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Uniform measures on braid monoids and dual braid monoids

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ABSTRACT

We aim at studying the asymptotic properties of typical *positive braids*, respectively *positive dual braids*. Denoting by μ_k the uniform distribution on positive (dual) braids of length k, we prove that the sequence $(\mu_k)_k$ converges to a unique probability measure μ_∞ on *infinite* positive (dual) braids. The key point is that the limiting measure μ_∞ has a Markovian structure which can be described explicitly using the combinatorial properties of braids encapsulated in the Möbius polynomial. As a by-product, we settle a conjecture by Gebhardt and Tawn (J. Algebra, 2014) on the shape of the Garside normal form of large uniform braids.

1. Introduction

Consider a given number of strands, say n, and the associated positive braid monoid B_n^+ defined by the following monoid presentation, known as the *Artin* presentation:

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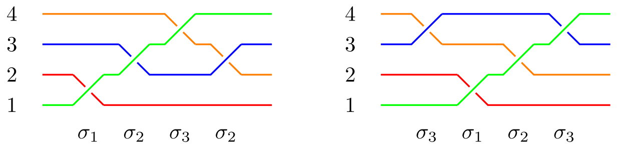

Fig. 1. Two isotopic braid diagrams representing a positive braid on 4 strands. Left: diagram corresponding to the word $\sigma_1\sigma_2\sigma_3\sigma_2$. Right: diagram corresponding to the word $\sigma_3\sigma_1\sigma_2\sigma_3$.

$$B_n^+ = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle^+ . \tag{1}$$

The elements of B_n^+, the *positive braids*, are therefore equivalence classes of words over the alphabet $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$. Alternatively, going back to the original geometric intuition, positive braids can be viewed as isotopy classes of *positive* braid diagrams, that is, braid diagrams in which the bottom strand always goes on top in a crossing, see Fig. 1.

We want to address the following question:

What does a typical complicated positive braid look like?

To make the question more precise, we need to clarify the meaning of "complicated" and "typical". First, let the complexity of a positive braid be measured by the length (number of letters) of any representative word. This is natural since it corresponds to the number of crossings between strings in any representative braid diagram. Therefore, a positive braid is "complicated" if its length is large.

Second, let us define a "typical" braid as a braid being picked at random according to some probability measure. The two natural candidates for such a probability measure are as follows. Fix a positive integer k.

- The first option consists in running a simple random walk on B_n^+: pick a sequence of random elements $x_i, i \geq 1$, independently and uniformly among the generators $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$, and consider the "typical" braid $X = x_1 \cdot x_2 \cdot \cdots \cdot x_k$. It corresponds to drawing a word uniformly in Σ^k and then considering the braid it induces.
- The second option consists in picking a "typical" braid of length k uniformly at random among all braids of length k.

The two approaches differ since the number of representative words varies among positive braids of the same length. For instance, in B_3^+ and for the length 3, the braid $\sigma_1 \cdot \sigma_2 \cdot \sigma_1$ $(= \sigma_2 \cdot \sigma_1 \cdot \sigma_2)$ will be picked with probability 2/8 in the first approach, and with probability 1/7 in the second one, while all the other braids of length 3 will be picked respectively with probabilities 1/8 and 1/7 in the two approaches. The random walk approach has been studied for instance in [26,30]; it is a special instance of random walks on (semi)groups, see [32]. In this paper, our focus is on the second approach, that is, on *uniform measures on positive braids.*

Let μ_k be the uniform probability measure on positive braids of B_n^+ of length k. The general question stated above can now be rephrased as follows: study μ_k for large k. Let us say that we are interested in some specific property, say, the number of occurrences of the Garside element Δ in a large random braid. To study it, a first approach consists in performing a numerical evaluation. To that purpose, the key ingredient is to have a *sampling algorithm*, that is, a random procedure which takes as input k and returns as output a random braid of distribution μ_k. Another, more intrinsic, approach consists first in defining a probability measure μ_∞ on *infinite* positive braids, encapsulating all the measures μ_k, and then in studying the asymptotics of the property *via* μ_∞. None of these two paths is easy to follow. The difficulty is that the probability measures $(\mu_k)_k$ are not consistent with one another. For instance, in B_3^+, we have:

$$1/4 = \mu_2(\sigma_1 \cdot \sigma_1) \neq \mu_3(\sigma_1 \cdot \sigma_1 \cdot \sigma_1) + \mu_3(\sigma_1 \cdot \sigma_1 \cdot \sigma_2) = 2/7\,. \tag{2}$$

Therefore, there is no obvious way to design a dynamic process to sample braids. As another consequence, the Kolmogorov consistency theorem does not apply, and there is no simple way to define a uniform probability measure on infinite positive braids. This is in sharp contrast with the simpler picture for the random walk approach described above.

To overcome the difficulties, the rich combinatorics of positive braids has to enter the scene. Going back to Garside [22] and Thurston [20], it is known that positive braids admit a *normal form*, that is a selection of a unique representative word for each braid, which is *regular*, that is recognized by a finite automaton. This so called *Garside normal form* enables to count positive braids in an effective way, see for instance Brazil [11], but a non-efficient one since the associated automaton has a large number of states, exponential in the number of strands n, see Dehornoy [16]. A breakthrough is provided by Bronfman [12] (see also [5]) who obtains, using an inclusion–exclusion principle, a simple recursive formula for counting positive braids. Based on this formula, a sampling algorithm whose time and space complexities are polynomial in both the number of strands n and the length k is proposed by Gebhardt and Gonzales-Meneses in [23]. Using the sampling procedure, extensive numerical evaluations are performed by Gebhardt and Tawn in [24], leading to the *stable region conjecture* on the shape of the Garside normal form of large uniform braids.

In the present paper, we complete the picture by proving the existence of a natural uniform probability measure μ_∞ on infinite positive braids. The measure induced by μ_∞ on braids of length k is not equal to μ_k, which is in line with the non-consistency illustrated in (2), but the sequence $(\mu_k)_k$ does converge weakly to μ_∞. The remarkable point is that the measure μ_∞ has a Markovian structure which can be described explicitly. It makes it possible to get precise information on μ_k for large k by using the limit μ_∞. For instance, we prove that the number of Δ in a random braid of B_n^+ is asymptotically geometric of parameter $q^{n(n-1)/2}$ where q is the unique root of smallest modulus of the Möbius polynomial of B_n^+. As another by-product of our results, we settle the stable region conjecture, proving one of the two statements in the conjecture, and refuting the other one. Our different results are achieved by strongly relying on refined properties of the combinatorics of positive braids, some of them new.

Mutatis mutandis, the results also hold in the Birman–Ko–Lee dual braid monoid [10]. We present the results in a unified way, with notations and conventions allowing to cover the braids and the dual braids at the same time. The prerequisites on these two monoids are recalled in Section 2, and the needed results on the combinatorics of braids are presented in Section 3. The main results are proved in Section 4, with applications in Section 5, including the clarification of the stable region conjecture. In Section 6, we provide explicit computations of the uniform measure μ_∞ for the braid monoid and the dual braid monoid on 4 strands. At last, analogs and extensions are evoked in Section 7. Indeed, our results on braid monoids form a counterpart to the results on trace monoids in [1,2], and, in a forthcoming paper [3], we plan to prove results in the same spirit for Artin–Tits monoids, a family encompassing both braids and traces.

2. Positive and dual positive braid monoids

In this section we introduce some basics on the monoid of positive braids and the monoid of positive dual braids. We recall the notions of simple braids for these monoids, as well of combinatorial representations of them.

2.1. Two distinct braid monoids

2.1.1. The braid group and two of its submonoids

For each integer $n \geq 2$, the *braid group* B_n is the group with the following group presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle . \tag{3}$$

Elements of B_n are called *braids*. Let $\mathbf{e}$ and "$\cdot$" denote respectively the unit element and the concatenation operation in B_n. It is well known since the work of Artin that elements of B_n correspond to isotopy classes of braid diagrams on n strands, as illustrated in Fig. 1; the elementary move where strand i crosses strand $i+1$ from above corresponds to generator σ_i, and the move where strand i crosses strand $i+1$ from behind to σ_i^{-1}.

We will be interested in two submonoids of B_n. The *positive braids monoid* B_n^+ is the submonoid of B_n generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$; and the *positive dual braid monoid* B_n^{+*} is the submonoid of B_n generated by $\{\sigma_{i,j} \mid 1 \leq i < j \leq n\}$, where $\sigma_{i,j}$ is defined by:

$$\begin{array}{ll} \sigma_{i,j} = \sigma_i\,, & \text{for } 1 \leq i < n \text{ and } j = i+1, \\ \sigma_{i,j} = \sigma_i\sigma_{i+1}\ldots\sigma_{j-1}\sigma_{j-2}^{-1}\sigma_{j-3}^{-1}\ldots\sigma_i^{-1}\,, & \text{for } 1 \leq i < n-1 \text{ and } i+2 \leq j \leq n\,. \end{array}$$

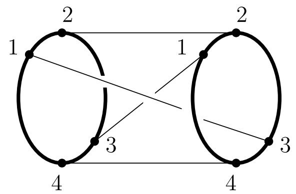

Fig. 2. Representation of $\sigma_{1,3}$ in B_4^{+*}.

Observe the inclusion $B_n^+ \subseteq B_n^{+*}$, since each generator σ_i of B_n^+ belongs to B_n^{+*}. The elements $\sigma_{i,j}$ are often called *Birman–Ko–Lee generators* in the literature, while the elements σ_i are called *Artin* generators.

Running examples for $n = 3$ Throughout the paper, we shall illustrate the notions and results on the most simple, yet non-trivial examples of braid monoids, namely on B_3^+ and on B_3^{+*}:

$$B_3^+ = \langle \sigma_1,\ \sigma_2 \rangle^+ ,$$

$$B_3^{+*} = \langle \sigma_{1,2},\ \sigma_{2,3},\ \sigma_{1,3} \rangle^+ \qquad \text{with } \sigma_{1,2} = \sigma_1 ,\sigma_{2,3} = \sigma_2 ,\sigma_{1,3} = \sigma_1 \cdot \sigma_2 \cdot \sigma_1^{-1} .$$

2.1.2. Presentations of the monoids

Defining B_n^+ and B_n^{+*} as submonoids of B_n, as we just did, is one way of introducing them. Another one is through generators and relations.

First, B_n^+ is isomorphic to the monoid with the monoid presentation (1), that is, the same presentation as B_n but viewed as a monoid presentation instead of a group presentation. Second, B_n^{+*} is isomorphic to the monoid with $n(n-1)/2$ generators $\sigma_{i,j}$ for $1 \leq i < j \leq n$ and the following relations, provided that the convention $\sigma_{j,i} = \sigma_{i,j}$ for $i < j$ is in force:

$$\begin{cases} \sigma_{i,j}\sigma_{j,k} = \sigma_{j,k}\sigma_{k,i} = \sigma_{k,i}\sigma_{i,j} & \text{for } 1 \leq i < j < k \leq n \\ \sigma_{i,j}\sigma_{k,\ell} = \sigma_{k,\ell}\sigma_{i,j} & \text{for } 1 \leq i < j < k < \ell \leq n \\ \sigma_{i,j}\sigma_{k,\ell} = \sigma_{k,\ell}\sigma_{i,j} & \text{for } 1 \leq i < k < \ell < j \leq n . \end{cases} \tag{4}$$

Elements of B_n^+ are called *positive braids*, they correspond to isotopy classes of braid diagrams involving only crossing of strands in the same direction, see Fig. 1. Elements of B_n^{+*} are called *dual positive braids*. They correspond to isotopy classes of chord diagrams [10]. This time, there are still n strands but they are arranged along a cylinder; the element $\sigma_{i,j}$ corresponds to a crossing of strands i and j. See Fig. 2.

The inclusion $B_n^+ \subseteq B_n^{+*}$ comes with the definition of B_n^+ and B_n^{+*} as submonoids of the braid group B_n. It can be obtained as follows when considering B_n^+ and B_n^{+*} as abstract monoids with generators and relations. Let $\iota : \{\sigma_1, \ldots, \sigma_n\} \to B_n^{+*}$ be defined by $\iota(\sigma_i) = \sigma_{i,i+1}$, and keep the notation ι to denote the natural extension on the free monoid $\iota : \{\sigma_1, \ldots, \sigma_n\}^* \to B_n^{+*}$. It is clear that ι is constant on congruence classes of positive braids, whence ι factors through $\iota : B_n^+ \to B_n^{+*}$. It can then be proved that this morphism is injective [22,10].

Remark 2.1. We emphasize that all the notions that we are about to define on B_n^+ and on B_n^{+*} may or may not coincide on $B_n^+ \cap B_n^{+*} = B_n^+$. Henceforth, it is probably clearer to keep in mind the point of view on these monoids through generators and relations, rather than as submonoids of B_n.

Running examples for $n = 3$ The presentations of the monoids B_3^+ and B_3^{+*} are the following:

$$B_3^+ = \langle \sigma_1,\ \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle^+$$

$$B_3^{+*} = \langle \sigma_{1,2},\ \sigma_{2,3},\ \sigma_{1,3} \mid \sigma_{1,2}\sigma_{2,3} = \sigma_{2,3}\sigma_{1,3} = \sigma_{1,3}\sigma_{1,2} \rangle^+$$

2.1.3. A common notation

We will consider simultaneously the monoids B_n^+ and B_n^{+*}. Henceforth, we will denote by $B_n^?$ a monoid which, unless stated otherwise, may be either the monoid B_n^+ or B_n^{+*}. The statements that we will prove for the monoid $B_n^?$ will then hold for both monoids B_n^+ and B_n^{+*}.

In addition, we will denote by Σ the set of generators of $B_n^?$, hence $\Sigma = \{\sigma_i \ :\ 1 \leq i \leq n-1\}$ if $B_n^? = B_n^+$ and $\Sigma = \{\sigma_{i,j} \ :\ 1 \leq i < j \leq n\}$ if $B_n^? = B_n^{+*}$.

2.1.4. Length and division relations. Mirror mapping

The above presentations (1) and (4) of $B_n^?$ are homogeneous, meaning that the relations involve words of the same lengths. Hence, the *length* of $x \in B_n^?$, denoted by $|x|$, is the length of any word in the equivalence class x, with respect to the congruence defining $B_n^?$.

Remark 2.2. The length is an example of a quantity which is defined on both B_n^+ and on B_n^{+*}, and which is invariant on B_n^+. That is to say, the length $|x|$ of a positive braid $x \in B_n^+$ is invariant whether x is considered as an element of B_n^+ or as an element of B_n^{+*}. Indeed, if x has length k as an element of B_n^+, then it necessarily writes as a product $x = \sigma_{\varphi(1)} \cdot \ldots \cdot \sigma_{\varphi(k)}$ for some function $\varphi : \{1, \ldots, k\} \to \{1, \ldots, n-1\}$. This entails that x, as an element of B_n^{+*}, writes as $x = \sigma_{\varphi(1),\varphi(1)+1} \cdot \ldots \cdot \sigma_{\varphi(k),\varphi(k)+1}$, and thus x also has length k as an element of B_n^{+*}.

The monoid $B_n^?$ is equipped with the *left* and with the *right divisibility* relations, denoted respectively $\leq_l$ and $\leq_r$, which are both partial orders on $B_n^?$, and are defined by:

$$x \leq_l y \iff \exists z \in B_n^? \quad y = x \cdot z, \qquad x \leq_r y \iff \exists z \in B_n^? \quad y = z \cdot x.$$

The mirror mapping, defined on words by $a_1 \dots a_k \mapsto a_k \dots a_1$, factorizes through $B_n^?$ and induces thus a *mirror mapping* on braids, denoted by $x \in B_n^? \mapsto x^* \in B_n^?$. It is an involutive anti-isomorphism of monoids, it preserves the length of braids and swaps the left and right divisibility relations:

$$\forall x \in B_n^? \quad |x^*| = |x|\,, \qquad \forall x, y \in B_n^? \quad x \leq_{\mathrm{r}} y \iff x^* \leq_{\mathrm{l}} y^*\,.$$

The mirror mapping being an isomorphism of partial orders $(B_n^?, \leq_{\mathrm{l}}) \to (B_n^?, \leq_{\mathrm{r}})$, we shall focus on the left divisibility relation $\leq_{\mathrm{l}}$ only.

Remark 2.3. Following Remark 2.2, it is clear that the left divisibility is also invariant on B_n^+: if $x, y \in B_n^+$ are such that $x \leq_{\mathrm{l}} y$ in B_n^+, then $x \leq_{\mathrm{l}} y$ also holds in B_n^{+*}. Observe however that the converse is not true. For instance, consider the case $n = 3$ and set $x = \sigma_2$ and $y = \sigma_1 \cdot \sigma_2$. In B_3^+, clearly, $x \leq_{\mathrm{l}} y$ does not hold. However, in B_3^{+*}, we have $x = \sigma_{2,3}$ and $y = \sigma_{1,2} \cdot \sigma_{2,3} = \sigma_{2,3} \cdot \sigma_{1,3}$, therefore $x \leq_{\mathrm{l}} y$ does hold.

2.2. Garside structure and simple braids

2.2.1. Garside structure

The monoid $B_n^?$ is known to be a *Garside monoid* [4,6,10]; that is to say:

(1) $B_n^?$ is a cancellative monoid;
(2) $B_n^?$ contains a *Garside element*, that is to say, an element whose set of left divisors coincides with its set of right divisors and contains the generating set Σ;
(3) Every finite subset X of $B_n^?$ has a least upper bound in $(B_n^?, \leq_{\mathrm{l}})$, and a greatest lower bound if $X \neq \emptyset$, respectively denoted $\bigvee_{\mathrm{l}} X$ and $\bigwedge_{\mathrm{l}} X$.

Let $\bigvee_{\mathrm{r}} X$ denote the least upper bound in $(B_n^?, \leq_{\mathrm{r}})$ of a subset $X \subseteq B_n^?$. Then, if X is a subset of Σ, it is known [10,22] that $\bigvee_{\mathrm{l}} X$ and $\bigvee_{\mathrm{r}} X$ coincide. We introduce therefore the notation Δ_X for:

$$\Delta_X = \bigvee\nolimits_{\mathrm{l}} X = \bigvee\nolimits_{\mathrm{r}} X\,, \qquad \text{for } X \subseteq \Sigma.$$

Moreover, Δ_X has the same left divisors and right divisors in $B_n^?$: $\{x \in B_n^? \ : \ x \leq_{\mathrm{l}} \Delta_X\} = \{x \in B_n^? \ : \ x \leq_{\mathrm{r}} \Delta_X\}$.

The element Δ_X one obtains when considering $X = \Sigma$ plays a special role in Garside theory. Indeed, based on the above remarks, it is not difficult to see that Δ_Σ is a Garside element of $B_n^?$, and is moreover the *smallest* Garside element of $B_n^?$. Defining the elements $\Delta_n \in B_n^+$ and $\delta_n \in B_n^{+*}$ by:

$$\Delta_n = (\sigma_1 \cdot \ldots \cdot \sigma_{n-1}) \cdot (\sigma_1 \cdot \ldots \cdot \sigma_{n-2}) \cdot \ldots \cdot (\sigma_1 \cdot \sigma_2) \cdot \sigma_1\,,$$

$$\delta_n = \sigma_{1,2} \cdot \sigma_{2,3} \cdot \ldots \cdot \sigma_{n-1,n}\,,$$

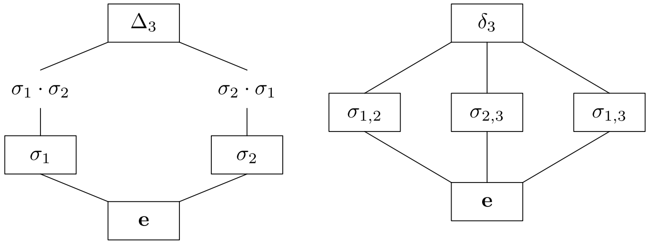

Fig. 3. Hasse diagrams of $(\mathcal{S}_3, \leq_l)$ for B_3^+ at left and for B_3^{+*} at right. Elements of $\mathcal{D}_3$ are framed.

we have $\Delta_\Sigma = \Delta_n$ if $B_n^? = B_n^+$ and $\Delta_\Sigma = \delta_n$ if $B_n^? = B_n^{+*}$. We adopt the single notation $\Delta = \Delta_\Sigma$ to denote either Δ_n or δ_n according to which monoid we consider.

2.2.2. Definition of simple braids

The set of all divisors of Δ is denoted by $\mathcal{S}_n$, and its elements are called *simple braids*. It is a bounded subset of $B_n^?$, with $\mathbf{e}$ as minimum and Δ as maximum, closed under $\bigvee_l$ and under $\bigwedge_l$. With the induced partial order, $(\mathcal{S}_n, \leq_l)$ is thus a lattice.

Consider the mapping $\Phi : \mathcal{P}(\Sigma) \to \mathcal{S}_n,\ X \mapsto \Delta_X$, and its image:

$$\mathcal{D}_n = \{\Delta_X \ : \ X \subseteq \Sigma\}\,.$$

Then $\Phi : \big(\mathcal{P}(\Sigma), \subseteq\big) \to (\mathcal{S}_n, \leq_l)$ is a lattice homomorphism, and $\mathcal{D}_n$ is thus a sub-lattice of $(\mathcal{S}_n, \leq_l)$. If $B_n^? = B_n^+$, the mapping Φ is injective, but *not onto* $\mathcal{S}_n$, and so $(\mathcal{D}_n, \leq_l)$ is isomorphic to $\big(\mathcal{P}(\Sigma), \subseteq\big)$. If $B_n^? = B_n^{+*}$, the mapping Φ is not injective, but it is onto $\mathcal{S}_n$, hence $\mathcal{D}_n = \mathcal{S}_n$.

Remark 2.4. Contrasting with the length discussed in Remark 2.2, the notion of simplicity is *not* invariant on B_n^+. For instance, the braid Δ_n is simple in B_n^+, but it is not simple as an element of B_n^{+*} as soon as $n \geq 3$. Indeed, since its length is $|\Delta_n| = n(n-1)/2$, it cannot be a divisor of δ_n which is of length $|\delta_n| = n - 1$.

Running examples for $n = 3$ The Garside elements of B_3^+ and of B_3^{+*} are:

$$\delta_3 = \sigma_{1,2} \cdot \sigma_{2,3}\,, \qquad \Delta_3 = \sigma_1 \cdot \sigma_2 \cdot \sigma_1\,.$$

The Hasse diagrams of $(\mathcal{S}_3, \leq_l)$ are pictured in Fig. 3. For B_3^+, the lattice $(\mathcal{D}_3, \leq_l)$ consists of the following four elements:

$$\Delta_\emptyset = \mathbf{e} \qquad \Delta_{\sigma_1} = \sigma_1 \qquad \Delta_{\sigma_2} = \sigma_2 \qquad \Delta_{\sigma_1,\sigma_2} = \sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \Delta$$

whereas the lattice $(\mathcal{S}_3, \leq_l)$ contains the two additional elements $\sigma_1 \cdot \sigma_2$ and $\sigma_2 \cdot \sigma_1$. For B_3^{+*}, the elements of $\mathcal{D}_3$ and $\mathcal{S}_3$ are:

$$\Delta_\emptyset = \mathbf{e} \qquad \Delta_{\{\sigma_{1,2}\}} = \sigma_{1,2} \qquad \Delta_{\{\sigma_{2,3}\}} = \sigma_{2,3} \qquad \Delta_{\{\sigma_{1,3}\}} = \sigma_{1,3}$$
$$\Delta_{\{\sigma_{1,2},\sigma_{2,3}\}} = \Delta_{\{\sigma_{2,3},\sigma_{1,3}\}} = \Delta_{\{\sigma_{1,2},\sigma_{1,3}\}} = \Delta_{\{\sigma_{1,2},\sigma_{2,3},\sigma_{1,3}\}} = \delta_3$$

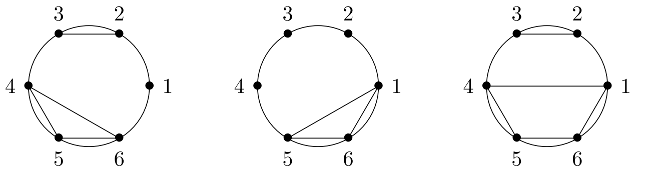

Fig. 4. Left, representation of $\sigma_{2,3} \cdot \sigma_{4,5} \cdot \sigma_{5,6}$ in B_6^{+*} as a non-crossing partition of $\{1, \ldots, 6\}$. Middle, representation of $\sigma_{1,5} \cdot \sigma_{5,6}$. Right, representation of their least upper bound $(\sigma_{2,3} \cdot \sigma_{4,5} \cdot \sigma_{5,6}) \vee_{l} (\sigma_{1,5} \cdot \sigma_{5,6}) = \sigma_{2,3} \cdot \sigma_{1,4} \cdot \sigma_{4,5} \cdot \sigma_{5,6}$.

2.2.3. *Combinatorial representations of simple braids*

The natural map that sends each generator σ_i of the braid group B_n to the transposition $(i, i+1)$ induces a morphism from B_n to $\mathfrak{S}_n$, the set of permutations of n elements. Hence, this map also induces morphisms from B_n^+ and from B_n^{+*} to $\mathfrak{S}_n$.

In the case of the braid monoid B_n^+, this morphism induces a bijection from $\mathcal{S}_n$ to $\mathfrak{S}_n$. Thus, $\mathcal{S}_n$ has cardinality $n!$. The element $\mathbf{e}$ corresponds to the identity permutation, and the element Δ_n to the mirror permutation $i \mapsto n + 1 - i$.

From the point of view of braid diagrams, such as the one pictured in Fig. 1, simple braids correspond to braids such that in any representative braid diagram, any two strands cross at most once.

In the case of the dual braid monoid B_n^{+*}, this morphism only induces an injection from $\mathcal{S}_n$ to $\mathfrak{S}_n$. It is thus customary to consider instead the following alternative representation. Recall that a partition $\{T^1, \ldots, T^m\}$ of $\{1, \ldots, n\}$ is called *non-crossing* if the sets $\{\exp(2\mathbf{i}k\pi/n) \ : \ k \in T^i\}$ have pairwise disjoint convex hulls in the complex plane. For instance, on the left of Fig. 4, is illustrated the fact that $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$ is a non-crossing partition of $\{1, 2, \ldots, 6\}$.

Now, for each subset T of $\{1, \ldots, n\}$, let x_T denote the dual braid $\sigma_{t_1,t_2} \cdot \sigma_{t_2,t_3} \cdot \ldots \cdot \sigma_{t_{k-1},t_k}$, where $t_1 < t_2 < \ldots < t_k$ are the elements of T. Then, for each non-crossing partition $\mathbf{T} = \{T^1, \ldots, T^m\}$ of $\{1, \ldots, n\}$, we denote by $x_{\mathbf{T}}$ the (commutative) product $x_{T^1} \cdot \ldots \cdot x_{T^m}$. It is known [10,7] that the mapping $\mathbf{T} \mapsto x_{\mathbf{T}}$ is a lattice isomorphism from the set of non-crossing partitions of $\{1, \ldots, n\}$ onto $\mathcal{S}_n$. Thus in particular, the cardinality of $\mathcal{S}_n$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. In this representation, $\mathbf{e}$ corresponds to the finest partition $\{\{1\}, \ldots, \{n\}\}$, and δ_n to the coarsest partition $\{\{1, \ldots, n\}\}$. See Fig. 4.

Running examples for $n = 3$ Let us consider the case $n = 3$. For B_3^+, the correspondence between simple braids and permutations of $\{1, 2, 3\}$ is the following:

$$\mathbf{e} = \mathrm{Id} \qquad \sigma_1 = (1,2) \qquad \sigma_2 = (2,3)$$
$$\sigma_1 \cdot \sigma_2 = (1,2,3) \qquad \sigma_2 \cdot \sigma_1 = (1,3,2) \qquad \Delta = (3,1)$$

Simple braids of B_3^{+*} correspond to non-crossing partitions of $\{1, 2, 3\}$, which in this case are simply all the partitions of $\{1, 2, 3\}$. The correspondence is the following, where singletons are omitted:

$$\mathbf{e} = \{\} \qquad \sigma_{1,2} = \{\{1,2\}\} \qquad \sigma_{2,3} = \{\{2,3\}\} \qquad \sigma_{1,3} = \{\{1,3\}\} \qquad \delta_3 = \{\{1,2,3\}\}$$

3. Garside normal form and combinatorics of braids

Braids are known to admit normal forms, that is to say, a unique combinatorial representation for each braid. Normal forms are the standard tool for several algorithmic problems related to braids, for instance the word problem to cite one of the most emblematic [17]. Among the several normal forms for braids introduced in the literature, we shall focus in this work on the Garside normal form which derives from the Garside structure attached to our braid monoids, as recalled above.

3.1. Garside normal form of braids

In the monoid $B_n^?$, and regardless on whether $B_n^? = B_n^+$ or $B_n^? = B_n^{+*}$, a sequence $(x_1, \ldots, x_k)$ of *simple braids* is said to be *normal* if $x_j = \Delta \wedge_{\mathrm{l}} (x_j \cdot \ldots \cdot x_k)$ holds for all $j = 1, \ldots, k$. Intuitively, this is a maximality property, meaning that no left divisor of $x_{j+1} \cdots x_k$ could be moved to x_j while remaining in the world of simple braids. We recall the two following well known facts concerning normal sequences of braids:

1. For $x, y \in \mathcal{S}_n$, let $x \to y$ denote the relation $\mathsf{R}(x) \supseteq \mathsf{L}(y)$, where the sets $\mathsf{R}(x)$ and $\mathsf{L}(y)$ are defined as follows:
$$\mathsf{R}(x) = \big\{\sigma \in \Sigma \ : \ x \cdot \sigma \notin \mathcal{S}_n\big\}, \qquad \mathsf{L}(y) = \big\{\sigma \in \Sigma \ : \ \sigma \leq_{\mathrm{l}} y\big\}. \tag{5}$$
Then a sequence $(x_1, \ldots, x_k)$ is normal if and only if $x_j \to x_{j+1}$ holds for all $j = 1, \ldots, k-1$, again meaning that left divisors of x_{j+1} are already present in x_j, and therefore cannot be pushed into x_j while keeping it simple.
2. For every non-unit braid $x \in B_n^?$, there exists a unique normal sequence $(x_1, \ldots, x_k)$ of non-unit simple braids such that $x = x_1 \cdot \ldots \cdot x_k$. This sequence is called the *Garside normal form* or *decomposition* of x.

In this work, the integer k is called the *height* of x (it is also called the *supremum* of x in the literature [20]). We denote it by $\tau(x)$.

Regarding the special elements $\mathbf{e}$ and Δ, the following dual relations hold, meaning that $\mathbf{e}$ is “final” whereas Δ is “initial”:

$$\begin{array}{ll} \forall x \in \mathcal{S}_n \quad x \to \mathbf{e} & \qquad \forall x \in \mathcal{S}_n \quad \mathbf{e} \to x \iff x = \mathbf{e} \\ \forall x \in \mathcal{S}_n \quad \Delta \to x & \qquad \forall x \in \mathcal{S}_n \quad x \to \Delta \iff x = \Delta \end{array}$$

Therefore, the Garside decomposition starts with a finite (possibly zero) number of Δs, and then is given by a finite path in the finite directed graph $(\mathcal{S}_n \setminus \{\Delta, \mathbf{e}\}, \to)$. By convention, we define the Garside normal form of the unit braid $\mathbf{e}$ as the sequence $(\mathbf{e})$, and we put $\tau(\mathbf{e}) = 1$. (It might seem that $\tau(\mathbf{e}) = 0$ would be a more natural convention, but it turns out that taking $\tau(\mathbf{e}) = 1$ is the good choice for convenient formulation of several results below as it encompasses the fact that, in normal forms, $\mathbf{e}$ can not be followed by any letter. For instance, Lemma 4.3 would not hold otherwise.) Then, it is a well-known property of Garside monoids that $\tau(x)$ is the least positive integer k such that x is a product of k simple braids.

Moreover, it will be convenient to complete the normal form of a braid with infinitely many factors $\mathbf{e}$. We call the infinite sequence $(x_k)_{k\geq 1}$ of simple braids thus obtained the *extended Garside decomposition* of the braid. The directed graph $(\mathcal{S}_n, \to)$ is then their accepting graph: extended Garside decompositions of braids correspond bijectively to infinite paths in $(\mathcal{S}_n, \to)$ that eventually hit $\mathbf{e}$, and then necessarily stay in $\mathbf{e}$ forever.

Remark 3.1. Following up on Remark 2.4, just as simplicity was observed not to be invariant on B_n^+, the height and the Garside normal forms are not invariant on B_n^+. For instance, the braid Δ_n has Garside normal form the sequence (Δ_n) itself in B_n^+, and the sequence $(\delta_n, \delta_{n-1}, \ldots, \delta_2)$ in B_n^{+*}. Its height is 1 in B_n^+ and $n-1$ in B_n^{+*}.

We gather in the following proposition some well-known properties of Garside normal forms [18] that we shall use later.

Proposition 3.2. *For all braids $x, y \in B_n^?$:*

1. *the height $\tau(x)$ is the smallest integer $k \geq 1$ such that $x \leq_l \Delta^k$;*
2. *if $(x_1, \ldots, x_k)$ is the normal form of x, then $x_1 \cdot \ldots \cdot x_j = x \wedge_l \Delta^j$ for all $j \in \{1, \ldots, k\}$;*
3. $x \leq_l y \iff x \leq y \wedge_l \Delta^{\tau(x)}$.

Remark 3.3. We should stress that, while the normal form is very convenient to enumerate braids, it behaves poorly with respect to multiplication: consider x with height k and Garside decomposition $(x_1, \ldots, x_k)$, and let σ be a generator. Then, $y = x \cdot \sigma$ has height in $\{k, k+1\}$, but if it has height k then the normal form $y = (y_1, \ldots, y_k)$ might be completely different from that of x (in the sense that $y_1 \neq x_1, \ldots, y_k \neq x_k$), although it is algorithmically computable.

Running examples for $n = 3$ Let us describe explicitly the accepting graphs $(\mathcal{S}_3, \to)$ for B_3^+ and for B_3^{+*}. Consider first the case of B_3^+. The subsets $\mathsf{L}(x)$ and $\mathsf{R}(x)$ are easily computed through their definition (5), from which the relation $\to$ is derived. The results of these computations are depicted in Fig. 5. The analogous computations for B_3^{+*} result in the data pictured in Fig. 6.

3.2. Combinatorics of braids

How many braids $x \in B_n^?$ of length k are there? Is there either an exact or an approximate formula? The aim of this subsection is to recall the classical answers to these questions, which we will do by analyzing the ordinary generating function, or

$\mathsf{L}(x)$	$x \in \mathcal{S}_3$	$\mathsf{R}(x)$	$\{y \in \mathcal{S}_3 \ : \ x \to y\}$
$\emptyset$	$\mathbf{e}$	$\emptyset$	$\{\mathbf{e}\}$
$\{\sigma_1\}$	σ_1	$\{\sigma_1\}$	$\{\mathbf{e}, \sigma_1, \sigma_1 \cdot \sigma_2\}$
$\{\sigma_2\}$	σ_2	$\{\sigma_2\}$	$\{\mathbf{e}, \sigma_2, \sigma_2 \cdot \sigma_1\}$
$\{\sigma_1\}$	$\sigma_1 \cdot \sigma_2$	$\{\sigma_2\}$	$\{\mathbf{e}, \sigma_2, \sigma_2 \cdot \sigma_1\}$
$\{\sigma_2\}$	$\sigma_2 \cdot \sigma_1$	$\{\sigma_1\}$	$\{\mathbf{e}, \sigma_1, \sigma_1 \cdot \sigma_2\}$
Σ	Δ_3	Σ	$\mathcal{S}_3$

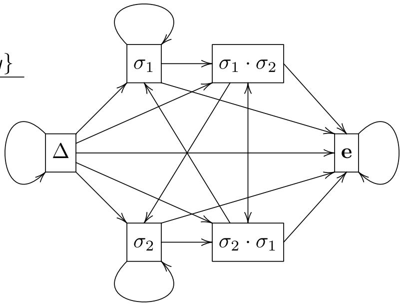

Fig. 5. Left hand side: subsets $\mathsf{L}(x)$ and $\mathsf{R}(x)$ for simple braids $x \in B_3^+$. Right hand side: the resulting accepting graph $(\mathcal{S}_3, \to)$ for B_3^+.

$\mathsf{L}(x)$	$x \in \mathcal{S}_3$	$\mathsf{R}(x)$	$\{y \in \mathcal{S}_3 \ : \ x \to y\}$
$\emptyset$	$\mathbf{e}$	$\emptyset$	$\{\mathbf{e}\}$
$\{\sigma_{1,2}\}$	$\sigma_{1,2}$	$\{\sigma_{1,2},\ \sigma_{1,3}\}$	$\{\mathbf{e}, \sigma_{1,2},\ \sigma_{1,3}\}$
$\{\sigma_{2,3}\}$	$\sigma_{2,3}$	$\{\sigma_{2,3},\ \sigma_{1,2}\}$	$\{\mathbf{e}, \sigma_{2,3},\ \sigma_{1,2}\}$
$\{\sigma_{1,3}\}$	$\sigma_{1,3}$	$\{\sigma_{1,3},\ \sigma_{2,3}\}$	$\{\mathbf{e}, \sigma_{1,3},\ \sigma_{2,3}\}$
Σ	δ_3	Σ	$\mathcal{S}_3$

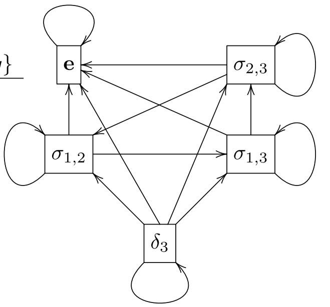

Fig. 6. Left hand side: subsets $\mathsf{L}(x)$ and $\mathsf{R}(x)$ for simple braids $x \in B_3^{+*}$. Right hand side: the resulting accepting graph $(\mathcal{S}_3, \to)$ for B_3^{+*}.

growth series, of $B_n^?$. This series turns out to be rational, and the Garside normal form is an efficient tool for the study of its coefficients.

3.2.1. Growth series of braid monoids and Möbius polynomial

Let $G_n(t)$ be the growth series of $B_n^?$. It is defined by:

$$G_n(t) = \sum_{x \in B_n^?} t^{|x|} .$$

According to a well-known result [19,15,12], the growth function of $B_n^?$ is rational, inverse of a polynomial:

$$G_n(t) = \frac{1}{H_n(t)} , \qquad \text{with} \quad H_n(t) = \sum_{X \subseteq \Sigma} (-1)^{|X|}\, t^{|\Delta_X|} . \tag{6}$$

There exist explicit or recursive formulæ allowing to compute effectively $H_n(t)$:

$$H_n(t) = \sum_{k=1}^{n} (-1)^{k+1} t^{\frac{k(k-1)}{2}} H_{n-k}(t) \qquad\qquad \text{if } B_n^? = B_n^+$$

$$H_n(t) = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1+k)!}{(n-1-k)!k!(k+1)!} t^k \qquad \text{if } B_n^? = B_n^{+*}$$

For reasons that will appear more clearly in a moment (see Subsection 3.3), the polynomial $H_n(t)$ deserves the name of *Möbius polynomial of* $B_n^?$.

Running examples for $n = 3$ For B_3^+, the computation of the Möbius polynomial may be done as follows:

$$H_3(t) = 1 - t^{|\sigma_1|} - t^{|\sigma_2|} + t^{|\sigma_1 \vee \sigma_2|} = 1 - 2t + t^3 \qquad \text{since } \sigma_1 \vee \sigma_2 = \Delta_3$$

and similarly for B_3^{+*}:

$$\begin{aligned} H_3(t) &= 1 - t^{|\sigma_{1,2}|} - t^{|\sigma_{2,3}|} - t^{|\sigma_{1,3}|} + t^{|\sigma_{1,2} \vee \sigma_{1,3}|} + t^{|\sigma_{1,2} \vee \sigma_{2,3}|} + t^{|\sigma_{2,3} \vee \sigma_{1,3}|} - t^{|\delta_3|} \\ &= 1 - 3t + 2t^2 \qquad \text{since } \sigma_{1,2} \vee \sigma_{1,3} = \sigma_{1,2} \vee \sigma_{2,3} = \sigma_{2,3} \vee \sigma_{1,3} = \delta_3 \end{aligned}$$

3.2.2. Connectivity of the Charney graph

The growth series $G_n(t)$ is a rational series with non-negative coefficients and with a finite positive radius of convergence, say q_n, which, by the Pringsheim theorem [21], is necessarily itself a pole of $G_n(t)$. Since $G_n(t) = 1/H_n(t)$ as recalled in (6), it follows that q_n is a root of minimal modulus of the polynomial $H_n(t)$. In order to evaluate the coefficients of $G_n(t)$, we shall prove that $G_n(t)$ has no other pole of modulus q_n, or equivalently, that $H_n(t)$ has no other root of modulus q_n. This is stated in Corollary 3.5 below.

To this end, we first study the connectivity of the *Charney graph*, which is the directed graph $\mathcal{G} = (V, E)$ with set of vertices $V = \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$ and set of edges $E = \{(x, y) \in V^2 \,:\, x \to y\}$. The connectivity of $\mathcal{G}$ is well known for $B_n^? = B_n^+$ [8,14,25], and actually the same result also holds for $B_n^? = B_n^{+*}$, although it does not seem to be found in the literature. We obtain thus the following result.

Proposition 3.4. *For $n \geq 3$, the Charney graph of $B_n^?$ is strongly connected and contains loops.*

Proof. First, observe that the graph $\mathcal{G}$ contains the loop $\sigma \to \sigma$ for every generator $\sigma \in \Sigma$. Since proofs of the strong connectivity of $\mathcal{G}$ are found in the literature when $B_n^? = B_n^+$, we focus on proving that $\mathcal{G}$ is strongly connected when $B_n^? = B_n^{+*}$.

Recall that simple braids are in bijection with non-crossing partitions of $\{1, \ldots, n\}$. For each subset T of $\{1, \ldots, n\}$, we denote by x_T the braid $\sigma_{t_1,t_2} \cdot \sigma_{t_2,t_3} \cdot \ldots \cdot \sigma_{t_{k-1},t_k}$, where $t_1 < t_2 < \ldots < t_k$ are the elements of T. Then, for each non-crossing partition $\mathbf{T} = \{T^1, \ldots, T^m\}$ of $\{1, \ldots, n\}$, we denote by $x_{\mathbf{T}}$ the (commutative) product $x_{T^1} \cdot \ldots \cdot x_{T^m}$. It is known [10] that

(1) the mapping $\mathbf{T} \mapsto x_{\mathbf{T}}$ is a bijection from the set of non-crossing partitions of $\{1, \ldots, n\}$ to $\mathcal{S}_n$, as mentioned in Subsection 2.2;
(2) the set $\mathsf{L}(x_{\mathbf{T}})$ is equal to $\{\sigma_{u,v} \ : \ \exists T \in \mathbf{T} \quad u, v \in T\}$;
(3) the set $\mathsf{R}(x_{\mathbf{T}})$ is equal to

$$\big\{\sigma_{u,v} \ : \ \exists T \in \mathbf{T} \quad T \cap \{u+1, \ldots, v\} \neq \emptyset \text{ and } T \cap \{1, \ldots, u, v+1, \ldots, n\} \neq \emptyset\big\}\,.$$

Hence, consider two braids $y, z \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$, and let $m = \lfloor \frac{n}{2} \rfloor$, as well as the set $A = \{\sigma_{1,2}, \ldots, \sigma_{n-1,n}, \sigma_{n,1}\}$. Since $z <_1 \Delta$, we know that $z = x_{\mathbf{Z}}$ where $\mathbf{Z}$ is a partition of $\{1, \ldots, n\}$ in at least two subsets. It follows that $\mathbf{Z}$ is a refinement of a non-crossing partition $\mathbf{V}$ of $\{1, \ldots, n\}$ in *exactly* two subsets. The map $\sigma_{i,j} \mapsto \sigma_{i+1,j+1}$ induces an automorphism of the dual braid monoid B_n^{+*}, hence we assume without loss of generality that $\mathbf{V} = \big\{\{1, \ldots, i, n\}, \{i+1, \ldots, n-1\}\big\}$ for some integer $i \in \{0, \ldots, m-1\}$.

Finally, consider some generator $\sigma_{a,b} \in \mathsf{L}(y)$, with $a < b$. Since $\mathsf{L}(y) \subseteq \mathsf{R}(y)$, it comes that $y \to \sigma_{a,b} \to \sigma_{1,a} \to \sigma_{1,n} \to \sigma_{2,n}$ if $1 < a$, or $y \to \sigma_{1,b} \to \sigma_{1,n} \to \sigma_{2,n}$ if $a = 1$. Since $2 \leq m + 1 < n$, we then observe that $\sigma_{2,n} \to \sigma_{1,m+1} \to x_{\mathbf{T}} \to x_{\mathbf{U}} \to x_{\mathbf{V}} \to x_{\mathbf{Z}} = z$, where

$$\mathbf{T} = \begin{cases} \big\{\{u, n+1-u\} \ : \ 1 \leq u \leq m\big\} & \text{if } n \text{ is even,} \\ \big\{\{u, n+1-u\} \ : \ 1 \leq u \leq m\big\} \cup \big\{\{m+1\}\big\} & \text{if } n \text{ is odd;} \end{cases}$$

$$\mathbf{U} = \big\{\{1, \ldots, m\}, \{m+1, \ldots, n\}\big\}.$$

This completes the proof. $\square$

The connectivity of the Charney graph stated in the above proposition has several consequences on the combinatorics of braids, which we gather in the following corollary, the result of which seems to have been unnoticed so far.

Corollary 3.5. *The Möbius polynomial $H_n(t)$ has a unique root of smallest modulus. This root, say q_n, is real and lies in $(0, 1)$ and it is simple. It coincides with the radius of convergence of the growth series $G_n(t)$.*

Furthermore, for each integer $k \geq 0$, put $\lambda_n(k) = \#\{x \in B_n^? \ : \ |x| = k\}$. Then, for $n \geq 3$, the following asymptotics hold for some constant $C_n > 0$:

$$\lambda_n(k) \sim_{k \to \infty} C_n q_n^{-k}\,. \tag{7}$$

Proof. Recall that we have already defined q_n, at the beginning of Subsection 3.2.2, as the radius of convergence of $G_n(t)$, and we know that q_n is a root of smallest modulus of $H_n(t)$. We will now derive the two statements of the corollary through an application of Perron–Frobenius theory for primitive matrices (see, *e.g.*, [28]).

Let I_Δ and $I_{\neg\Delta}$ denote the sets

$$I_\Delta = \{(i, \Delta) \ : \ 1 \leq i \leq |\Delta|\}\,, \qquad I_{\neg\Delta} = \{(i, \sigma) \ : \ \sigma \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\} \text{ and } 1 \leq i \leq |\sigma|\}\,,$$

and let I denote the disjoint union $I_\Delta \cup I_{\neg\Delta}$. Let $M = (M_{x,y})_{x,y\in I}$ be the non-negative matrix defined as follows:

$$M_{(i,\sigma),(j,\tau)} = \begin{cases} 1, & \text{if } j = i+1 \text{ and } \sigma = \tau, \\ 1, & \text{if } i = |\sigma| \text{ and } j = 1 \text{ and } \sigma \to \tau, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, M is a block triangular matrix $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ where A, B and C are the restrictions of M to the respective sets of indices $I_\Delta \times I_\Delta$, $I_\Delta \times I_{\neg\Delta}$ and $I_{\neg\Delta} \times I_{\neg\Delta}$.

Since the Charney graph is strongly connected and contains loops according to Proposition 3.4, and since it contains at least $n-1$ vertices (the elements of Σ), it follows that C is a primitive matrix with Perron eigenvalue $\rho > 1$. By construction, we know that $A^{|\Delta|} = \mathbf{Id}_{|\Delta|}$, hence that the eigenvalues of A have modulus 1. Consequently, ρ is a simple eigenvalue of M, and has a strictly greater modulus than all other eigenvalues of M. Hence, there exist left and right eigenvectors $\mathbf{l}$ and $\mathbf{r}$ of M for the eigenvalue ρ with non-negative entries, whose restrictions $(\mathbf{l}_x)_{x\in I_{\neg\Delta}}$ and $(\mathbf{r}_x)_{x\in I_{\neg\Delta}}$ only have positive entries, and such that $\mathbf{l}\cdot\mathbf{r} = 1$.

Then, observe that $\lambda_n(k) = \mathbf{u}\cdot M^{k-1}\cdot\mathbf{v}$ for all $k \geq 1$, where $\mathbf{u}$ is the row vector defined by $\mathbf{u}_{(i,\sigma)} = \mathbf{1}(i = 1)$ and $\mathbf{v}$ is the column vector defined by $\mathbf{v}_{(i,\sigma)} = \mathbf{1}(i = |\sigma|)$. Indeed, this follows at once from the existence and uniqueness of the Garside normal form for braids, and from the construction of the matrix M.

Both vectors $\mathbf{u}$ and $\mathbf{v}$ have some non-zero entries in $I_{\neg\Delta}$, and therefore

$$\lambda_n(k) = \mathbf{u}\cdot M^{k-1}\cdot\mathbf{v} \sim_{k\to\infty} \rho^{k-1}(\mathbf{u}\cdot\mathbf{r})(\mathbf{l}\cdot\mathbf{v}) \tag{8}$$

Hence, ρ^{-1} is the radius of convergence of the generating series $G_n(t) = \sum_{k\geq 0}\lambda_n(k)$, and thus $\rho^{-1} = q_n$.

To complete the proof, consider the decomposition of $H_n(t)$ as a product of the form:

$$H_n(t) = (1 - t/q_n)\cdot(1 - t/r_1)\cdot\ldots\cdot(1 - t/r_i)\cdot(1 - t/a_1)\cdot\ldots\cdot(1 - t/a_j)\,,$$

where $r_1,\ldots,r_i$ are the other complex roots of $H_n(t)$ of modulus q_n, including q_n if its multiplicity is > 1, and $a_1,\ldots,a_j$ are the remaining complex roots of $H_n(t)$, hence of modulus greater than q_n. Since we know that $G_n(t) = 1/H_n(t)$, and considering the series expansion of $1/H_n(t)$, one sees that the equivalent found in (8) for the coefficients $\lambda_n(\cdot)$ of $G_n(t)$ cannot hold if $i > 0$, whence the result. □

3.3. Two Möbius transforms

This last subsection is devoted to the study of Möbius transforms. In § 3.3.1, we particularize to our braid monoids $B_n^?$ the classical Möbius transform, as defined for general classes of partial orders [27,29]. We prove the Möbius inversion formula for our particular case, although it could be derived from more general results. Next, we introduce in § 3.3.2 a variant, called graded Möbius transform, which will prove to be most useful later for the probabilistic analysis.

We will use extensively the notation $\mathbf{1}(A)$ for the characteristic function of A, equal to 1 if A is true and to 0 is A is false.

3.3.1. The standard Möbius transform

In the framework of braid monoids, the usual Möbius transform is defined as follows and leads to the next proposition.

Definition 3.6. Given a real-valued function $f : B_n^? \mapsto \mathbb{R}$, its *Möbius transform* is the function $h : B_n^? \mapsto \mathbb{R}$ defined by:

$$h(x) = \sum_{X \subseteq \Sigma} (-1)^{|X|} f(x \cdot \Delta_X) \text{ for all } x \in B_n^?. \tag{9}$$

Proposition 3.7. *Let $f, h : B_n^? \to \mathbb{R}$ be two functions such that the series $\sum_{x \in B_n^?} |f(x)|$ and $\sum_{x \in B_n^?} |h(x)|$ are convergent. Then h is the Möbius transform of f if and only if*

$$\forall x \in B_n^? \quad f(x) = \sum_{y \in B_n^?} h(x \cdot y)\,. \tag{10}$$

Proof. For every non-unit braid $z \in B_n^?$, the sets $\mathsf{L}(z)$ and $\mathsf{L}(z^*)$ are non-empty, hence the powersets $\mathcal{P}\mathsf{L}(z)$ and $\mathcal{P}\mathsf{L}(z^*)$ are non-trivial Boolean lattices. It follows from the equality $\Delta_X = \bigvee_{\mathrm{l}} X$ that:

$$\sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(\Delta_X \leq_{\mathrm{l}} z) = \sum_{X \subseteq \mathsf{L}(z)} (-1)^{|X|} = 0\,.$$

And, similarly, it follows from the equality $\Delta_X = \bigvee_{\mathrm{r}} X$ that:

$$\sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(\Delta_X \leq_{\mathrm{r}} z) = \sum_{X \subseteq \mathsf{L}(z^*)} (-1)^{|X|} = 0\,.$$

Consider now two functions f and h such that the series $\sum_{x \in B_n^?} |f(x)|$ and $\sum_{x \in B_n^?} |h(x)|$ are convergent. Assume first that h is the Möbius transform of f. Using the change of variable $v = y \cdot \Delta_X$, we derive from (9) the following:

$$\begin{aligned} f(x) &= \sum_{v \in B_n^?} \Big(\sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(\Delta_X \leq_{\mathrm{r}} v) \Big) f(x \cdot v) \\ &= \sum_{y \in B_n^?} \sum_{X \subseteq \Sigma} (-1)^{|X|} f(x \cdot y \cdot \Delta_X) = \sum_{y \in B_n^?} h(x \cdot y), \end{aligned}$$

proving (10).

Table 1
Values of the Möbius transform $h : \mathcal{S}_3 \to \mathbb{R}$ of the function $f : \mathcal{S}_3 \to \mathbb{R}$ defined by $f(x) = p^{|x|}$ for B_3^+ (left hand side) and for B_3^{+*} (right hand side).

$x \in \mathcal{S}_3$	$h(x)$
$\mathbf{e}$	$1 - 2p + p^3 = H_3(p)$
σ_1	$p - p^2$
σ_2	$p - p^2$
$\sigma_1 \cdot \sigma_2$	$p^2 - p^3$
$\sigma_2 \cdot \sigma_1$	$p^2 - p^3$
Δ_3	p^3

$x \in \mathcal{S}_3$	$h(x)$
$\mathbf{e}$	$1 - 3p + 2p^2 = H_3(p)$
$\sigma_{1,2}$	$p - p^2$
$\sigma_{2,3}$	$p - p^2$
$\sigma_{1,3}$	$p - p^2$
δ_3	p^2

Conversely, if (10) holds, we use the change of variable $u = \Delta_X \cdot y$ to derive:

$$\begin{aligned} h(x) &= \sum_{u \in B_n^?} \Big(\sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(\Delta_X \leq_l u) \Big) h(x \cdot u) \\ &= \sum_{y \in B_n^?} \sum_{X \subseteq \Sigma} (-1)^{|X|} h(x \cdot \Delta_X \cdot y) = \sum_{X \subseteq \Sigma} (-1)^{|X|} f(x \cdot \Delta_X). \end{aligned}$$

This shows that h is the Möbius transform of f, completing the proof. $\square$

In particular, observe that, if a function f has support in $\mathcal{S}_n$, then so does its Möbius transform h. Hence, we also define the notion of Möbius transform of real-valued functions $f : \mathcal{S}_n \to \mathbb{R}$ in a natural way. In that narrower context, Proposition 3.7 formulates as follows.

Corollary 3.8. *Let $f, h : \mathcal{S}_n \to \mathbb{R}$ be two functions. Then the two statements:*

$$\forall x \in \mathcal{S}_n \quad f(x) = \sum_{y \in \mathcal{S}_n} \mathbf{1}(x \cdot y \in \mathcal{S}_n) h(x \cdot y) \tag{11}$$

$$\forall x \in \mathcal{S}_n \quad h(x) = \sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(x \cdot \Delta_X \in \mathcal{S}_n) f(x \cdot \Delta_X) \tag{12}$$

are equivalent.

In particular, by comparing the expressions (6) of H_n and (12) of the Möbius transform of $f : \mathcal{S}_n \to \mathbb{R}$, we observe that if p is a real number, and if $f : \mathcal{S}_n \to \mathbb{R}$ is defined by $f(x) = p^{|x|}$, then its Möbius transform h satisfies:

$$h(\mathbf{e}) = H_n(p). \tag{13}$$

Running examples for $n = 3$ We tabulate in Table 1 the values of the Möbius transform of the function $p^{|x|}$ defined on $\mathcal{S}_3$, for B_3^+ and for B_3^{+*}. It is easily computed based on the elements found in Figs. 5 and 6 respectively.

3.3.2. The graded Möbius transform

The above relation between real-valued functions $f : B_n^? \mapsto \mathbb{R}$ and their Möbius transforms works only when the Möbius transform is summable. In order to deal with all functions defined on $B_n^?$, we introduce a variant of those transforms, which is the notion of *graded Möbius transform*. To this end, for each finite braid $x \in B_n^?$, we define $B_n^?[x]$ as the following subset:

$$B_n^?[x] = \{y \in B_n^? \ : \ \tau(x \cdot y) = \tau(x)\}\,,$$

Definition 3.9. Given a real-valued function $f : B_n^? \mapsto \mathbb{R}$, its *graded Möbius transform* is the function $h : B_n^? \mapsto \mathbb{R}$ defined by:

$$\forall x \in B_n^? \quad h(x) = \sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(\Delta_X \in B_n^?[x]) f(x \cdot \Delta_X)\,. \tag{14}$$

For functions that vanish outside of $\mathcal{S}_n$, the notions of Möbius transform and graded Möbius transform coincide, while this is not the case in general.

The generalization of the summation formula (10) stands in the following result.

Theorem 3.10. *Let $f, h : B_n^? \to \mathbb{R}$ be two functions. Then h is the graded Möbius transform of f if and only if*

$$\forall x \in B_n^? \quad f(x) = \sum_{y \in B_n^?[x]} h(x \cdot y)\,. \tag{15}$$

Note that, in formula (15), the braids $y \in B_n^?[x]$ may have normal forms that differ completely from that of x. This relates with Remark 3.3.

Proof. For a generic braid $x \in B_n^?$ of height $k = \tau(x)$, we denote by $(x_1, \ldots, x_k)$ the Garside decomposition of x. Observe that, for all $x, y, z \in B_n^?$, we have:

$$y \in B_n^?[x] \wedge z \in B_n^?[x \cdot y] \iff y \cdot z \in B_n^?[x]. \tag{16}$$

Indeed, $y \in B_n^?[x] \wedge z \in B_n^?[x \cdot y] \iff \tau(x) = \tau(x \cdot y) = \tau(x \cdot y \cdot z) \iff \tau(x) = \tau(x \cdot y \cdot z) \iff y \cdot z \in B_n^?[x]$.

Hence, if h is the graded Möbius transform of f, then:

$$\begin{aligned}
\sum_{y \in B_n^?[x]} h(x \cdot y) &= \sum_{y \in B_n^?} \sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(y \in B_n^?[x]) \mathbf{1}(\Delta_X \in B_n^?[x \cdot y]) f(x \cdot y \cdot \Delta_X) \\
&\qquad \text{by (14)} \\
&= \sum_{v \in B_n^?} \sum_{y \in B_n^?} \sum_{X \subseteq \Sigma} (-1)^{|X|} \mathbf{1}(v \in B_n^?[x]) \mathbf{1}(v = y\Delta_X) f(x \cdot v) \\
&\qquad \text{by (16) with } z = \Delta_X
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in B_n^?} \sum_{X\subseteq\Sigma} (-1)^{|X|} \mathbf{1}\big(v \in B_n^?[x]\big) \mathbf{1}\big(\Delta_X \leq_{\mathrm{r}} v\big) f(x\cdot v) \\
&= \sum_{v\in B_n^?} \Big(\sum_{X\subseteq \mathsf{L}(v^*)} (-1)^{|X|} \Big) \mathbf{1}\big(v \in B_n^?[x]\big) f(x\cdot v) \\
&= f(x).
\end{aligned}
$$

Conversely, if (15) holds, then:

$$
\begin{aligned}
&\sum_{X\subseteq\Sigma} (-1)^{|X|} \mathbf{1}\big(\Delta_X \in B_n^?[x]\big) f(x\cdot\Delta_X) \\
&\qquad = \sum_{X\subseteq\Sigma} \sum_{z\in B_n^?} (-1)^{|X|} \mathbf{1}\big(\Delta_X \in B_n^?[x]\big) \mathbf{1}\big(z \in B_n^?[x\cdot\Delta_X]\big) h(x\cdot\Delta_X\cdot z) \\
&\qquad = \sum_{u\in B_n^?} \sum_{X\subseteq\Sigma} \sum_{z\in B_n^?} (-1)^{|X|} \mathbf{1}\big(u \in B_n^?[x]\big) \mathbf{1}\big(u = \Delta_X\cdot z\big) h(x\cdot u) \\
&\qquad = \sum_{u\in B_n^?[x]} \sum_{X\subseteq\Sigma} (-1)^{|X|} \mathbf{1}\big(\Delta_X \leq_{\mathrm{l}} u\big) h(x\cdot u) \\
&\qquad = \sum_{u\in B_n^?[x]} \Big(\sum_{X\subseteq \mathsf{L}(u)} (-1)^{|X|} \Big) h(x\cdot u) \\
&\qquad = h(x).
\end{aligned}
$$

This completes the proof. □

3.3.3. Additional properties of Möbius transforms

Finally, we state in this subsection a couple of lemmas which we will use in next section for the probabilistic study.

Lemma 3.11. *For p a real number, let $f : \mathcal{S}_n \to \mathbb{R}$ be defined by $f(x) = p^{|x|}$, and let $h : \mathcal{S}_n \to \mathbb{R}$ be the Möbius transform of f. Let also $g : \mathcal{S}_n \to \mathbb{R}$ be defined by:*

$$
g(x) = \sum_{y\in\mathcal{S}_n} \mathbf{1}\big(x \to y\big) h(y)\,. \tag{17}
$$

Then the identity $h(x) = f(x)g(x)$ holds for all $x \in \mathcal{S}_n$.

Proof. Let $P = \mathcal{P}(\Sigma)$, and consider the two functions $F, G : P \to \mathbb{R}$ defined, for $A \in P$, by:

$$
\begin{aligned}
F(A) &= \sum_{I\in P} (-1)^{|I|} \mathbf{1}\big(I \subseteq \mathsf{L}(\Delta_{\Sigma\setminus A})\big) f(\Delta_I)\,, \\
G(A) &= \sum_{y\in\mathcal{S}_n} \mathbf{1}\big(\mathsf{L}(y) \cap \mathsf{L}(\Delta_{\Sigma\setminus A}) = \emptyset\big) h(y)\,.
\end{aligned}
$$

Then we claim that $F = G$.

Let us prove the claim. For every $I \in P$ and for every $y \in \mathcal{S}_n$, we have:

$$I \subseteq \mathsf{L}(y) \iff \Delta_I \leq_1 y \iff \mathsf{L}(\Delta_I) \subseteq \mathsf{L}(y).$$

Therefore, according to the Möbius summation formula (10), we have:

$$f(\Delta_I) = \sum_{y \in \mathcal{S}_n} \mathbf{1}(I \subseteq \mathsf{L}(y))h(y).$$

Reporting the right hand member above in the sum defining $F(A)$, and inverting the order of summations, yields:

$$\begin{aligned} F(A) &= \sum_{y \in \mathcal{S}_n} \Big(\sum_{I \in P} (-1)^{|I|} \mathbf{1}(I \subseteq \mathsf{L}(\Delta_{\Sigma \setminus A}) \cap \mathsf{L}(y)) \Big) h(y) \\ &= \sum_{y \in \mathcal{S}_n} \mathbf{1}(\mathsf{L}(\Delta_{\Sigma \setminus A}) \cap \mathsf{L}(y) = \emptyset) h(y) = G(A), \end{aligned}$$

which proves the claim.

Now observe that, for every $x \in \mathcal{S}_n$, we have

$$x \cdot \Delta_{\Sigma \setminus \mathsf{R}(x)} = \bigvee\nolimits_1 \{x \cdot \sigma \ :\ \sigma \in \Sigma \setminus \mathsf{R}(x)\} \leq \bigvee\nolimits_1 \mathcal{S}_n = \Delta_\Sigma,$$

and therefore $\mathsf{L}(\Delta_{\Sigma \setminus \mathsf{R}(x)}) = \Sigma \setminus \mathsf{R}(x)$. This proves that

$$x \cdot \Delta_I \in \mathcal{S}_n \iff I \cap \mathsf{R}(x) = \emptyset \iff I \subseteq \mathsf{L}(\Delta_{\Sigma \setminus \mathsf{R}(x)}) \tag{18}$$

$$x \to y \iff \mathsf{L}(y) \subseteq \mathsf{R}(x) \iff \mathsf{L}(y) \cap \mathsf{L}(\Delta_{\Sigma \setminus \mathsf{R}(x)}) = \emptyset. \tag{19}$$

Hence, using the multiplicativity of f, we have simultaneously

$$h(x) = \sum_{I \in P} (-1)^{|I|} \mathbf{1}(I \subseteq \mathsf{L}(\Delta_{\Sigma \setminus \mathsf{R}(x)})) f(x \cdot \Delta_I) = f(x) F(\mathsf{R}(x))$$

$$\text{and } g(x) = \sum_{y \in \mathcal{S}_n} \mathbf{1}(\mathsf{L}(y) \cap \mathsf{L}(\Delta_{\Sigma \setminus \mathsf{R}(x)} = \emptyset) h(y) = G(\mathsf{R}(x))$$

Since $F = G$, it implies $h(x) = f(x)g(x)$, which completes the proof of the lemma. □

Lemma 3.12. *Let $(x_1, \ldots, x_k)$ be the Garside decomposition of a braid $x \in B_n^?$ and let X be a subset of Σ. We have:*

$$\Delta_X \in B_n^?[x] \iff \Delta_X \in B_n^?[x_k]. \tag{20}$$

Proof. The result is immediate if $x = \mathbf{e}$. Moreover, if $x \neq \mathbf{e}$ and if $\Delta_X \in B_n^?[x_k]$, we observe that $x_k \Delta_X$ is a simple braid, and therefore that $x_1 \cdot \ldots \cdot x_{k-1} \cdot (x_k \Delta_X)$ is a factorization of x into k simple braids, whence $\tau(x\Delta_X) \leq \tau(x)$. Since the Garside length is non-decreasing for $\leq_l$, it follows that $\Delta_X \in B_n^?[x]$.

Conversely, if $x \neq \mathbf{e}$ and if $\Delta_X \notin B_n^?[x_k]$, since the set $\mathcal{S}_n$ is closed under $\vee_l$, there must exist some generator $\sigma \in X \setminus B_n^?[x_k]$. Hence, we have $x_1 \to \ldots x_k \to \sigma$, and therefore $\tau(x \cdot \Delta_X) \geq \tau(x \cdot \sigma) = k + 1$, i.e. $\Delta_X \notin B_n^?[x]$. □

Corollary 3.13. *Let $f : B_n^? \to \mathbb{R}$ be the function defined by $f(x) = p^{|x|}$. Then the graded Möbius transform $h : B_n^? \to \mathbb{R}$ of f satisfies the following property:*

$$h(x) = p^{|x_1|+\ldots+|x_{k-1}|} h(x_k)\,, \tag{21}$$

where $(x_1, \ldots, x_k)$ is the Garside decomposition of x.

Proof. It follows directly from the definition (9) of the graded Möbius transform, together with Lemma 3.12. □

4. Uniform measures on braid monoids

We are now equipped with adequate tools to study uniform measures on braids. Consider the following (vague) questions: how can we pick a braid uniformly at random? How can we pick a large braid uniformly at random? What are the characteristics of such random braids?

Since there are countably many braids, these questions cannot be given immediately a consistent meaning. However, for each fixed integer $k \geq 0$, there are finitely many braids of size k, and it is thus meaningful to pick uniformly a braid of size k at random. Please notice the difference between picking a braid of size k uniformly at random, and picking a word uniformly in Σ^k and then considering the braid it induces. The later corresponds to the uniform random walk on $B_n^?$, but not the former.

A possible way of picking a braid at random is thus the following: first pick the size k at random, and then pick a braid uniformly among those of size k. The problem remains of how to draw k in a "natural" way. It is the topic of this section to demonstrate that there is indeed a natural family, indexed by a real parameter p, of conducting this random procedure. Furthermore, the parameter p is bound to vary in the interval $(0, q_n)$, where q_n is the root of $H_n(t)$ introduced earlier; and letting p tend to q_n by inferior values, the distributions induced on braids weight large braids more and more, such that at the limit we obtain a natural uniform measure on "infinite braids". In turn, we shall derive in the next section information on large random braids, that is to say, on random braids of size k when k is large enough, based on the notion of uniform measure on infinite braids.

4.1. Generalized braids

Considering the extended Garside decomposition of braids, one sees that elements of $B_n^?$ are in bijection with *infinite paths in* $(\mathcal{S}_n, \to)$ *that eventually hit* $\mathbf{e}$. Therefore, it is natural to define a compactification $\overline{B_n^?}$ as the set of all infinite paths in this graph. As a subset of $\mathcal{S}_n^{\mathbb{N}^*}$, it is endowed with a canonical topology, for which it is compact. Moreover, the restriction of this topology to $B_n^?$ is the discrete topology, and $\overline{B_n^?}$ is the closure of $B_n^?$. This is the set of *generalized braids*. We endow the set $\overline{B_n^?}$ with its Borel σ-algebra. All measures we shall consider on $\overline{B_n^?}$ will be finite and Borelian.

We may refer to elements of $B_n^?$ as *finite braids*, to emphasize their status as elements of $\overline{B_n^?}$. We define the *boundary* $\partial B_n^?$ by:

$$\partial B_n^? = \overline{B_n^?} \setminus B_n^? \,.$$

Elements in ∂B_n^+ correspond to infinite paths in $(\mathcal{S}_n, \to)$ that never hit $\mathbf{e}$, we may thus think of them as *infinite braids*.

If $(x_1, \ldots, x_p)$ is a finite path in the graph $(\mathcal{S}_n, \to)$, the corresponding cylinder set $\mathcal{D}_{(x_1,\ldots,x_p)}$ is defined as the set of paths starting with vertices $(x_1, \ldots, x_p)$. Cylinder sets are both open and closed, and they generate the topology on $\overline{B_n^?}$.

Definition 4.1. For $x \in B_n^?$ of Garside decomposition $(x_1, \ldots, x_p)$, we define the *Garside cylinder*, and we denote by $\mathcal{C}_x$, the cylinder subset of $\overline{B_n^?}$ given by $\mathcal{C}_x = \mathcal{D}_{(x_1,\ldots,x_p)}$.

Garside cylinders only reach those cylinders sets of the form $D = \mathcal{D}_{(x_1,\ldots,x_p)}$ with $x_p \neq \mathbf{e}$. But if $x_p = e$, then D reduces to the singleton $\{x\}$, with $x = x_1 \cdot \ldots \cdot x_p$. And then, denoting by q the greatest integer with $x_q \neq \mathbf{e}$ and writing $y = x_1 \cdot \ldots \cdot x_q$, one has:

$$\{x\} = \mathcal{C}_y \setminus \bigcup_{z \in \mathcal{S}_n \setminus \{\mathbf{e}\} \,:\, x_q \to z} \mathcal{C}_{y \cdot z} \,.$$

It follows that *Garside cylinders generate the topology on* $\overline{B_n^?}$, which implies the following result.

Proposition 4.2. *Any finite measure on the space $\overline{B_n^?}$ of generalized braids is entirely determined by its values on Garside cylinders. In other words, if ν and ν' are two finite measures on $\overline{B_n^?}$ such that $\nu(\mathcal{C}_x) = \nu'(\mathcal{C}_x)$ for all $x \in B_n^?$, then $\nu = \nu'$.*

Proof. We have already seen that Garside cylinders generate the topology, and thus the Borel σ-algebra of $\overline{B_n^?}$. The collection of Garside cylinders, augmented with the empty set, is obviously stable by intersection:

$$\forall x, y \in B_n^? \quad \text{either } \mathcal{C}_x \subseteq \mathcal{C}_y \text{ or } \mathcal{C}_x \supseteq \mathcal{C}_y \text{ or } \mathcal{C}_x \cap \mathcal{C}_y = \emptyset \,,$$

and forms thus a so-called π-system. The result follows from classical measure theory [9, Th. 3.3]. $\square$

Garside cylinders are very natural from the point of view of the normal forms, however they are somewhat unnatural from the algebraic point of view as they discard most of the divisibility information (*cf.* Remark 3.3). A more natural notion is that of *visual cylinder*, which corresponds, for a given finite braid $x \in B_n^?$, to the subset of those generalized braids which are "left divisible" by x. It will be useful to differentiate between generalized braids and infinite braids, therefore we introduce both the *full visual cylinder* $\Uparrow x$ and the *visual cylinder* $\uparrow x$, as follows:

$$\Uparrow x = \mathrm{Closure}\big(\{x \cdot z \ : \ z \in B_n^?\}\big)\,, \qquad \uparrow x = \Uparrow x \cap \partial B_n^?\,,$$

where $\mathrm{Closure}(A)$ denotes the topological closure of the set A.

The relationship between Garside cylinders and visual cylinders is given by the following result.

Lemma 4.3. *For each finite braid $x \in B_n^?$, the full visual cylinder $\Uparrow x$ is the following disjoint union of Garside cylinders:*

$$\Uparrow x = \bigcup_{y \in B_n^?[x]} \mathcal{C}_{x \cdot y}. \tag{22}$$

Proof. We first observe that $\Uparrow x \cap B_n^? = \bigcup_{y \in B_n^?[x]} (B_n^? \cap \mathcal{C}_{x \cdot y})$. Indeed, the $\supseteq$ inclusion is obvious, while the converse one is a consequence of points 2 and 3 of Proposition 3.2. Since $\Uparrow x$ and $\bigcup_{y \in B_n^?[x]} \mathcal{C}_{x \cdot y}$ are the respective topological closures of $\Uparrow x \cap B_n^?$ and of $\bigcup_{y \in B_n^?[x]} (B_n^? \cap \mathcal{C}_{x \cdot y})$ in $\overline{B_n^?}$, the result follows. $\square$

Hence, $\Uparrow x$, as a finite union of Garside cylinders, is also open and closed in $\overline{B_n^?}$. In the same way, $\uparrow x$ is open and closed in $\partial B_n^?$.

4.2. Studying finite measures on generalized braids via the graded Möbius transform

In this subsection, we study finite measures on the set $\overline{B_n^?}$ of generalized braids.

Assume that ν is some finite measure on $\overline{B_n^?}$. Then for practical purposes, we are mostly interested in the values of ν on Garside cylinders $\nu(\mathcal{C}_x)$. However, most natural measures will enjoy good properties with respect to the full visual cylinders $\Uparrow x$, which is not surprising as these sets are most natural from the point of view of divisibility properties. For instance, the limits ν of uniform measures on the set of braids of length k will satisfy $\nu(\Uparrow x) = p^{|x|}$ for some p, see Definition 4.6 and Theorem 5.1 below.

Henceforth, to understand these measures, we need to relate $\nu(\mathcal{C}_x)$ and $\nu(\Uparrow x)$ in an explicit way, and this is where the graded Möbius transform of Subsection 3.3.2 plays a key role, as shown by Proposition 4.4 below. In turn, Proposition 4.4 provides a nice probabilistic interpretation of the graded Möbius transform.

Proposition 4.4. *Let ν be a finite measure on $\overline{B_n^?}$. Let $f : B_n^? \mapsto \mathbb{R}$ be defined by $f(x) = \nu(\Uparrow x)$, and let $h : B_n^? \mapsto \mathbb{R}$ be the graded Möbius transform of f. Then, for every integer $k \geq 1$ and for every finite braid y of height k, holds:*

$$\nu(\mathcal{C}_y) = h(y). \tag{23}$$

Proof. The decomposition (22) of a full visual cylinder as a disjoint union of Garside cylinders shows that

$$\nu(\Uparrow x) = \sum_{y \in B_n^?[x]} \nu(\mathcal{C}_{x \cdot y}).$$

Thus, the characterization (15) of the graded Möbius transforms shows that $y \mapsto \nu(\mathcal{C}_y)$ is the graded Möbius transform of $x \mapsto \nu(\Uparrow x)$, as claimed. □

Corollary 4.5. *A finite measure ν on $\overline{B_n^?}$ is entirely determined by its values $\nu(\Uparrow x)$ on the countable collection of full visual cylinders.*

Proof. According to Proposition 4.2, a finite measure ν is entirely determined by its values on Garside cylinders. Hence the result follows from Proposition 4.4. □

4.3. Uniform measures

Our ultimate goal is to understand the uniform measure $\mu_{n,k}$ on braids in $B_n^?$ of a given length k, when k tends to infinity. We will see below in Theorem 5.1 that this sequence of measures converges to a measure on $\partial B_n^?$ which behaves nicely on the visual cylinders $\uparrow x$ (this is not surprising as these are the natural objects from the point of view of the monoid structure on $B_n^?$).

Therefore, it is good methodology to study the general class of measures which do behave nicely on *full* visual cylinders. Our usual conventions and notations are in force throughout this subsection, in particular concerning $B_n^?$ which may be either B_n^+ or B_n^{+*}.

Definition 4.6. A *uniform measure for braids* of parameter $p \geq 0$ is a probability measure ν_p on $\overline{B_n^?}$ satisfying:

$$\forall x \in B_n^? \quad \nu_p(\Uparrow x) = p^{|x|}.$$

Although not apparent from this definition, we will see in Theorem 4.8 below that such a measure either weights $B_n^?$ or $\partial B_n^?$, but not both. Theorem 4.8 will describe quite precisely all uniform measures. It will allow us to define the *uniform measure at infinity* as the unique non-trivial uniform measure supported by the boundary ∂B_n^+, see Definition 4.9. A realization result for uniform measures will be the topic of Subsection 4.4.

Before coming to the theorem, we state a key lemma.

Lemma 4.7. *Let ν be a uniform measure of parameter $p < 1$. Assume that ν is concentrated at infinity,* i.e., *$\nu(\partial B_n^?) = 1$. Then $H_n(p) = 0$.*

Furthermore, let $B = (B_{x,x'})$ be the non-negative matrix indexed by pairs of simple braids (x, x') such that $x, x' \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$, and defined by:

$$B_{x,x'} = \mathbf{1}(x \to x')p^{|x'|}\,.$$

Then B is a primitive matrix of spectral radius 1. The Perron right eigenvector of B is the restriction to $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$ of the vector g defined in (17).

Proof. Let $f(x) = p^{|x|}$, and let $h : \mathcal{S}_n \to \mathbb{R}$ be the graded Möbius transform of f.

According to Proposition 4.4, we have $h(\mathbf{e}) = \nu(\mathcal{C}_\mathbf{e}) = \nu(\{\mathbf{e}\})$. Since it is assumed that $\nu(B_n^?) = 0$, it follows that $h(\mathbf{e}) = 0$. But $H_n(p) = h(\mathbf{e})$, as previously stated in (13), hence $H_n(p) = 0$, proving the first claim of the lemma.

Let g be defined on $\mathcal{S}_n$ as in (17), and let $\widetilde{g}$ be the restriction of g to $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$. It follows from Lemma 3.11 that $h(x) = p^{|x|}g(x)$ holds on $\mathcal{S}_n$. Therefore the computation of $B\widetilde{g}$ goes as follows, for $x \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$:

$$(B\widetilde{g})_x = \sum_{y \in \mathcal{S}_n \setminus \{\mathbf{e},\Delta\}} \mathbf{1}(x \to y)p^{|y|}g(y) = \sum_{y \in \mathcal{S}_n \setminus \{\mathbf{e},\Delta\}} \mathbf{1}(x \to y)h(y).$$

But $h(\mathbf{e}) = 0$ on the one hand; and on the other hand, $x \to \Delta$ does not hold since $x \neq \Delta$. Hence the above equality rewrites as:

$$(B\widetilde{g})_x = \sum_{y \in \mathcal{S}_n} \mathbf{1}(x \to y)h(y) = \widetilde{g}(x).$$

We have proved that $\widetilde{g}$ is right invariant for B. Let us prove that $\widetilde{g}$ is non-identically zero. Observe that h is non-negative, as a consequence of Proposition 4.4. Therefore g is non-negative as well. If $\widetilde{g}$ were identically zero on $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$, so would be h on $\mathcal{S}_n \setminus \{\Delta\}$. The Möbius summation formula (11) would imply that f is constant, equal to $f(\Delta)$ on $\mathcal{S}_n$, which is not the case since we assumed $p \neq 1$. Hence $\widetilde{g}$ is not identically zero.

But B is also aperiodic and irreducible, hence primitive. Therefore Perron–Frobenius theory [28, Chapter 1] implies that $\widetilde{g}$ is actually *the* right Perron eigenvector of B, and B is thus of spectral radius 1. □

Theorem 4.8. *For each braid monoid $B_n^?$, uniform measures ν_p on $\overline{B_n^?}$ are parametrized by the parameter p ranging exactly over the closed set of reals $[0, q_n] \cup \{1\}$.*

1. *For $p = 0$, ν_0 is the Dirac measure at* $\mathbf{e}$*.*
2. *For $p = 1$, ν_1 is the Dirac measure on the element Δ^∞ defined by its infinite Garside decomposition: $(\Delta \cdot \Delta \cdot \ldots)$.*
3. *For $p \in (0, q_n)$, the support of ν_p is $B_n^?$, and it is equivalently characterized by:*

$$\nu_p\bigl(\{x\}\bigr) = H_n(p) \cdot p^{|x|} \qquad \textit{or} \qquad \nu_p(\Uparrow x) = p^{|x|} \tag{24}$$

   for x ranging over $B_n^?$.
4. *For $p = q_n$, the support of ν_{q_n} is $\partial B_n^?$, and it is characterized by:*

$$\forall x \in B_n^? \quad \nu_{q_n}(\uparrow x) = q_n^{|x|} . \tag{25}$$

It follows from this statement that, except for the degenerated measure ν_1, there exists a unique uniform measure on the boundary $\partial B_n^?$. It is thus natural to introduce the following definition.

Definition 4.9. The uniform measure on $\partial B_n^?$ which is characterized by $\nu_{q_n}(\uparrow x) = q_n^{|x|}$ for $x \in B_n^?$, is called the *uniform measure at infinity.*

Proof of Theorem 4.8. The statements 1–4 contains actually three parts: the existence of ν_p for each $p \in [0, q_n] \cup \{1\}$, the uniqueness of the measures satisfying the stated characterizations, and that $[0, q_n] \cup \{1\}$ is the only possible range for p.

Existence and uniqueness of ν_p for $p \in [0, q_n] \cup \{1\}$. The cases $p = 0$ and $p = 1$ (points 1 and 2) are trivial. For $p \in (0, q_n)$ (point 3), let ν_p be the discrete distribution on $B_n^?$ defined by the left hand side of (24). Since $p < q_n$, the series $G_n(p)$ is convergent, and it implies that the following formula is valid in the field of real numbers:

$$G_n(p) \cdot H_n(p) = 1 .$$

It implies in particular that $H_n(p) > 0$, and thus:

$$\sum_{x \in B_n^?} \nu_p\bigl(\{x\}\bigr) = 1 ,$$

and therefore ν_p is a probability distribution on $B_n^?$.

It remains to prove that ν_p is indeed uniform with parameter p. Since $B_n^?$ is left cancellative, we notice that, for each $x \in B_n^?$, the mapping $y \in B_n^? \mapsto x \cdot y$ is a bijection of $B_n^?$ onto $\Uparrow x \cap B_n^?$. Whence:

$$\nu_p(\Uparrow x) = H_n(p) \cdot p^{|x|} \cdot \Bigl(\sum_{y \in B_n^?} p^{|y|}\Bigr) = p^{|x|} .$$

Conversely, if ν is a probability measure on $\overline{B_n^?}$ such that $\nu(\Uparrow x) = p^{|x|}$, then ν and ν_p agree on full visual cylinders, hence $\nu = \nu_p$ according to Corollary 4.5.

We now treat the case of point 4, corresponding to $p = q_n$. For this, let $(p_j)_{j\geq 1}$ be any sequence of reals $p_j < q_n$ such that $\lim_{j\to\infty} p_j = q_n$, and such that $(\nu_{p_j})_{j\geq 1}$ is a weakly convergent sequence of probability measures. Such a sequence exists since $\overline{B_n^?}$ is a compact metric space. Let ν be the weak limit of $(\nu_{p_j})_{j\geq 1}$.

Obviously, for each braid x fixed:

$$\lim_{j\to\infty} \nu_{p_j}(\Uparrow x) = q_n^{|x|}\,.$$

But $\Uparrow x$ is both open and closed in $\overline{B_n^?}$, it has thus an empty topological boundary. The Portemanteau theorem [9] implies that the above limit coincides with $\nu(\Uparrow x)$, hence $\nu(\Uparrow x) = q_n^{|x|}$ for all $x \in B_n^?$. The same reasoning applied to every singleton $\{x\}$, for $x \in B_n^?$, yields:

$$\nu(\{x\}) = \lim_{j\to\infty} \nu_{p_j}(\{x\}) = \lim_{j\to\infty} \frac{p_j^{|x|}}{G_n(p_j)} = 0\,,$$

the later equality since $\lim_{t\to q_n^-} G_n(t) = +\infty$. Since $B_n^?$ is countable, it follows that ν puts weight on $\partial B_n^?$ only, and thus:

$$\forall x \in B_n^? \quad \nu(\uparrow x) = \nu(\Uparrow x) = q_n^{|x|}\,.$$

If ν' is a probability measure on $\overline{B_n^?}$ satisfying $\nu'(\uparrow x) = q_n^{|x|}$ for every $x \in B_n^?$, then we observe first that ν' is concentrated on the boundary, since $\nu'(\partial B_n^?) = \nu'(\uparrow \mathbf{e}) = 1$. And since ν and ν' coincide on all visual cylinders $\uparrow x$, for x ranging over $B_n^?$, it follows from Corollary 4.5 that $\nu = \nu'$.

Range of p. It remains only to prove that, if ν is a uniform probability measure on $\overline{B_n^?}$ of parameter p, then $p = 1$ or $p \leq q_n$. Seeking a contradiction, assume on the contrary that $p > q_n$ and $p \neq 1$ holds.

We first show the following claim:

$$\nu(\partial B_n^?) = 1\,. \tag{26}$$

Assume on the contrary $\nu(\partial B_n^?) < 1$, hence $\nu(B_n^?) > 0$. Then we claim that the inclusion–exclusion principle yields:

$$\forall x \in B_n^? \quad \nu\big(\{x\}\big) = H_n(p)\cdot p^{|x|}\,. \tag{27}$$

Indeed, for any braid $x \in B_n^?$, the singleton $\{x\}$ decomposes as:

$$\{x\} = \Uparrow x \setminus \bigcup_{\sigma\in\Sigma} \Uparrow (x\cdot\sigma)$$

and therefore:

$$\nu\left(\{x\}\right) = \sum_{I\subseteq\Sigma}(-1)^{|I|}\nu\Big(\bigcap_{\sigma\in I}\Uparrow(x\cdot\sigma)\Big) = \sum_{I\subseteq\Sigma}(-1)^{|I|}\nu\big(\Uparrow(x\cdot\Delta_I)\big)\,.$$

Note that the above equality is valid for any finite measure on $\overline{B_n^?}$. Since ν is assumed to be uniform of parameter p, it specializes to the following:

$$\nu\left(\{x\}\right) = \sum_{I\subseteq\Sigma}(-1)^{|I|}p^{|x|+|\Delta_I|} = p^{|x|}\cdot H_n(p)\,,$$

given the form (6) for $H_n(p)$. This proves our claim (27).

Together with $\nu(B_n^?) > 0$, it implies $H_n(p) > 0$. Consequently, summing up $\nu(\{x\})$ for x ranging over $B_n^?$ yields $G_n(p) < \infty$. Hence $p < q_n$, which is a contradiction since we assumed $p > q_n$. This proves the claim (26).

Next, consider the two matrices B and B' indexed by all braids $x \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$ and defined by:

$$B_{x,x'} = \mathbf{1}(x\to x')p^{|x'|} \quad \text{and} \quad B'_{x,x'} = \mathbf{1}(x\to x')q_n^{|x'|}.$$

They are both non-negative and primitive, and of spectral radius 1 according to Lemma 4.7 (which applies since $p \neq 1$ by assumption). According to Perron–Frobenius theory [28, Chapter 1], there cannot exist a strict ordering relation between them. Yet, this is implied by $p > q_n$, hence the latter is impossible. The proof is complete. $\square$

Remark 4.10 *(Multiplicative measures).* Since the length of braids is additive, any uniform measure is multiplicative, *i.e.*, it satisfies: $\nu_p(\Uparrow(x\cdot y)) = \nu_p(\Uparrow x)\cdot\nu_p(\Uparrow y)$.

Conversely, assume that ν is a multiplicative probability measure on $\overline{B_n^?}$. Then ν is entirely determined by the values $p_\sigma = \nu(\Uparrow\sigma)$ for $\sigma \in \Sigma$.

If $B_n^? = B_n^+$, let us write p_i instead of p_{σ_i}. The braid relations $\sigma_i\cdot\sigma_{i+1}\cdot\sigma_i = \sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1}$ entail: $p_ip_{i+1}(p_i - p_{i+1}) = 0$. Hence, if any two consecutive p_i, p_{i+1} are non-zero, they must be equal. Removing the generators σ_i for which $p_i = 0$, the braid monoid splits into a direct product of sub-braid monoids, each one equipped with a uniform measure.

Similarly, if $B_n^? = B_n^{+*}$, let us write $p_{i,j}$ instead of p_{σ_i,σ_j}. Then the dual braid relations $\sigma_{i,j}\cdot\sigma_{j,k} = \sigma_{j,k}\cdot\sigma_{k,i} = \sigma_{k,i}\cdot\sigma_{i,j}$ (if $i < j < k$) yield the following three relations: $p_{j,k}(p_{i,j} - p_{k,i}) = 0$, $p_{i,k}(p_{i,j} - p_{j,k}) = 0$ and $p_{i,j}(p_{j,k} - p_{i,k}) = 0$. Therefore the following implication holds for all $i < j < k$: $(p_{i,j} > 0 \wedge p_{j,k} > 0) \implies p_{i,j} = p_{j,k} = p_{i,k}$. Removing the generators $\sigma_{i,j}$ for which $p_{i,j} = 0$, the dual braid monoid splits thus into a direct product of sub-dual braid monoids, each one equipped with a uniform measure.

Therefore, without loss of generality, the study of multiplicative measures for braid monoids reduces to the study of uniform measures. This contrasts with other kinds of monoids, such as heap monoids, see [2] and the discussion in Section 7.

4.4. Markov chain realization of uniform measures

Recall that generalized braids $\xi \in \overline{B_n^?}$ are given by infinite sequences of linked vertices in the graph $(\mathcal{S}_n, \to)$. For each integer $k \geq 1$, let $X_k(\xi)$ denote the kth vertex appearing in an infinite path $\xi \in \overline{B_n^?}$. This defines a sequence of measurable mappings

$$X_k : \overline{B_n^?} \to \mathcal{S}_n\,,$$

which we may interpret as random variables when equipping $\overline{B_n^?}$ with a probability measure, say for instance a uniform measure ν_p.

It turns out that, under any uniform measure ν_p, the process $(X_k)_{k\geq 1}$ has a quite simple form, namely that of a Markov chain. This *realization result* is the topic of the following theorem (the trivial cases $p = 0$ and $p = 1$ are excluded from the discussion).

Theorem 4.11. *Let $p \in (0, q_n]$, and let ν_p be the uniform measure of parameter p on $\overline{B_n^?}$. Let $h : \mathcal{S}_n \to \mathbb{R}$ be the Möbius transform of $x \in \mathcal{S}_n \mapsto p^{|x|}$.*

1. *Under ν_p, the process $(X_k)_{k\geq 1}$ of simple braids is a Markov chain, taking its values in $\mathcal{S}_n$ if $p < q_n$, and in $\mathcal{S}_n \setminus \{\mathbf{e}\}$ if $p = q_n$.*
2. *The initial measure of the chain coincides with h, which is a probability distribution on $\mathcal{S}_n$. The initial distribution puts positive weight on every non-unit simple braid, and on the unit $\mathbf{e}$ if and only if $p < q_n$.*
3. *The transition matrix P of the chain $(X_k)_{k\geq 1}$ is the following:*

$$P_{x,x'} = \mathbf{1}(x \to x')p^{|x|}\frac{h(x')}{h(x)}, \tag{28}$$

where x and x' range over $\mathcal{S}_n$ for $p < q_n$ or over $\mathcal{S}_n \setminus \{\mathbf{e}\}$ for $p = q_n$.

Proof. Let $f : B_n^? \to \mathbb{R}$ be defined by $f(x) = p^{|x|}$.

We first show that $h > 0$ on $\mathcal{S}_n$ if $p < q_n$, and that $h > 0$ on $\mathcal{S}_n \setminus \{\mathbf{e}\}$ if $p = q_n$. Obviously, it follows from Proposition 4.4 that h is non-negative on $\mathcal{S}_n$, and even on $B_n^?$.

1. *Case $p < q_n$.* Then $H_n(p) > 0$ and therefore, according to Theorem 4.8 and Proposition 4.4, we obtain:

$$h(x) = \nu_p(\mathcal{C}_x) \geq \nu_p(\{x\}) = H_n(p)\cdot p^{|x|} > 0 \text{ for all } x \in \mathcal{S}_n,$$

which was to be shown.

2. *Case $p = q_n$* . Consider the matrix B indexed by pairs (x, x') of simple braids distinct from $\mathbf{e}$ and from Δ, and defined by $B_{x,x'} = \mathbf{1}(x \to x')q_n^{|x'|}$. According to Lemma 4.7, the restriction of g to $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$ is the Perron right eigenvector of B, where g has been defined in (17). Therefore $g > 0$ on $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$. But $h(x) = q_n^{|x|}g(x)$ holds on

$\mathcal{S}_n$ according to Lemma 3.11, therefore $h > 0$ on $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$. As for Δ, one has $h(\Delta) = p^{|\Delta|} > 0$. Hence $h > 0$ on $\mathcal{S}_n \setminus \{\mathbf{e}\}$, as claimed.

It follows in particular from the above discussion that the matrix P defined in the statement of the theorem is well defined. Now, let $(x_1, \ldots, x_k)$ be any sequence of simple braids (including maybe the unit braid). Let δ and δ' denote the following quantities:

$$\delta = \nu_p(X_1 = x_1, \ldots, X_k = x_k)$$
$$\delta' = h(x_1) \cdot P_{x_1,x_2} \cdot \ldots \cdot P_{x_{k-1},x_k} .$$

We prove that $\delta = \delta'$.

We observe first that both δ and δ' are zero if the sequence $(x_1, \ldots, x_k)$ is not normal. Hence, without loss of generality, we restrict our analysis to the case where $(x_1, \ldots, x_k)$ is a normal sequence of simple braids.

Consider the braid $y = x_1 \cdot \ldots \cdot x_k$. By the uniqueness of the Garside normal form, the following equality holds:

$$\{X_1 = x_1, \ldots, X_k = x_k\} = \{X_1 \cdot \ldots \cdot X_k = y\} .$$

Applying successively Proposition 4.4 and Corollary 3.13, we have thus:

$$\delta = h(y) = p^{|x_1|+\ldots+|x_{k-1}|} h(x_k) .$$

On the other hand, we have:

$$\delta' = h(x_1) \cdot p^{|x_1|} \frac{h(x_2)}{h(x_1)} \cdot \ldots \cdot p^{|x_{k-1}|} \frac{h(x_k)}{h(x_{k-1})} = p^{|x_1|+\ldots+|x_{k-1}|} h(x_k)$$

which completes the proof of the equality $\delta = \delta'$. It follows that $(X_k)_{k\geq 1}$ is indeed a Markov chain with the specified initial distribution and transition matrix.

If $p = q_n$, then we have already observed in the proof of Lemma 4.7 that $h(\mathbf{e}) = 0$. It implies both $\nu(X_1 = \mathbf{e}) = 0$ and $P_{x,\mathbf{e}} = 0$ for all $x \in \mathcal{S}_n \setminus \{\mathbf{e}\}$. We conclude that $(X_k)_{k\geq 1}$ does never reach $\mathbf{e}$, which completes the proof of point 1, and of the theorem. □

Running examples for $n = 3$ We characterize the uniform measure at infinity both for B_3^+ and for B_3^{+*}. For this, we first determine the root of smallest modulus of the Möbius polynomial, which we determined in Subsection 3.2.1: $q_3 = (\sqrt{5} - 1)/2$ for B_3^+ and $q_3 = 1/2$ for B_3^{+*}.

The Markov chain of simple braids induced by the uniform measure at infinity takes its values in $\mathcal{S}_3 \setminus \{\mathbf{e}\}$, which has 5 elements for B_3^+ and 4 elements for B_3^{+*}. Since the Möbius transform of $q_3^{|x|}$ is tabulated in Table 1, we are in the position to compute both the initial distribution and the transition matrix of the chain by an application of Theorem 4.11, yielding the results given in Table 2.

Table 2
Transition matrix of the Markov chain on simple braids induced by the uniform measure at infinity for B_3^+ (up) and for B_3^{+*} (down). The initial distribution of the chain can be read on the first line of each matrix.

$$\begin{array}{c} \Delta_3 \\ \sigma_1 \\ \sigma_2 \\ \sigma_1 \cdot \sigma_2 \\ \sigma_2 \cdot \sigma_1 \end{array} \begin{pmatrix} \sqrt{5}-2 & \sqrt{5}-2 & \sqrt{5}-2 & (7-3\sqrt{5})/2 & (7-3\sqrt{5})/2 \\ 0 & (\sqrt{5}-1)/2 & 0 & (3-\sqrt{5})/2 & 0 \\ 0 & 0 & (\sqrt{5}-1)/2 & 0 & (3-\sqrt{5})/2 \\ 0 & 0 & (\sqrt{5}-1)/2 & 0 & (3-\sqrt{5})/2 \\ 0 & (\sqrt{5}-1)/2 & 0 & (3-\sqrt{5})/2 & 0 \end{pmatrix}$$

$$\begin{array}{c} \delta_3 \\ \sigma_{1,2} \\ \sigma_{2,3} \\ \sigma_{1,3} \end{array} \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

5. Applications to finite uniform distributions

5.1. Weak convergence of finite uniform distributions

The following result states a relationship between the finite uniform distributions, and the uniform measure at infinity. If one were only interested in finite uniform distributions, that would be a justification for studying uniform measures as defined previously.

Theorem 5.1. *The uniform measure at infinity ν_{q_n} is the weak limit of the sequence $(\mu_{n,k})_{k\geq 0}$ as $k \to \infty$, where $\mu_{n,k}$ is for each integer $k \geq 0$ the uniform distribution on the finite set $B_n^?(k)$ defined by:*

$$B_n^?(k) = \{x \in B_n^? \ : \ |x| = k\} .$$

Proof. Recall that $B_n^?(k)$, as a subset of $B_n^?$, is identified with its image in $\overline{B_n^?}$, and thus $\mu_{n,k}$ identifies with a discrete probability distribution on $\overline{B_n^?}$. We denote $\lambda_n(k) = \#B_n^?(k)$. For a fixed braid $x \in B_n^?$, the map $y \mapsto x \cdot y$ is a bijection between $B_n^?$ and $\Uparrow x \cap B_n^?$, a fact that we already used in the proof of Theorem 4.8, point 4. Hence, for any $k \geq |x|$, and using the asymptotics found in Corollary 3.5, one has:

$$\mu_{n,k}(\Uparrow x) = \frac{\lambda_n(k-|x|)}{\lambda_n(k)} \to_{k\to\infty} q_n^{|x|} .$$

Invoking the Portemanteau theorem as in the proof of Theorem 4.8, we deduce that any weak cluster value ν of $(\mu_{n,k})_{k\geq 0}$ satisfies $\nu(\Uparrow x) = q_n^{|x|}$ for any full visual cylinder $\Uparrow x$. Theorem 4.8 implies $\nu = \nu_{q_n}$. By compactness of $\overline{B_n^?}$, it follows that $(\mu_{n,k})_{k\geq 0}$ converges toward ν_{q_n}. $\square$

A practical interest of Theorem 5.1 lies in the following corollary. Define $X_i : \overline{B_n^?} \to \mathcal{S}_n$ by $X_i(\xi) = x_i$, where $(x_k)_{k\geq 1}$ is the extended Garside normal form of ξ.

Corollary 5.2. *Let $j \geq 1$ be an integer. Then the joint distribution of the first j simple braids appearing in the extended Garside decomposition of a uniformly random braid of*

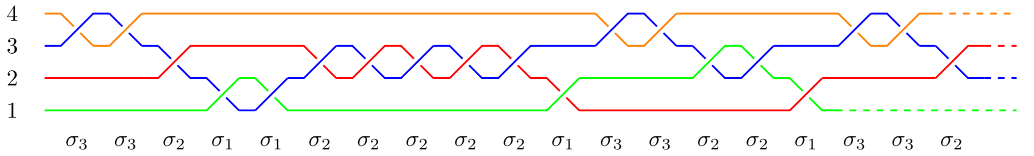

Fig. 7. A random uniform braid on four strands. The first elements of its Garside decomposition are: $(\sigma_3)\cdot(\sigma_3\sigma_2\sigma_1)\cdot(\sigma_1\sigma_2)\cdot(\sigma_2)\cdot(\sigma_2)\cdot(\sigma_2)\cdot(\sigma_2\sigma_1\sigma_3)\cdot(\sigma_3\sigma_2)\cdot(\sigma_2\sigma_1\sigma_3)\cdot(\sigma_3\sigma_2)$.

size k converges, as $k\to\infty$, toward the joint distribution of $(X_1,\ldots,X_j)$ under the uniform measure at infinity.

Proof. By definition of the topology on $\overline{B_n^?}$, the mapping $\xi\in\overline{B_n^?}\mapsto(X_1,\ldots,X_j)$ is continuous for each integer $j\geq 1$. The result follows thus from Theorem 5.1. □

Example for $n=4$ In anticipation of the computations to be performed in Section 6, we depict in Fig. 7 the beginning of a "truly random infinite braid" on $n=4$ strands, up to height $k=10$. These are "typical 10 first elements" in the decomposition of a large random braid on four strands. Observe the absence of Δ; the numerical values found in next subsection make it quite likely.

5.2. A geometric number of Δs

The Δ element only appears at the beginning of normal sequences of simple braids. Accordingly, under the uniform probability measure ν_p, the occurrences of Δ in the Markov chain $(X_k)_{k\geq 1}$ are only observed in the first indices, if any, and their number is geometrically distributed.

This behavior is quite easy to quantify, as the probabilistic parameters associated with Δ have simple expressions:

$$\nu_p(X_1=\Delta)=h(\Delta)=p^{|\Delta|}=\begin{cases}p^{\frac{n(n-1)}{2}} & \text{if } B_n^?=B_n^+\\ p^{n-1} & \text{if } B_n^?=B_n^{+*}\end{cases},\qquad P_{\Delta,\Delta}=p^{|\Delta|}\,.$$

It follows that the number of Δs appearing in the normal form of a random braid, possibly infinite and distributed according to a uniform measure of parameter $p\in(0,q_n]$, is geometric of parameter $p^{|\Delta|}$.

As a consequence of Theorem 5.1, we obtain this corollary.

Corollary 5.3. *Let $T_k:B_n^?(k)\to\mathbb{N}$ denote the number of Δs in the Garside decomposition of a random braid of size k. Then $(T_k)_{k\geq 0}$ converges in distribution, as $k\to\infty$, toward a geometric distribution of parameter $q_n^{\frac{n(n-1)}{2}}$ if $B_n^?=B_n^+$, or q_n^{n-1} if $B_n^?=B_n^{+*}$.*

Authors are sometimes only interested by the elements of the Garside decomposition of a "large" braid that appear *after* the last occurrence of Δ. The notion of uniform measure at infinity allows also to derive information on these, as we shall see next.

Table 3
For each monoid, we tabulate on the left hand side the parameter of the geometric distribution of the number of Δs appearing in a random infinite braid. On the right hand side, we tabulate the associated probability of occurrence of at least one Δ.

	Monoid B_n^+		Monoid B_n^{+*}	
	parameter	prob. of occ. of Δ	parameter	prob. of occ. of Δ
$n=3$	$\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^3 \approx 0.236$	$\approx .309$	$\frac{1}{4}$	$\frac{1}{3}$
$n=4$	≈ 0.0121	≈ 0.0122	$\left(\frac{1}{2}-\frac{\sqrt{5}}{10}\right)^3 \approx 0.021$	≈ 0.022

Examples for $n = 3$ and $n = 4$ Exact or numerical values for the parameter of the geometrical distribution are easily computed for $n = 3$ and for $n = 4$, based on our previous computations for $n = 3$ and on the computations of Section 6 for $n = 4$; see Table 3.

5.3. *On a conjecture by Gebhardt and Tawn*

This subsection only deals with positive braids monoids B_n^+. In their *Stable region Conjecture*, the authors of [24] suggest the following, based on a thorough experimental analysis. For each integer $i \geq 1$, let $\lambda_{i*}(\mu_k)$ be the distribution of the ith factor in the extended Garside normal that occurs after the last Δ, for braids drawn at random uniformly among braids of length k. Then two facts are suspected to hold, according to [24, Conjecture 3.1]:

Presumed Fact 1. For each integer $i \geq 1$, the sequence $(\lambda_{i*}(\mu_k))_{k\geq 1}$ is convergent as $k \to \infty$.

Presumed Fact 2. There exists a probability measure, say μ on $\mathcal{S}_n\backslash\{\mathbf{e}, \Delta\}$, and a constant $C > 0$ such that holds:

$$\forall i > C \quad \lambda_{i*}(\mu_k) \to \mu \quad \text{as } k \to \infty. \tag{29}$$

Theorems 5.1 and 4.11 translate the problem of the limiting behavior of factors of the normal form within the familiar field of Markov chains with a finite number of states. This brings a simple way of determining the status of the above conjecture.

It follows from Theorem 5.1 that the distribution of the kth element in the extended Garside decomposition of a random braid (including all the starting Δs), distributed uniformly among finite braids of size k, converges toward the distribution of the kth element in the extended Garside decomposition of an *infinite* braid, distributed according to the unique uniform measure at infinity. And, according to Theorem 4.11, this is the distribution of a Markov chain at time k, with the prescribed initial distribution and transition matrix.

As for $\lambda_{i*}(\mu_k)$ it converges thus toward the distribution of the same chain, i steps after having left the state Δ. Hence we can affirm *the veracity of Fact 1*. Using the notations of Theorem 4.11, we may also describe the limit, say $\lambda_{i*} = \lim_{k\to\infty} \lambda_{i*}(\mu_k)$, as follows:

$$\forall s \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\} \quad \lambda_{1*}(s) = \frac{h(s)}{1 - q_n^{\frac{n(n-1)}{2}}},$$

where the denominator comes from the conditioning on $s \neq \Delta$. The next values for λ_{i*} are obtained recursively:

$$\forall i \geq 1 \quad \lambda_{i*} = \lambda_{1*} P^{i-1}, \tag{30}$$

where P is the transition matrix of the chain described in Theorem 4.11.

On the contrary, *Fact 2 is incorrect.* Indeed, keeping the notation $\lambda_{i*} = \lim_{k\to\infty} \lambda_{i*}(\mu_k)$, if (29) was true, then $\mu = \lambda_{i*}$ for i large enough, would be the invariant measure of the Markov chain according to (30). But that would imply that the chain is stationary. We prove below that this is not the case when $n > 3$. What is true however, is that $(\lambda_{i*})_{i\geq 1}$ converges toward the stationary measure of the chain when $i \to \infty$.

Assume, seeking a contradiction, that the chain $(X_k)_{k\geq 1}$ is stationary. That would imply that the Möbius transform h of the function $f(x) = q_n^{|x|}$, identified with a vector indexed by $\mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$, is left invariant for the transition matrix P. Hence, for $y \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\}$:

$$h(y) = (hP)_y = \sum_{x \in \mathcal{S}_n \setminus \{\mathbf{e},\Delta\} \,:\, x \to y} h(x) \frac{f(x)h(y)}{h(x)}.$$

We deduce, since $h > 0$ on $\mathcal{S}_n \setminus \{\mathbf{e}\}$:

$$\forall y \in \mathcal{S}_n \setminus \{\mathbf{e}, \Delta\} \qquad \sum_{x \in \mathcal{S}_n \setminus \{\mathbf{e},\Delta\} \,:\, x \to y} q_n^{|x|} = 1. \tag{31}$$

Consider $y = \sigma_1$ and $y' = \sigma_1 \cdot \sigma_2 \cdot \sigma_1$. One has:

$$\mathsf{L}(y) = \{\sigma_1\} \subsetneq \mathsf{L}(y') = \{\sigma_1, \sigma_2\},$$

which entails:

$$\{x \in \mathcal{S}_n \;:\; (x \neq \mathbf{e},\, \Delta) \wedge x \to y\} \supsetneq \{x \in \mathcal{S}_n \;:\; (x \neq \mathbf{e},\, \Delta) \wedge x \to y'\}.$$

It follows that the equality stated in (31) cannot hold both for y and for y', which is the sought contradiction.

6. Explicit computations

We gather in this section the computations needed, for $n = 4$, to characterize the uniform measure at infinity, both for B_4^+ and for B_4^{+*}.

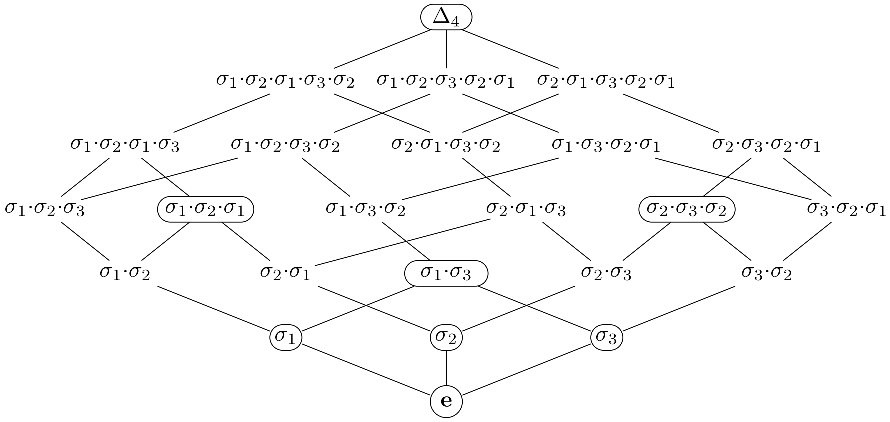

Fig. 8. Hasse diagram of $\mathcal{S}_4$ for B_4^+. Elements of $\mathcal{D}_4$ are circled.

6.1. Computations for B_4^+

The monoid B_4^+ has the following presentation:

$$B_4^+ = \left\langle \sigma_1,\ \sigma_2,\ \sigma_3 \;\middle|\; \sigma_1\sigma_3 = \sigma_3\sigma_1,\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_1,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \right\rangle^+ .$$

In order to shorten notations, we denote a product of generators σ_i simply by the corresponding sequence of indices. So for instance, the Garside element is denoted: $\Delta_4 = 123121$.

The lattice $\mathcal{D}_4 = \{\Delta_X \mid X \subseteq \Sigma_4\}$ has $2^3 = 8$ elements, and is isomorphic to the lattice of subsets of $\{1,2,3\}$, whereas $\mathcal{S}_4$ has $4! = 24$ elements. The Hasse diagram of $\mathcal{S}_4$ is depicted in Fig. 8.

In order to compute the Möbius transform h of the function $f(x) = p^{|x|}$ defined on $\mathcal{S}_4$, we refer to the expression (12):

$$h(x) = \sum_{X \subseteq \Sigma\ :\ x\cdot\Delta_X \in \mathcal{S}_4} (-1)^{|X|} f(x \cdot \Delta_X)$$

Furthermore, recalling the property $x \cdot \Delta_X \in \mathcal{S}_4 \iff X \subseteq \mathsf{L}(\Delta_{\Sigma\setminus\mathsf{R}(x)})$ proved earlier in (18), the range of those $X \subseteq \Sigma$ such that $x \cdot \Delta_X \in \mathcal{S}_4$ is directly derived from the knowledge of the sets $\mathsf{L}(y)$ and $\mathsf{R}(y)$. All these elements are gathered in Table 4.

The Möbius polynomial $H_4(t)$ can be obtained, for instance, by evaluating on $\mathbf{e}$ the Möbius transform of the function $x \mapsto t^{|x|}$. From the first line of Table 4, we read:

$$H_4(t) = (1-t)(1 - 2t - t^2 + t^3 + t^4 + t^5)$$

Let $p = q_4$ be the smallest root of $H_4(t)$. We illustrate on the example $x = 12$ the computation of a line $P_{x,\bullet}$ of the transition matrix corresponding to the uniform measure at infinity. From Table 4, we read the list of non-zero values of the corresponding line of

Table 4
Characteristic elements for B_4^+. Elements of $\mathcal{D}_4$ are framed. The column $h(x)$ tabulates the Möbius transform of the function $f(x) = p^{|x|}$ on $\mathcal{S}_4$.

$\mathsf{L}(x)$	$x \in \mathcal{S}_4$	$\mathsf{R}(x)$	$\{y \in \mathcal{S}_4 \ : \ x \to y\}$	$h(x)$
$\emptyset$	$\boxed{\mathbf{e}}$	$\emptyset$	$\mathbf{e}$	$1 - 3p + p^2 + 2p^3 - p^6$
1	$\boxed{1}$	1	1, 12, 123	$p - 2p^2 + p^4$
2	$\boxed{2}$	2	2, 21, 23, 213, 2132	$p - 2p^2 + p^3$
3	$\boxed{3}$	3	3, 32, 321	$p - 2p^2 + p^4$
1	12	2	2, 21, 23, 213, 2132	$p^2 - 2p^3 + p^4$
3, 1	$\boxed{13}$	1, 3	1, 3, 12, 13, 32, 123, 132, 321, 1232, 1321, 12321	$p^2 - p^3$
2	21	1	1, 12, 123	$p^2 - 2p^3 + p^5$
2	23	3	3, 32, 321	$p^2 - 2p^3 + p^5$
3	32	2	2, 21, 23, 213, 2132	$p^2 - 2p^3 + p^4$
2, 1	$\boxed{121}$	1, 2	1, 2, 12, 21, 23, 121, 123, 213, 1213, 2132, 21323	$p^3 - p^4$
1	123	3	3, 32, 321	$p^3 - 2p^4 + p^6$
3, 1	132	2	2, 21, 23, 213, 2132	$p^3 - 2p^4 + p^5$
2	213	1, 3	1, 3, 12, 13, 32, 123, 132, 321, 1232, 1321, 12321	$p^3 - p^4$
3, 2	$\boxed{232}$	2, 3	2, 3, 21, 23, 32, 213, 232, 321, 2132, 2321, 21321	$p^3 - p^4$
3	321	1	1, 12, 123	$p^3 - 2p^4 + p^6$
2, 1	1213	1, 3	1, 3, 12, 13, 32, 123, 132, 321, 1232, 1321, 12321	$p^4 - p^5$
3, 1	1232	2, 3	2, 3, 21, 23, 32, 213, 232, 321, 2132, 2321, 21321	$p^4 - p^5$
3, 1	1321	1, 2	1, 2, 12, 21, 23, 121, 123, 213, 1213, 2132, 21323	$p^4 - p^5$
2	2132	2	2, 21, 23, 213, 2132	$p^4 - 2p^5 + p^6$
3, 2	2321	1, 3	1, 3, 12, 13, 32, 123, 132, 321, 1232, 1321, 12321	$p^4 - p^5$
3, 1	12321	1, 3	1, 3, 12, 13, 32, 123, 132, 321, 1232, 1321, 12321	$p^5 - p^6$
3, 2	21321	1, 2	1, 2, 12, 21, 23, 121, 123, 213, 1213, 2132, 21323	$p^5 - p^6$
2, 1	21323	2, 3	2, 3, 21, 23, 32, 213, 232, 321, 2132, 2321, 21321	$p^5 - p^6$
3, 2, 1	$\boxed{\Delta_4}$	1, 2, 3	$\mathcal{S}_4$	p^6

the matrix, which are for this case: 2, 21, 23, 213 and 2132. According to Theorem 4.11, for x fixed, the entries $P_{x,y}$ of the matrix are proportional to $h(y)$, and the normalization factor is $p^{-|x|}h(x)$. Reading the values of h in Table 4, we use the relation $1 - 2p - p^2 + p^3 + p^4 + p^5 = 0$ to write the coefficients as polynomials in p, yielding:

$$P_{12,2} = p\,, \qquad P_{12,213} = -1 + p + 2p^2 + 2p^3 + p^4\,,$$
$$P_{12,21} = P_{12,23} = p(1 - 2p^2 - 2p^3 - p^4)\,, \qquad P_{12,2132} = p^4\,.$$

6.2. Computations for B_4^{+*}

We now treat the case of the B_4^{+*}. In order to simplify the notations, we write (ij) for the generator $\sigma_{i,j}$, so for instance: $\delta_4 = (12)\cdot(23)\cdot(34)$. The monoid B_4^{+*} has the six generators (ij) for $1 \le i < j \le 4$, subject to the following relations:

$$(12)\cdot(23) = (23)\cdot(13) = (13)\cdot(12) \qquad (12)\cdot(24) = (24)\cdot(14) = (14)\cdot(12)$$
$$(13)\cdot(34) = (34)\cdot(14) = (14)\cdot(13) \qquad (23)\cdot(34) = (34)\cdot(24) = (24)\cdot(23)$$
$$(12)\cdot(34) = (34)\cdot(12) \qquad (23)\cdot(14) = (14)\cdot(23)$$

Table 5
Characteristic elements for the dual braid monoid B_4^{+*}. The column $h(x)$ gives the Möbius transform on $\mathcal{S}_4$ of the function $f(x) = p^{|x|}$. The column $\rho(x)$ evaluates the same quantity for the particular case $p = q_4$.

$x \in \mathcal{S}_4$	$\{y \in \mathcal{S}_4 \,:\, x \to y\}$	$h(x)$	$\rho(x)$
$\mathbf{e}$	$\mathbf{e}$	$1 - 6p + 10p^2 - 5p^3$	0
δ_4	$\mathcal{S}_4$	p^3	$1/5 - 2\sqrt{5}/25$
(12)	$(12),\ (13),\ (14)$	$p(1-3p+2p^2)$	$\sqrt{5}/25$
(13)	$(13),\ (14),\ (23),\ (24),\ (14)\cdot(23)$	$p(1-2p+p^2)$	$1/10 + \sqrt{5}/50$
(14)	$(14),\ (24),\ (34)$	$p(1-3p+2p^2)$	$\sqrt{5}/25$
(23)	$(12),\ (23),\ (24)$	$p(1-3p+2p^2)$	$\sqrt{5}/25$
(24)	$(12),\ (13),\ (24),\ (34),\ (12)\cdot(34)$	$p(1-2p+p^2)$	$1/10 + \sqrt{5}/50$
(34)	$(13),\ (23),\ (34)$	$p(1-3p+2p^2)$	$\sqrt{5}/25$
$(12)\cdot(23)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (34),\ (23)\cdot(34),\ (13)\cdot(34),\ (12)\cdot(34)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$
$(12)\cdot(24)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (23),\ (12)\cdot(23),\ (23)\cdot(34),\ (14)\cdot(23)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$
$(23)\cdot(34)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (14),\ (12)\cdot(24),\ (13)\cdot(34),\ (14)\cdot(23)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$
$(13)\cdot(34)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (12),\ (12)\cdot(23),\ (12)\cdot(24),\ (12)\cdot(34)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$
$(14)\cdot(23)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (13),\ (12)\cdot(23),\ (13)\cdot(34)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$
$(12)\cdot(34)$	$\mathcal{S}_4 \setminus \{\delta_4,\ (24),\ (12)\cdot(24),\ (23)\cdot(34)\}$	$p^2(1-p)$	$1/10 - \sqrt{5}/50$

The set of simple braids $\mathcal{S}_4$ has 14 elements, which we organize below according to the type of partition of the integer 4 that the associated non-crossing partition of $\{1, 2, 3, 4\}$ induces (see Subsection 2.2.3):

$$\begin{aligned}
\mathcal{S}_4 = \big\{ &\mathbf{e}, && \text{1 partition of type } 1+1+1+1\\
&\delta_4, && \text{1 partition of type } 4\\
&(12),\ (13),\ (14),\ (23),\ (24),\ (34), && \text{6 partitions of type } 2+1+1\\
&(12)\cdot(23),\ (12)\cdot(24),\ (13)\cdot(34),\ (23)\cdot(34), && \text{4 partitions of type } 3+1\\
&(13)\cdot(24),\ (12)\cdot(34)\big\} && \text{2 partitions of type } 2+2
\end{aligned}$$

Following the same scheme as for B_3^{+*}, we gather in Table 5 the characteristic elements for B_4^{+*}. In particular, the first line gives the Möbius polynomial, from which the characteristic value q_4 is derived:

$$H_4(t) = (1-t)(1-5t+5t^2)\,, \qquad q_4 = \frac{1}{2} - \frac{\sqrt{5}}{10}\,.$$

The computation of the transition matrix of the chain of simple braids induced by the uniform measure at infinity yields the values reported in Table 6, where the line corresponding to δ_4 has been omitted. This line, which also corresponds to the initial measure of the chain, is given by the function $\rho(x)$ tabulated in Table 5.

7. Extensions

There are various other questions of interest concerning the asymptotic behavior of random braids, uniformly distributed among braids of length k. For instance, what is

Table 6
Transition matrix for the uniform measure at infinity for B_4^{+*}, restricted to its unique ergodic component $\mathcal{S}_4 \setminus \{\mathbf{e}, \delta_4\}$, and where we have put $\theta = \sqrt{5}/10$.

	(12)	(13)	(14)	(23)	(24)	(34)	$\cdots$
(12)	$1/2-\theta$	2θ	$1/2-\theta$	0	0	0	$\cdots$
(13)	0	$1/2-\theta$	$-1/2+3\theta$	$-1/2+3\theta$	$1/2-\theta$	0	$\cdots$
(14)	0	0	$1/2-\theta$	0	2θ	$1/2-\theta$	$\cdots$
(23)	$1/2-\theta$	0	0	$1/2-\theta$	2θ	0	$\cdots$
(24)	$-1/2+3\theta$	$1/2-\theta$	0	0	$1/2-\theta$	$-1/2+3\theta$	$\cdots$
(34)	0	2θ	0	$1/2-\theta$	0	$1/2-\theta$	$\cdots$
$(12)\cdot(23)$	$-1/10+\theta$	$1/5$	$-1/10+\theta$	$-1/10+\theta$	$1/5$	0	$\cdots$
$(12)\cdot(24)$	$-1/10+\theta$	$1/5$	$-1/10+\theta$	0	$1/5$	$-1/10+\theta$	$\cdots$
$(23)\cdot(34)$	$-1/10+\theta$	$1/5$	0	$-1/10+\theta$	$1/5$	$-1/10+\theta$	$\cdots$
$(13)\cdot(34)$	0	$1/5$	$-1/10+\theta$	$-1/10+\theta$	$1/5$	$-1/10+\theta$	$\cdots$
$(14)\cdot(23)$	$-1/10+\theta$	0	$-1/10+\theta$	$-1/10+\theta$	$1/5$	$-1/10+\theta$	$\cdots$
$(12)\cdot(34)$	$-1/10+\theta$	$1/5$	$-1/10+\theta$	$-1/10+\theta$	0	$-1/10+\theta$	$\cdots$

$\cdots$	$(12)\cdot(23)$	$(12)\cdot(24)$	$(23)\cdot(34)$	$(13)\cdot(34)$	$(14)\cdot(23)$	$(12)\cdot(34)$	
$\cdots$	0	0	0	0	0	0	(12)
$\cdots$	0	0	0	0	$1-4\theta$	0	(13)
$\cdots$	0	0	0	0	0	0	(14)
$\cdots$	0	0	0	0	0	0	(23)
$\cdots$	0	0	0	0	0	$1-4\theta$	(24)
$\cdots$	0	0	0	0	0	0	(34)
$\cdots$	$3/10-\theta$	$3/10-\theta$	0	0	$3/10-\theta$	0	$(12)\cdot(23)$
$\cdots$	0	$3/10-\theta$	0	$3/10-\theta$	0	$3/10-\theta$	$(12)\cdot(24)$
$\cdots$	$3/10-\theta$	0	$3/10-\theta$	0	0	$3/10-\theta$	$(23)\cdot(34)$
$\cdots$	0	0	$3/10-\theta$	$3/10-\theta$	$3/10-\theta$	0	$(13)\cdot(34)$
$\cdots$	0	$3/10-\theta$	$3/10-\theta$	0	$3/10-\theta$	$3/10-\theta$	$(14)\cdot(23)$
$\cdots$	$3/10-\theta$	0	0	$3/10-\theta$	$3/10-\theta$	$3/10-\theta$	$(12)\cdot(34)$

the asymptotic value of the height of a large braid? In other words, how do the Garside length and the Artin length compare to each other for large braids?

The height of braids gives rise to a sequence of integer random variables $\tau_k : B_n^?(k) \to \mathbb{N}$, indexed by k, where $B_n^?(k) = \{x \in B_n^? \;:\; |x| = k\}$ is equipped with the uniform distribution. Since the ratios height over length are uniformly bounded and bounded away from zero, performing the correct normalization leads to considering the sequence of real random variables $\rho_k : B_n^?(k) \to \mathbb{R}$ defined by:

$$\forall x \in B_n^?(k) \qquad \rho_k(x) = \frac{\tau(x)}{|x|} = \frac{\tau(x)}{k}\,,$$

which takes values in the fixed interval $[1/|\Delta|, 1]$. Since all these random variables are defined on different probability spaces, the natural way of studying their asymptotic behavior is by studying their convergence in distribution.

A first result one may wish to establish is a *concentration result*: one aims to prove that $(\rho_k)_{k\geq 1}$ converges in distribution toward a single value, say ρ. Hence one expects a convergence in distribution of the following form, where δ_ρ denotes the Dirac probability measure on the singleton $\{\rho\}$:

$$\frac{\tau(\cdot)}{k} \xrightarrow[k\to\infty]{\mathrm{d}} \delta_\rho\,, \tag{32}$$

where ρ is some real number in the open interval $(1/|\Delta|, 1)$. The number ρ would appear as a *limit average rate*: most of braids of Artin size k would have, for k large enough, a Garside size close to ρk. If ρ can furthermore be simply related to the quantities we have introduced earlier, such as the characteristic parameter q_n, it is reasonable to expect that ρ would be an algebraic number.

Once this would have been established, the next step would consist in studying a Central Limit Theorem: upon normalization, is the distribution of ρ_k Gaussian around its limit value ρ? Hence, one expects a convergence in distribution of the following form, for some constant $\sigma_n^2 > 0$ and where $\mathcal{N}(0, \sigma^2)$ denotes the Normal distribution of zero mean and variance σ^2:

$$\sqrt{k}\Big(\frac{\tau(\cdot)}{k} - \rho\Big) \xrightarrow[k\to\infty]{\mathrm{d}} \mathcal{N}(0, \sigma_n^2) \tag{33}$$

It turns out that both results (32) and (33) hold indeed. Because of space constraints, we postpone their proofs to a forthcoming work [3].

This concerned an extension of the results established in this paper. Generalizations to other monoids are also possible, which we intend to expose in [3]. Braid monoids fall into the wider class of Artin–Tits monoids, investigated by several authors since the 1960's, including Tits, Deligne, Sato, Brieskorn, Garside, Charney, Dehornoy. Several results established in this paper for braid monoids admit generalizations to Artin–Tits monoids, and analogues of the convergences (32) and (33) also hold.

Among Artin–Tits monoids, one class in particular has retained the attention of the authors: the class of trace monoids, also called partially commutative monoids [13]. In trace monoids, the only relations between generators are commutativity relations (there is no braid relations); they correspond to Viennot's *heap monoids* [31]. Trace monoids differ from braid monoids for several reasons, for instance there is no lattice structure and their associated Coxeter group is not finite. From the point of view adopted in this paper, the main difference lies in the existence of a *continuum of multiplicative measures*, among which the uniform measure is a particular case. Recall that we have observed in Remark 4.10 that the uniform measure for infinite braids is the only instance of multiplicative measures, so the situation for braids presents a sharp contrast with trace monoids. The investigation of multiplicative measures for trace monoids has been the topic of [1].

We shall prove in [3] that, from this perspective, there are essentially only two types of Artin–Tits monoids: the trace type and the braid type, corresponding respectively to the type with a continuum of multiplicative measures, and the type where multiplicative measures reduce to the uniform measure only. For the trace type, multiplicative measures are parametrized by a sub-manifold of $\mathbb{R}^m$, diffeomorphic to the standard $(m-1)$-simplex, where m is the minimal number of generators of the monoid.

References

[1] S. Abbes, J. Mairesse, Uniform and Bernoulli measures on the boundary of trace monoids, J. Combin. Theory Ser. A 135 (2015) 201–236.
[2] S. Abbes, J. Mairesse, Uniform generation in trace monoids, in: G. Italiano, G. Pighizzini, D. Sannella (Eds.), Math. Found. Comput. Sc. 2015 (MFCS 2015), part 1, in: Lecture Notes in Comput. Sci., vol. 9234, Springer, 2015, pp. 63–75.
[3] S. Abbes, S. Gouëzel, V. Jugé, J. Mairesse, Uniform and multiplicative measures at infinity for Artin–Tits monoids and applications, in preparation.
[4] S.I. Adian, Fragments of the word δ in the braid group, Mat. Zametki 36 (1) (1984) 25–34.
[5] M. Albenque, P. Nadeau, Growth function for a class of monoids, in: 21st Internat. Conf. Formal Power Series Algebraic Combin., FPSAC 2009, in: DMTCS Proceedings, 2009, pp. 25–38.
[6] D. Bessis, The dual braid monoid, Ann. Sci. Éc. Norm. Supér. (4) 36 (2003) 647–683.
[7] D. Bessis, F. Digne, J. Michel, Springer theory in braid groups and the Birman–Ko–Lee monoid, Pacific J. Math. 205 (2) (2002) 287–309.
[8] M. Bestvina, Non-positively curved aspects of Artin groups of finite type, Geom. Topol. 3 (1) (1999) 269–302.
[9] P. Billingsley, Probability and Measure, 3rd edition, Wiley, 1995.
[10] J. Birman, K.H. Ko, S.J. Lee, A new approach to the word and conjugacy problems in the braid groups, Adv. Math. 139 (2) (1998) 322–353.
[11] M. Brazil, Monoid growth functions for braid groups, Internat. J. Algebra Comput. 1 (2) (1991) 201–205.
[12] A. Bronfman, Growth functions of a class of monoids, preprint, 2001.
[13] P. Cartier, D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Math., vol. 85, Springer, 1969.
[14] S. Caruso, On the genericity of pseudo-Anosov braids I: rigid braids, arXiv preprint, arXiv:1306.3757, 2013.
[15] R. Charney, Geodesic automation and growth functions for Artin groups of finite type, Math. Ann. 301 (1995) 307–324.
[16] P. Dehornoy, Combinatorics of normal sequences of braids, J. Combin. Theory Ser. A 114 (3) (2007) 389–409.
[17] P. Dehornoy, Efficient solutions to the braid isotopy problem, Discrete Appl. Math. 156 (2008) 3094–3112.
[18] P. Dehornoy, F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of Garside theory, arXiv preprint, arXiv:1309.0796, 2013.
[19] P. Deligne, Les immeubles des groupes de tresses généralisés, Invent. Math. 17 (4) (1972) 273–302.
[20] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, Word Processing in Groups, Jones and Bartlett, Boston, 1992.
[21] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge Univ. Press, 2009.
[22] F. Garside, The braid group and other groups, Q. J. Math. 20 (1) (1969) 235–254.
[23] V. Gebhardt, J. Gonzáles-Meneses, Generating random braids, J. Combin. Theory Ser. A 120 (1) (2013) 111–128.
[24] V. Gebhardt, S. Tawn, Normal forms of random braids, J. Algebra 408 (2014) 115–137.
[25] V. Gebhardt, S. Tawn, On the penetration distance in Garside monoids, J. Algebra 451 (2016) 544–576.
[26] J. Mairesse, F. Mathéus, Randomly growing braid on three strands and the manta ray, Ann. Appl. Probab. 17 (2) (2007) 502–536.
[27] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, Z. Wahrscheinlichkeitstheor. 2 (1964) 340–368.
[28] E. Seneta, Non-negative Matrices and Markov Chains, Revised printing, Springer, 1981.
[29] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, 1997.
[30] A. Vershik, S. Nechaev, R. Bikbov, Statistical properties of locally free groups with applications to braid groups and growth of random heaps, Comm. Math. Phys. 212 (2) (2000) 469–501.
[31] X. Viennot, Heaps of pieces, I: basic definitions and combinatorial lemmas, in: Combinatoire énumérative, in: Lecture Notes in Math., vol. 1234, Springer, 1986, pp. 321–350.
[32] W. Woess, Random Walks on Infinite Graphs and Groups, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, 2000.