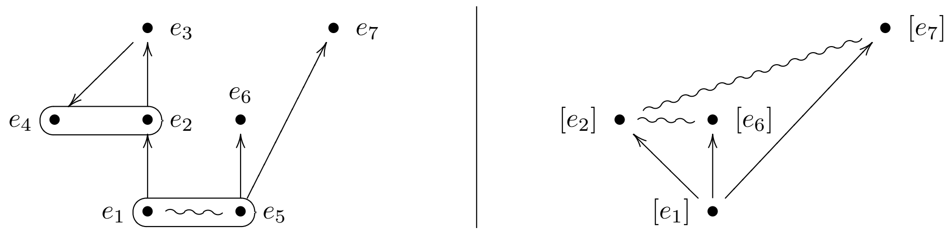

Fig. 2: *Quotient of an event structure. Remark that e_3 is identified with e_2 and e_4 because of the cycle* $[e_2]\overline{\leq}[e_3]\overline{\leq}[e_4]=[e_2]$.

the later implication thanks to (6). We have shown that $g : F \to G$ is a morphism of event structures, as claimed. This completes the proof of the universal property of the pair (F,π). The uniqueness of the pair (F,π), up to a unique isomorphism, follows from the usual argument from category theory: if (H,ρ) satisfies the same universal property, then we apply both universal properties to $\pi : E \to F$ and to $\rho : E \to H$ to get the existence and uniqueness of two arrows g and h making the following diagrams commutative:

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\rho"'] & F \arrow[dl, "g"] \\ H & \end{tikzcd} \qquad \begin{tikzcd} H \arrow[dr, "h"'] & E \arrow[l, "\rho"'] \arrow[d, "\pi"] \\ & F \end{tikzcd}$$

From $\pi = h\circ\rho$ and $\rho = g\circ\pi$ we obtain $\pi = h\circ g\circ\pi$, whence $\mathrm{Id}_F = h\circ g$. Conversely, $\mathrm{Id}_H = g\circ h$ holds. Hence (F,π) is uniquely determined, up to a unique isomorphism. □

Example. Let $(E,\leq,\#)$ be the event structure $E=\{e_1,e_2,\ldots,e_7\}$ with causality defined by:

$$e_1\leq e_2\leq e_3\leq e_4,\quad e_5\leq e_6,\quad e_5\leq e_7\,,$$

and conflict relation such that $e_1\#e_5$ (and other conflicts obtained by inheritance). The event structure is depicted in Figure 2, left. Let also $\sim$ be the reflexive transitive closure of the relation $e_1\sim e_5$, $e_2\sim e_4$. The quotient is depicted in Figure 2, right (see below the note about the drawing conventions of quotients).

Graphical convention for drawing quotients. We depict equivalence relations on event structures as follows (see Figure 2, left). Events belonging to a same equivalence classes are circled within a curved frame. Singleton classes are not depicted. The quotient of the event structure is depicted at right, following our usual conventions for drawing event structures.

In the following subsection, we analyze a class of event structures for which the conflict relation can be simply described. The example of the compression of event structures, detailed in Section 4, falls into this class. This example also shows that the quotient operation may not preserve the finitary condition of event structures.

3.3 Max-synchronous Properties of Quotients.

Maybe surprisingly, the natural projection $\pi : E \to F$ from an event structure E to its quotient F with respect to an equivalence relation is not max-synchronous in general. Figure 3 depicts the example of

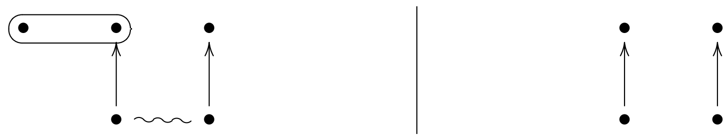

Fig. 3: *A quotient of event structure with a non max-synchronous projection. An event structure E is depicted at left together with an equivalence relation. The quotient event structure F is depicted at right. Remark that $\#_F = \emptyset$. The configuration consisting of the three left events of E is maximal in E, but its image (the two left events) is not maximal in the quotient.*

such a quotient. We can however state a sufficient condition for the projection to be max-synchronous (Corollary 3.1 below).

We begin with the following notion. Let $(E, \leq, \#)$ be an event structure, and let $\sim$ be an equivalence relation on E. Say that the conflict relation $\#$ is a $\sim$*-congruence* if the following holds:

$$\forall e_0, e_1, e \in E, \qquad \left.\begin{array}{c} e_0 \# e_1 \\ e_0 \sim e \end{array}\right\} \Rightarrow e \# e_1. \tag{7}$$

In other words, the conflict is a $\sim$-congruence if it is closed under the equivalence relation $\sim$.

The main property of congruent conflicts is the following:

Proposition 3.2 *Let $(E, \leq, \#)$ be an event structure, let $\sim$ be an equivalence relation on E, and let $(F, \preceq, \#_F)$ be the quotient event structure with $\pi : E \to F$ the natural projection. Then $\#$ is $\sim$-congruent if and only if the quotient conflict relation $\#_F$ satisfies:*

$$\forall e, e' \in E, \qquad \pi(e) \#_F \, \pi(e') \iff e \# e'. \tag{8}$$

Proof: It is clear that, if $\#_F$ satisfies (8), then $\#$ is $\sim$-congruent.

To prove the converse, assume that $\#$ is $\sim$-congruent. In (8), the implication $\Rightarrow$ is trivial since it follows from the definition of morphisms, so we focus on the implication $\Leftarrow$.

Consider the conflict relation $\overline{\#}$ on F defined by:

$$\forall u, v \in F, \quad u \overline{\#} v \iff (\forall e, e' \in E, \quad [e]_{\mathcal{R}} = u, \, [e']_{\mathcal{R}} = u' \Rightarrow e \# e').$$

We claim that $(F, \preceq, \overline{\#})$ is an event structure. Indeed, $\overline{\#}$ is obviously symmetric and irreflexive. We prove the inheritance axiom. Assume that u, u', u'' are three elements in F with $u \overline{\#} u'$ and $u' \preceq u''$, and let $e \in u$ and $e'' \in u''$. Pick $e' \in u'$. Then $e \# e'$ since $u \overline{\#} u'$, and $e' \overline{\leq} e''$ since $u' \preceq u''$. But, since $\#$ is $\sim$-congruent, this implies that $e \# e''$, and this shows that $u \overline{\#} u''$. Hence $(F, \preceq, \overline{\#})$ is an event structure. Obviously, $\pi : (E, \leq, \#) \to (F, \preceq, \overline{\#})$ is then a morphism, so that $\overline{\#} \in \mathcal{C}_\pi$. In turn, this implies that $e \# e' \Rightarrow \pi(e) \#_F \pi(e')$ for any two events $e, e' \in E$. This completes the proof of the proposition. □

As a consequence, we have the following sufficient condition for $\pi : E \to F$ to be max-synchronous:

Corollary 3.1 *Let $(E, \leq, \#)$ be an event structure. Let $\sim$ be an equivalence relation on E, and let $(F, \preceq, \#_F)$ be the quotient event structure. If $\#$ is a $\sim$-congruence, then the canonical projection $\pi : E \to F$ is max-synchronous and $\widehat{\pi}|_{\Omega_E} : \Omega_E \to \Omega_F$ is a homeomorphism.*

Proof: By construction, $\pi : E \to F$ is onto. Since $\#$ is assumed to be $\sim$-congruent, it follows from Proposition 3.2 that π satisfies both conditions of Theorem 2.2. The conclusion follows. □

4 Compression of Event Structures

We introduce the compression of event structures as an example of quotient. It is a canonical example in the sense that it can be applied to any event structure. It completes the study of the category $\mathbf{ES_{max}}$ introduced in §2.5 by giving for each event structure E a canonical event structure sharing the same space of maximal elements as E, and minimal in some sense.

4.1 The compression equivalence.

To begin with, we introduce the following equivalence relation:

Definition 4.1 *Let $(E, \leq, \#)$ be an event structure, with Ω_E the space of maximal configurations of E. Let $\sim$ be the equivalence relation on E defined by:*

$$\forall e, e' \in E, \qquad e \sim e' \iff (\forall v \in \Omega_E, \quad e \in v \iff e' \in v). \tag{9}$$

We call $\sim$ the compression equivalence*. The quotient event structure is called the* compression *of E.*

The relation $\sim$ is obviously an equivalence relation. It identifies events that cannot be distinguished from the point of view of maximal configurations. An equivalent (proof left to the reader), more operational definition of the compression equivalence is the following: for any two events $e, e' \in E$, $e \sim e'$ if and only if

$$\forall e'' \in E, \quad e\#e'' \iff e'\#e''.$$

We give in Proposition 4.1 below some elements that describe more explicitly the quotient event structure. Using the notations of §3.2, we have:

Proposition 4.1 *Let $(E, \leq, \#)$ be an event structure, and let $(F, \preceq, \#_F)$ be the compression of E.*

1. *The equivalence relation $\mathcal{R}$ coincides with the compression equivalence.*
2. *The conflict relation $\#$ is a $\sim$-congruence. Therefore, $\#_F$ is given by:*

$$\forall e, e' \in E, \quad [e]\#_F\,[e'] \iff e\#e', \tag{10}$$

the projection $\pi : E \to F$ is max-synchronous, and $\widehat{\pi}|_{\Omega_E} : \Omega_E \to \Omega_F$ is a homeomorphism.

Proof: 1. It is enough to show that for any $e, e' \in E$, the following holds: $e\mathcal{R}e' \Rightarrow e \sim e'$, since the converse implication is trivial. Hence let $e, e' \in E$ with $e\mathcal{R}e'$, i.e., $e\overline{\leq}e'\,\overline{\leq}e$. By definition of $\overline{\leq}$, there are events $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ of E such that:

$$e \sim a_0 \leq a_1 \sim a_2 \leq \cdots \leq a_n \sim e' \sim b_0 \leq b_1 \sim b_2 \leq \cdots \leq b_m \sim e.$$

Pick $v \in \Omega_E$ with $e \in v$. Then $b_m \in v$ since $b_m \sim e$. Therefore $b_{m-1} \in v$ since $b_{m-1} \leq b_m$, and thus $b_{m-2} \in v$ since $b_{m-2} \sim b_{m-1}$. Iterating m times, we finally obtain that $b_0 \in v$, and thus $e' \in v$ since $e' \sim b_0$. This shows that $e \in v \Rightarrow e' \in v$ holds for any $v \in \Omega_E$. Exchanging the roles of e and e', we obtain the converse implication and thus $e \sim e'$.

2. We show that $\#$ is $\sim$-congruent. Let $e, e' \in E$ such that $e\#e'$, and let $a \sim e$. If $\neg(a\#e')$ holds, then there is a maximal configuration v that contains both a and e'. But v also contains e since $a \sim e'$, a contradiction with $e\#e'$. The remaining properties follow from Proposition 3.2 and Corollary 3.1. □

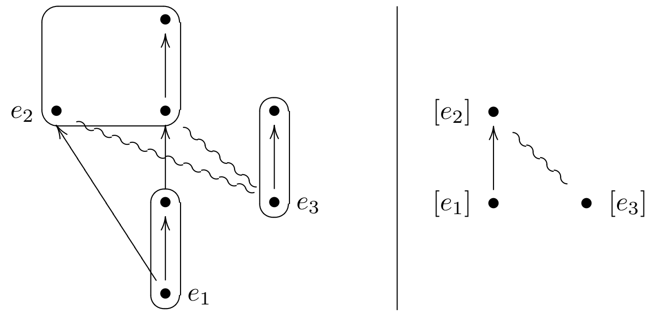

Fig. 4: *Example of compression of an event structure. An event structure is depicted at left, together with its compression equivalence. The compressed event structure is depicted at right.*

Example. Figure 4 depicts an example of compression.

Point 2 of Proposition 4.1 shows that the compression of E is a particular event structure sharing the same space of maximal configurations, up to homeomorphism, as E. The fact that it is a canonical event structure with this property follows from the study of the compression seen as an idempotent functor, which is our next topic.

4.2 The compression functor.

The properties of the natural projection $\pi : E \to F$ from an event structure to its compression motivate the introduction of the following sub-category of $\mathbf{ES_{max}}$:

Definition 4.2 *We denote by $\mathbf{ES'_{max}}$ the sub-category of $\mathbf{ES_{max}}$ with event structures as objects, and with arrows from E to F those max-synchronous morphisms $f : E \to F$ which are onto and preserve conflict, i.e.:*

$$\forall e, e' \in E, \quad e\#e' \iff f(e)\#f(e').$$

Observe that, according to Theorem 2.2, point 2, it is enough for a morphism $f : E \to F$ to be onto and to satisfy $f(\#) = \#$ in order to be max-synchronous. It turns out that $\mathbf{ES'_{max}}$ is the right category to consider in order to see the compression operation as a functor:

Theorem 4.1 *Let $f : E \to G$ be a morphism in $\mathbf{ES'_{max}}$, and let F, H be the respective compressions of E and G, with natural projections $\pi : E \to F$ and $\rho : G \to H$. Then there is a unique morphism $\mathbf{C}(f)$ in $\mathbf{ES'_{max}}$ such that the following diagram is commutative:*

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "f"'] & F \arrow[d, "\mathbf{C}(f)"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd} \tag{11}$$

There is a functor $\mathbf{C} : \mathbf{ES'_{max}} \to \mathbf{ES'_{max}}$ defined as follows: for E an event structure, $\mathbf{C}(E)$ is the compression of E, and for $f : E \to F$ a morphism in $\mathbf{ES'_{max}}$, $\mathbf{C}(f)$ is the morphism defined above. The functor $\mathbf{C}$ is idempotent, i.e., $\mathbf{C} \circ \mathbf{C} = \mathbf{C}$.

Proof: Let $f : E \to G$ be a morphism in $\mathbf{ES}'_{\mathbf{max}}$. Let $\sim_E$ and $\sim_G$ denote respectively the compression equivalences in E and in G. We first show the following:

$$\forall e, e' \in E, \quad e \sim_E e' \Rightarrow f(e) \sim_G f(e'). \tag{12}$$

Let $e, e' \in E$ such that $e \sim_E e'$. For any $\xi \in \Omega_G$ such that $f(e) \in \xi$, there exists $v \in \Omega_E$ such that $\widehat{f}(v) = \xi$—according to Theorem 2.2, point 2, this unique v is given by $v = \widehat{f}|_{\Omega_E}^{-1}(\xi)$. The event e is compatible with v. Otherwise, there would be an event $e_0 \in v$ such that $e_0 \# e$. But this would imply $f(e) \# f(e_0)$, contradicting that both $f(e)$ and $f(e_0)$ belong to configuration ξ. Hence e is indeed compatible with v. Since v is maximal, this implies that $e \in v$, which in turns imply that $e' \in v$ since $e \sim_E e'$, and thus $f(e') \in \xi$. Hence we have shown $f(e) \in \xi \Rightarrow f(e') \in \xi$. By symmetry, we would show the converse implication. Therefore, $f(e) \sim_G f(e')$, which completes the proof of (12).

According to the universal property of quotients (Theorem 3.1) applied to the mapping $\rho \circ f$, there is thus a unique morphism of event structures $g : F \to H$ such that the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "\rho\circ f"'] & F \arrow[d, "g"] \\ & H \end{tikzcd}$$

g is onto, since $\rho \circ f$ is itself onto as a composite of surjective mappings. Therefore, to show that g is an arrow of $\mathbf{ES}'_{\mathbf{max}}$, it remains only to show that $g(\#_F) \subseteq \#_H$. And indeed, if $a, a' \in F$ are such that $a \#_F a'$, we pick $e, e' \in E$ such that $\pi(e) = a$ and $\pi(e') = a'$. Then e and e' are in conflict in E since $\pi : E \to F$ is a morphism of event structures. Since both ρ and f are morphisms of $\mathbf{ES}'_{\mathbf{max}}$, this implies that $\rho \circ f(e) \#_H \rho \circ f(e')$, i.e., $g(a) \#_H g(a')$, as claimed. Therefore $g = \mathbf{C}(f)$ is the requested morphism of diagram (11). We use the usual argument from category theory shows that $\mathbf{C}$ is a functor.

We now show that $\mathbf{C}$ is idempotent. Let E be an event structure, and let F be the compression of E. We show that the compression equivalence $\sim_F$ of F is the identity relation. Let $a, a' \in F$ such that $a \sim_F a'$, we claim that $a = a'$. Indeed, let $e, e' \in E$ such that $\pi(e) = a$, $\pi(e') = a'$, and we show that $e \sim_E e'$. Let $v \in \Omega_E$ such that $e \in v$. Then $\widehat{\pi}(v) \in \Omega_F$ and $a \in \widehat{\pi}(v)$, hence $a' \in \widehat{\pi}(v)$. This implies that e' is compatible with v, and therefore $e' \in v$ since v is maximal. We have thus shown: $e \in v \Rightarrow e' \in v$. The converse implication is shown in the same manner, hence $e \sim_E e'$, and thus $a = a'$, as claimed. This shows that the compression of F is F itself. We have shown that $\mathbf{C} \circ \mathbf{C}(E) = E$ for any event structure E. We now show that $\mathbf{C}$ is idempotent on morphisms. Let $f : E \to G$ be a morphism of $\mathbf{ES}'_{\mathbf{max}}$, and let F and H be the respective compressions of E and G. Then, applying the compression functor $\mathbf{C}$ to $\mathbf{C}(f) : F \to H$ gives the following commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f"'] & F \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathbf{C}(f)"'] & F \arrow[d, "\mathbf{C}\circ\mathbf{C}(f)"] \\ G \arrow[r] & H \arrow[r, "\mathrm{Id}"] & H \end{tikzcd}$$

This shows that $\mathbf{C} \circ \mathbf{C}(f) = \mathbf{C}(f)$, and completes the proof. $\square$

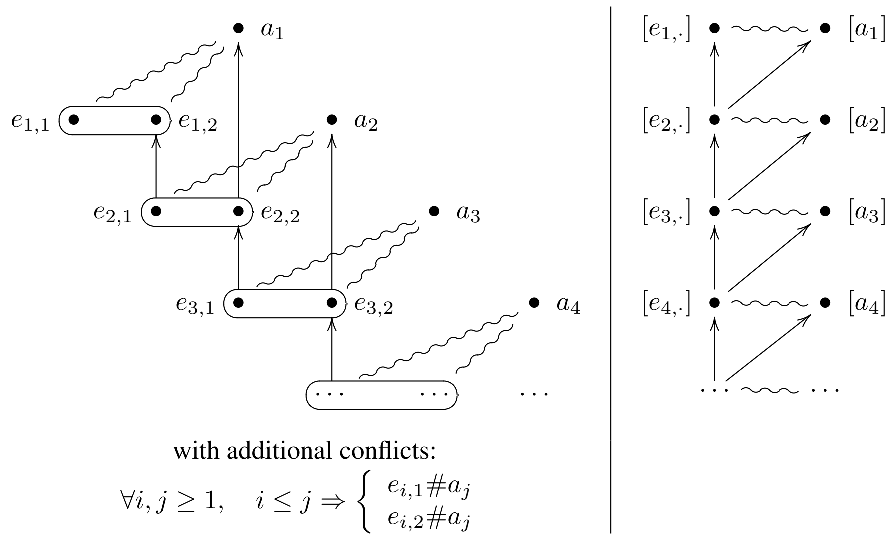

Fig. 5: Example of a finitary event structure (left), the compression of which (right) is not finitary.

4.3 Compression of Finitary Event Structures.

In general, the compression of a finitary event structure may not be finitary. Figure 5 (left) depicts an example of such an event structure. On the other hand, some non finitary event structure may have a finitary compression. Think for example of the event structure $(\mathbb{R}, \leq, \emptyset)$, where $\mathbb{R}$ denotes the set of real numbers, equipped with the usual ordering relation $\leq$ and empty conflict relation. Then $\mathbb{R}$ is not countable, neither well-founded, and thus it is not finitary. But its compression is a singleton, and so is finitary.

Although we couldn't give a criterion to guarantee that a compression is finitary, Theorem 4.2 below proposes some elements in this direction. We need the following definition.

Definition 4.3 *Let $(E, \leq, \#)$ be an event structure. Say that two events e, e' are* concurrent *if neither $e \leq e'$ nor $e' \leq e$ nor $e \# e'$ holds. We denote by* $\mathbf{co}$ *the concurrency relation thus defined. Say that the event structure $(E, \leq, \#)$ has* finite concurrency width *if there is no infinite concurrency clique. That is to say, if $A \subseteq E$ satisfies:*

$$\forall e, e' \in A, \quad e \neq e' \Rightarrow e \,\mathbf{co}\, e',$$

then A is finite.

Event structures with finite concurrency width are very common in practice. They represent systems where only finitely many events can occur concurrently. For example, unfoldings of safe Petri nets [11] have finite concurrency width. Observe that the event structure of Fig. 5 (top-left) does not have finite concurrency width.

If E is a countable event structure, and if E is not finitary, E is at least in one of the following situations:

1. There is in E an infinite strictly descending sequence;
2. There is an event $e \in E$ and an infinite concurrency clique in $\downarrow e$;

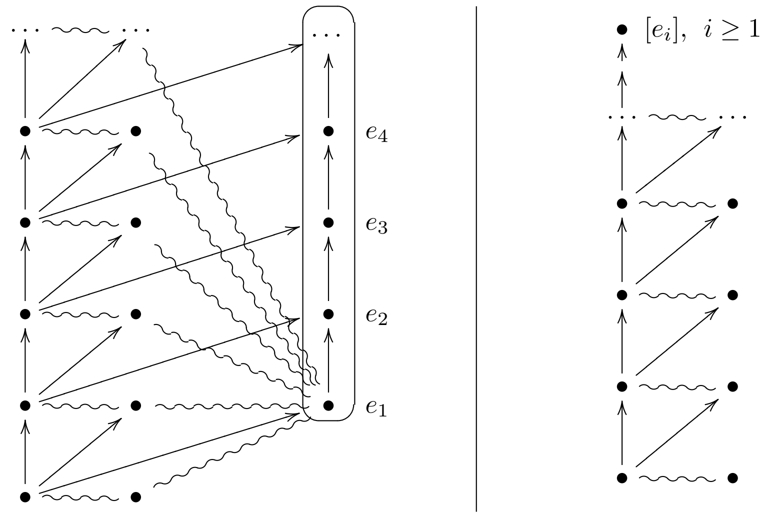

Fig. 6: Example of a finitary event structure with finite concurrency width (left), the compression of which (right) is not finitary. Observe the infinite ascending sequence of events in the compressed event structure.

3. There is an event $e \in E$ and an infinite strictly increasing sequence of events in $\downarrow e$.

The following result shows that the compression of a finitary event structure with finite concurrency width can at most be in case 3. Figure 6 (left) depicts a finitary event structure with finite concurrency width, which compression is actually in case 3.

Theorem 4.2 *If $(E, \leq, \#)$ is a finitary event structure with finite concurrency width, then the compression of E is countable, is well-founded (i.e., there is no infinite strictly decreasing sequence) and has finite concurrency width.*

Proof: Let $(E, \leq, \#)$ be a finitary event structure with finite concurrency width, and let $(F, \preceq, \#_F)$ be the compression of E, with $\pi : E \to F$ the natural projection.

We show that $(F, \preceq)$ is well founded. Assume it is not. Then there is an infinite sequence $a_1 \succeq a_2 \succeq a_3 \ldots$ in F, with all a_i pairwise distinct, $i \geq 1$. For each $i \geq 1$, pick $e_i \in E$ such that $[e_i] = a_i$. Observe that:

$$\forall i, j \geq 1, \quad i < j \Rightarrow \neg(e_i \leq e_j). \tag{13}$$

For, if $e_i \leq e_j$ with $i < j$, then $a_i \preceq a_j$ since $\pi : E \to F$ is order preserving. But we also have $a_j \preceq a_i$ by construction, and thus $a_i = a_j$, contradicting that all a_i are pairwise distinct. This proves (13). Note that (13) implies that the e_i are pairwise distinct, $i \geq 1$. Moreover, we have:

$$\forall i, j \geq 1, \quad \neg(e_i \# e_j). \tag{14}$$

Indeed, if $e_i \# e_j$ for some $i, j \geq 1$, it follows from Proposition 4.1, point 2, that $a_i \# a_j$, and this contradicts that a_i and a_j are causally related.

Consider now the following inductive construction. Let $i_1 = 1$. Since the set $\downarrow e_{i_1}$ is finite, and since the e_i are pairwise distinct, there are only finitely many integers $i \geq 1$ such that $e_i \leq e_{i_1}$. Hence there is an integer $i_2 > i_1$ such that:

$$\forall i \geq i_2, \quad \neg(e_i \leq e_{i_1}).$$

For the same reasons, there is an integer $i_3 > i_2$ such that:

$$\forall i \geq i_3, \quad \neg(e_i \leq e_{i_2}).$$

Continuing inductively, we construct a strictly increasing sequence $(i_k)_{k\geq 1}$ of integers such that:

$$\forall k, k' \geq 1, \quad k < k' \Rightarrow \neg(e_{i_{k'}} \leq e_{i_k}). \tag{15}$$

Set $A = \{e_{i_k},\, k \geq 1\}$. It follows from (13), (14) and (15) that A is a concurrency clique. But, since $(i_k)_{k\geq 1}$ is strictly increasing, and since the e_i are pairwise distinct, the e_{i_k} are pairwise distinct, $k \geq 1$. Hence A is an infinite concurrency clique, and this contradicts that E has finite concurrency width. This shows that $(F, \preceq)$ is well founded.

F has finite concurrency width. Indeed, let B be a concurrency clique of F. Pick for every $a \in B$ an event $e \in E$ such that $[e_a] = a$. Then we show as above that $A = \{e_a,\, a \in B\}$ is a concurrency clique of E. Therefore A is finite. Since $\pi|_A : A \to B$ is onto, it follows that B is finite. This proves the claim, and completes the proof of the theorem. □

5 Conclusion

We have introduced a cartesian closed category of event structures. Although the set-theoretic action of a morphism $f : E \to F$ on a configuration of E may not be a configuration, morphisms naturally defined an action from configurations of E to configurations of F. This action respects prime elements.

We have defined a notion of quotient event structure within this category. We have introduced the compression of event structure as an example of quotient. The compression of an event structure E is a quotient of E sharing the same topological space of maximal configurations as E.

Several extensions are to be considered for the notion of quotient. From the mathematical viewpoint, the quotient of an event structure induces a quotient in the domain of configurations. A first question would be to analyze, conversely, under which condition the quotient of a domain can be realized as the quotient of the associated event structure. In particular, is the so-called observational equivalence of processes of this type?

Since we have analyzed the cartesian closure properties of the category **ES** of event structures, an extension of this work would be to consider the domain of valuations defined on the domain of configurations of event structures. In particular, does the probabilistic powerdomain [7] of an event structure factorize through its compression?

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