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A Cut-Invariant Law of Large Numbers for Random Heaps

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Abstract We consider the framework of Bernoulli measures for heap monoids. We introduce in this framework the notion of asynchronous stopping time, which generalizes the notion of stopping time for classical probabilistic processes. A strong Bernoulli property is proved. A notion of cut-invariance is formulated for convergent ergodic means. Then, a version of the strong law of large numbers is proved for heap monoids with Bernoulli measures. We study a sub-additive version of the law of large numbers in this framework.

Keywords Random heaps · Law of large numbers · Subadditive ergodic theorem

Mathematics Subject Classification (2010) 60F15 · 68Q87

1 Introduction

1.1 Heap Monoids

Dimer models are widely used models, in Statistical Physics as a growth model [13,22], and in Combinatorics where they occurred for their application to the enumeration of directed animals for instance [4]. Dimer models belong to the larger class of *heap monoids* models, studied, among others, by Cartier and Foata [5] and by Viennot [23]. Heap monoids have also been used for the analysis of distributed databases [9] and more generally as a model of concurrent systems [15,24].

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The simplest way to define heap monoids is to consider their algebraic presentation, which is of the following form:

$$\mathcal{M} = \langle \Sigma \mid ab = ba \ \text{ for } \ (a, b) \in I,$$

where Σ is a finite and non-empty set, the elements of which are called *pieces*, and I is an irreflexive and symmetric relation on Σ. Elements of $\mathcal{M}$ are called *heaps*. A heap is thus an equivalence class of Σ-words, with respect to the congruence which relates two words that can be reached from one another by finitely many elementary transformations of the following form:

$$a_1 \ldots a_k \, ab \, a_{k+1} \ldots a_{k+j} \longrightarrow a_1 \ldots a_k \, ba \, a_{k+1} \ldots a_{k+j}, \quad \text{for } k, j \geq 0 \text{ and } (a, b) \in I.$$

The combinatorial interpretation of this algebraic presentation is discussed later.

1.2 Probabilistic Models and Bernoulli Measures

On the probabilistic side, at least two probabilistic frameworks for heap monoids can be considered. A first natural framework is that of a random walk: pick a piece at random, and add it by right multiplication in the monoid to the previously constructed heap. Random walks have a direct relation with the uniform distribution on words: for the uniform random walk for instance, at time n, the distribution of the random heap of size n thus constructed is the law of the equivalence class $[x]$, where x is a random word uniformly distributed among words of size n. Limit theorems for this framework are particular instances of classical results.

Another natural framework is the following. Let $n \geq 0$ be an integer, and let $\mathcal{M}_n$ be the set of heaps in $\mathcal{M}$ of size n—that is to say, those heaps containing exactly n pieces. Since $\mathcal{M}_n$ is finite, it is natural to consider the uniform distribution m_n on $\mathcal{M}_n$. It turns out that, for n going to ∞, the sequence of finite distributions $(m_n)_{n\geq 0}$ converges weakly toward a probability measure on the space of *infinite heaps*. This probability measure, called the *uniform measure*, is a particular instance of a *Bernoulli measure for heap monoids*, introduced in a previous work [2]. It is the aim of this paper to study a law of large numbers for these measures.

1.3 Cut-Invariant Law of Large Numbers

For limit theorems for Bernoulli measures, two approaches can be considered. For the uniform measure for instance, which is the weak limit of finite uniform distributions, it is natural to evaluate ergodic sums along finite heaps of size n and to form ergodic means by dividing by n and then to study the limit of these when n goes to infinity. This approach is the topic of a paper in preparation [1].

This approach, however, does not answer the following type of questions. Assume that $\mathbb{P}$ is a Bernoulli measure on the space of infinite heaps, and let a be some fixed piece. Let ξ be a random infinite heap distributed according to $\mathbb{P}$. With probability 1, the heap ξ contains infinitely many occurrences of the piece a. To each of these occurrences, corresponds a sub-heap of ξ, namely the smallest sub-heap of ξ that contains it. Ergodic sums can thus be formed along these finite sub-heaps; dividing the ergodic sums by their size yields ergodic means, for which the limit is of interest. Furthermore, one might expect that the limit is *independent of the chosen piece a*. This is the main result proved in this paper, justifying the name of cut-invariant for the law of large numbers that we obtain.

1.4 Contributions

The previous discussion illustrates a procedure for recursively cutting sub-heaps out of an infinite heap. We introduce the notion of asynchronous stopping time (AST), of which the previous example is a particular instance. The notion of AST generalizes that of stopping time for classical probabilistic processes. For instance, we recognize in the former example the counterpart of the first hitting time of a state.

To each AST is associated a shift operator on the space of infinite heaps, which we prove to be ergodic under mild conditions on the AST. AST and the associated shift operators are the basic elements for stating a strong Bernoulli property, analogous to the strong Markov property in our framework.

AST allow to define ergodic sums and ergodic means. The cut-invariant law of large numbers states the convergence of these ergodic means on the one hand, and that this limit is independent of the AST that has been considered on the other hand. It is proved for additive cost functions and under additional conditions for sub-additive cost functions as well. In both cases, the main point is to prove that the limit of the ergodic means is independent of the chosen AST.

1.5 Organization of the Paper

We have included a preliminary section which illustrates most of the notions on a very simple example, allowing to do all the calculations by hand. In particular, the cut-invariance is demonstrated by performing simple computations using only geometric laws. Later in the paper, all the specific computations for this example will be reinterpreted under the light of Bernoulli measures on heap monoids. Introducing more theoretical material is mandatory, since the basic analysis performed in Sect. 2 relies on an easy description of all possible heaps for this example—an intractable task in general.

The paper is organized as follows. Section 2 is the preliminary example section, which relies on no theoretical material at all. Section 3 introduces the background on heap monoids and Bernoulli measures. Section 4 introduces asynchronous stopping times for heap monoids. Section 5 studies the iteration of asynchronous stopping times. Section 6 states and proves the cut-invariant law of large numbers. Section 7 is devoted to the sub-additive variant of the law of large numbers.

2 Cut-Invariance on an Example

The purpose of this preliminary section is twofold. First, it will help motivating the model of Bernoulli measures on heap monoids, which will appear as a natural probabilistic model for systems involving asynchronous actions. Second, it will illustrate that asymptotic quantities relative to the model may be computed according to different presentations, corresponding to different cut shapes of random heaps.

Consider two communicating devices A and B. Device A may perform some actions on its own, called actions of type a. Similarly, device B may perform actions of type b on its own. Finally, both devices may perform together a synchronizing action of type c, involving communication on both sides—a sort of *check-hand* action.

Consider the following simple probabilistic protocol, involving two fixed probabilistic parameters $\lambda, \lambda' \in (0, 1)$.

1. Device A and device B perform actions of type a and b, respectively, in an asynchronous and probabilistically independent way. The number N_a of occurrences of type a actions and the number N_b of occurrences of type b actions follow geometric laws with parameters λ and λ', respectively. Hence:

$$\forall k, k' \geq 0 \qquad \mathbb{P}(N_a = k,\ N_b = k') = \lambda(1-\lambda)^k \cdot \lambda'(1-\lambda')^{k'}. \tag{1}$$

2 Devices A and B perform a synchronizing action of type c, acknowledging that they have completed their local actions.
3. Go to 1.

We say that Steps 1–2 form a *round* of the protocol. The successive geometric variables that will occur when executing several rounds of the protocol are assumed to be independent.

The question that will guide us throughout this study is the following: what are the asymptotic densities of actions of type a, b, and c? Hence, we are looking for non-negative quantities $\gamma_a, \gamma_b, \gamma_c$ such that $\gamma_a + \gamma_b + \gamma_c = 1$, and that represent the average ratio of each type of action among all three possible types.

Before we suggest possible definitions for the asymptotic density vector, it is customary to interpret the executions of the above protocol with a *heap of pieces* model. For this, associate dominoes to each type of action a, b and c. The occurrence of an action corresponds to a domino of the associated type falling from top to bottom until it reaches either the ground or a previously piled domino. Asynchrony of types a and b actions is rendered by letting dominoes of type a and b falling according to parallel, separated lanes, whereas dominoes of type c are blocking for dominoes of types a and b, which renders the synchronization role of type c actions. A typical first round of the protocol, in the heap model, corresponds to a heap as depicted in Fig. 1a for $N_a = 1$ and $N_b = 2$. The execution of several rounds of the protocol makes the heap growing up, as depicted in Fig. 1b.

Letting the protocol execute without limit of time yields random *infinite heaps*. Let $\mathbb{P}$ denotes the probability measure that equips the canonical space associated with the execution of infinitely many rounds of the protocol. The measure $\mathbb{P}$ can also be seen as the law of the infinite heap resulting from the execution of the protocol.

It is interesting to observe that the law $\mathbb{P}$ cannot be reached by the execution of any Markov chain with three states a, b, c (proof left to the reader). In particular, the estimation of the asymptotic quantities that we perform below do not result from a straightforward translation into a Markov chain model. However, we will see that there

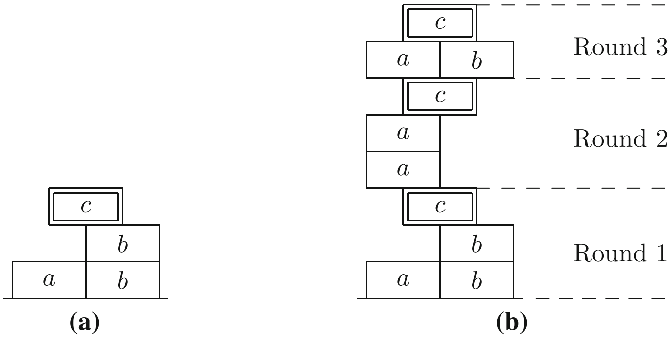

Fig. 1 Heaps of pieces corresponding to the execution of: (*a*) the first round of the protocol, (*b*) the three first rounds of the protocol

is a natural interpretation of $\mathbb{P}$ as the law of trajectories of a finite Markov chain with *four* states, see Sect. 3.8.

If N denotes the total number of pieces at Round 1 of the protocol, one has $N = N_a + N_b + N_c$ with $N_c = 1$. Each round of the protocol corresponding to a fresh pair (N_a, N_b), it is natural to define the asymptotic density vector γ as:

$$\gamma_a = \frac{\mathbb{E}N_a}{\mathbb{E}N}, \quad \gamma_b = \frac{\mathbb{E}N_b}{\mathbb{E}N}, \quad \gamma_c = \frac{\mathbb{E}N_c}{\mathbb{E}N} = \frac{1}{\mathbb{E}N}, \tag{2}$$

where $\mathbb{E}$ denotes the expectation with respect to probability $\mathbb{P}$.

Since N_a and N_b follow geometric laws on the one hand, and since $\mathbb{E}N = 1 + \mathbb{E}N_a + \mathbb{E}N_b$ on the other hand, the computation of $\gamma = \big(\gamma_a\ \gamma_b\ \gamma_c\big)$ is immediate and yields:

$$\gamma_a = \frac{\lambda'(1-\lambda)}{\lambda + \lambda' - \lambda\lambda'}, \quad \gamma_b = \frac{\lambda(1-\lambda')}{\lambda + \lambda' - \lambda\lambda'}, \quad \gamma_c = \frac{\lambda\lambda'}{\lambda + \lambda' - \lambda\lambda'}. \tag{3}$$

The description we have given of the protocol has naturally lead us to the Definition (2) for the density vector γ. However, abstracting from the description of the protocol and focusing on the heap model only, we realize that an equivalent description of the probabilistic protocol is the following:

3. Consider the probability distribution $\mathbb{P}$ over random heaps.
4. Recursively cut an infinite random heap, say ξ distributed according to $\mathbb{P}$, by selecting the successive occurrences of type c dominoes in ξ, and cutting apart the associated sub-heaps, as in Fig. 1b.

In this new formulation, point 3 is now intrinsic, while only point 4 relies on a special cut shape. Henceforth, the following questions are natural: if we change the cut shape in point 4, and if we compute the new associated densities of pieces, say $\gamma' = \big(\gamma'_a\ \gamma'_b\ \gamma'_c\big)$, is it true that $\gamma = \gamma'$?

Let us consider for instance the variant where heaps are cut at "first occurrence" of type a dominoes. We illustrate in Fig. 2 the successive new rounds, corresponding

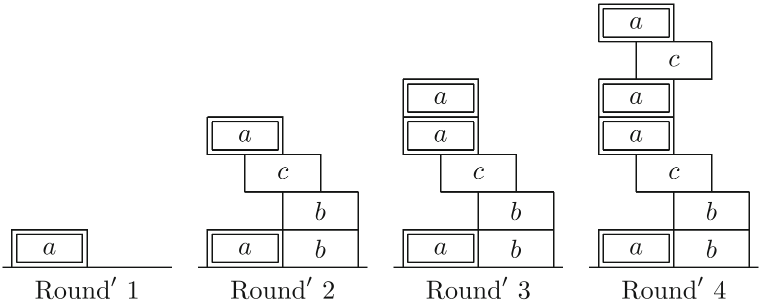

Fig. 2 Cutting heaps relatively to type a dominoes

to the same heap that we already depicted in Fig. 1. Observe that each new round involves a *finite but unbounded number of rounds* of the original protocol.

Let V' denotes the random heap obtained by cutting an infinite heap at the first occurrence of a domino of type a, which is defined with $\mathbb{P}$-probability 1. Denote by $|V'|$ the number of pieces in V'. Denote also by $|V'|_a$ the number of occurrences of piece a in V', and so on for $|V'|_b$ and for $|V'|_c$, so that $|V'| = |V'|_a + |V'|_b + |V'|_c$ holds. By construction, $|V'|_a = 1$ holds $\mathbb{P}$-almost surely. We define the new density vector $\gamma' = \begin{pmatrix}\gamma'_a & \gamma'_b & \gamma'_c\end{pmatrix}$ by:

$$\gamma'_a = \frac{\mathbb{E}|V'|_a}{\mathbb{E}|V'|} = \frac{1}{\mathbb{E}|V'|}, \quad \gamma'_b = \frac{\mathbb{E}|V'|_b}{\mathbb{E}|V'|}, \quad \gamma'_c = \frac{\mathbb{E}|V'|_c}{\mathbb{E}|V'|}.$$

The computation of γ' is easy. A typical random heap V' ends up with a, after having crossed, say, k occurrences of c. Immediately before the j^{th} occurrence of c, for $j \leq k$, there has been an arbitrary number, say l_j, of occurrences of b. Hence, $V' = b^{l_1} \cdot c \cdot \cdots \cdot b^{l_k} \cdot c \cdot a$, with $k \geq 0$ and $l_1, \ldots, l_k \geq 0$. Referring to the definition of the probability $\mathbb{P}$, one has:

$$\forall k, l_1, \ldots, l_k \geq 0, \quad \mathbb{P}(V' = b^{l_1} \cdot c \cdot \cdots \cdot b^{l_k} \cdot c \cdot a) = \lambda^k (1-\lambda)\lambda'^k (1-\lambda')^{l_1+\cdots+l_k}.$$

Since $|V'|_b = l_1 + \cdots + l_k$ and $|V'|_c = k$, the computation of the various expectations is straightforward:

$$\mathbb{E}|V'|_b = \sum_{k,l_1,\ldots,l_k \geq 0} (l_1 + \cdots + l_k)\lambda^k (1-\lambda)\lambda'^k (1-\lambda')^{l_1+\cdots+l_k} = \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}$$

$$\mathbb{E}|V'|_c = \sum_{k,l_1,\ldots,l_k \geq 0} k\lambda^k (1-\lambda)\lambda'^k (1-\lambda')^{l_1+\cdots+l_k} = \frac{\lambda}{1-\lambda}.$$

$$\mathbb{E}|V'| = 1 + \mathbb{E}|V'|_b + \mathbb{E}|V'|_c = \frac{\lambda + \lambda' - \lambda\lambda'}{\lambda'(1-\lambda)}$$

We obtain the density vector γ':

$$\gamma'_a = \frac{\lambda'(1-\lambda)}{\lambda+\lambda'-\lambda\lambda'}, \quad \gamma'_b = \frac{\lambda(1-\lambda')}{\lambda+\lambda'-\lambda\lambda'}, \quad \gamma'_c = \frac{\lambda\lambda'}{\lambda+\lambda'-\lambda\lambda'}.$$

Comparing with (3), we observe the announced equality $\gamma = \gamma'$.

3 Heap Monoids and Bernoulli Measures

In this section, we collect the needed material on heap monoids and on associated Bernoulli measures. Classical references on heap monoids are [5,9,10,23]. For infinite heaps and Bernoulli measures, see [2].

3.1 Independence Pairs and Heap Monoids

Let Σ be a finite, non-empty set of cardinal > 1. Elements of Σ are called *pieces*. We say that the pair (Σ, I) is an *independence pair* if I is a symmetric and irreflexive relation on Σ, called *independence relation*. We will furthermore always assume that the following irreducibility assumption is in force: the associated *dependence relation* D on Σ, defined by $D = (\Sigma \times \Sigma)\backslash I$, makes the graph (Σ, D) connected.

The free monoid generated by Σ is denoted by Σ^*, it consists of all Σ-words. The congruence $\mathcal{I}$ is defined as the smallest congruence on Σ^* that contains all pairs of the form (ab, ba) for (a, b) ranging over I. The *heap monoid* $\mathcal{M} = \mathcal{M}(\Sigma, I)$ is defined as the quotient monoid $\mathcal{M} = \Sigma^*/\mathcal{I}$. Hence, $\mathcal{M}$ is the presented monoid:

$$\mathcal{M} = \langle \Sigma \mid ab = ba, \quad \text{for} \quad (a, b) \in I \rangle.$$

Elements of a heap monoid are called *heaps*. In the literature, heaps are also called *traces*; heap monoids are also called *free partially commutative monoids*.

We denote by the dot "·" the concatenation of heaps, and by 0 the empty heap.

A graphical interpretation of heaps is obtained by letting pieces fall as dominoes on a ground, in such a way that (1) dominoes corresponding to different occurrences of the same piece follow the same lane; and (2)two dominoes corresponding to pieces a and b are blocking with respect to each other if and only if $(a, b) \notin I$. This is illustrated in Fig. 3 for the heap monoid on three generators $\mathcal{T} = \langle a, b, c \mid ab = ba \rangle$. This will be our running example throughout the paper.

3.2 Length and Ordering

The congruence $\mathcal{I}$ coincides with the reflexive and transitive closure of the *immediate equivalence*, which relates any two Σ-words of the form $xaby$ and $xbay$, where $x, y \in \Sigma^*$ and $(a, b) \in I$. In particular, the length of congruent words is invariant, which defines a mapping $|\cdot| : \mathcal{M} \to \mathbb{N}$. For any heap $x \in \mathcal{M}$, the integer $|x|$ is

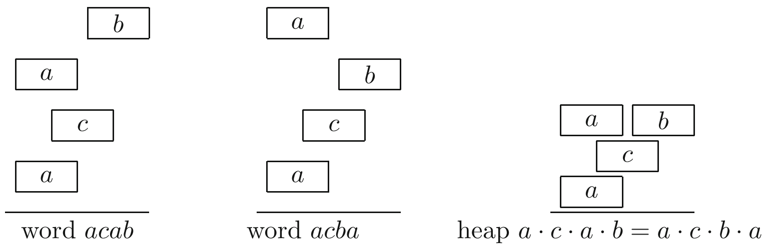

Fig. 3 Two congruent words and the resulting heap

called the *length* of x. Obviously, the length is additive on $\mathcal{M}$: $|x \cdot y| = |x| + |y|$ for all heaps $x, y \in \mathcal{M}$.

The left divisibility relation "$\leq$" is defined on $\mathcal{M}$ by:

$$\forall x, y \in \mathcal{M} \quad x \leq y \iff \exists z \in \mathcal{M} \quad y = x \cdot z. \tag{4}$$

It defines a partial ordering relation on $\mathcal{M}$. If $x \leq y$, we say that x is a *sub-heap* of y.

Visually, $x \leq y$ means that x is a heap that can be seen at the bottom of y. But, contrary to words, a given heap might for instance have several sub-heaps of length 1. Indeed, in the example monoid $\mathcal{T}$ defined above, one has both $a \leq a \cdot b$ and $b \leq a \cdot b$ since $a \cdot b = b \cdot a$ in $\mathcal{T}$.

Heap monoids are known to be *cancellative*, meaning:

$$\forall x, y, u, u' \in \mathcal{M} \quad x \cdot u \cdot y = x \cdot u' \cdot y \implies u = u'.$$

This implies in particular that, if x, y are heaps such that $x \leq y$ holds, then the heap z in (4) is unique. We denote it by: $z = y - x$.

3.3 Cliques and Cartier-Foata Normal Form

Recall that a *clique* of a graph is a subgraph which is complete as a graph—this includes the empty graph. The independence pair (Σ, I) may be seen as a graph. The cliques of (Σ, I) are called the *independence cliques*, or simply the *cliques* of the heap monoid $\mathcal{M}$.

Each clique γ, with set of vertices $\{a_1, \ldots, a_n\}$, identifies with the heap $a_1 \cdot \cdots \cdot a_n \in \mathcal{M}$, which, by commutativity, is independent of the sequence $(a_1, \ldots, a_n)$ enumerating the vertices of γ. In the graphical representation of heaps, cliques correspond to horizontal layers of pieces. Note that any piece is by itself a clique of length 1. We denote by $\mathscr{C}$ the set of cliques of the heap monoid $\mathcal{M}$, and by $\mathfrak{C} = \mathscr{C} \setminus \{0\}$ the set of non-empty cliques. For the running example monoid $\mathcal{T}$, there are 4 non-empty cliques: $\mathfrak{C} = \{a, b, c, a \cdot b\}$.

It is visually intuitive that heaps can be uniquely written as a succession of horizontal layers, hence of cliques. More precisely, define the relation $\to$ on $\mathscr{C}$ as follows:

$$\forall \gamma, \gamma' \in \mathscr{C} \quad \gamma \to \gamma' \iff \forall b \in \gamma' \quad \exists a \in \gamma \quad (a, b) \notin I.$$

The relation $\gamma \to \gamma'$ means that γ "supports" γ', in the sense that no piece of γ' can fall when piled upon γ.

A sequence $\gamma_1, \ldots, \gamma_n$ of cliques is said to be *Cartier–Foata admissible* if $\gamma_i \to \gamma_{i+1}$ holds for all $i \in \{1, \ldots, n-1\}$. For every *non-empty heap* $x \in \mathcal{M}$, there exists a unique integer $n \geq 1$ and a unique Cartier–Foata admissible sequence $(\gamma_1, \ldots, \gamma_n)$ of *non-empty cliques* such that $x = \gamma_1 \cdot \cdots \cdot \gamma_n$.

This unique sequence of cliques is called the *Cartier–Foata normal form* or *decomposition* of x (CF for short). The integer n is called the *height* of x, denoted by $n = \tau(x)$. By convention, we set $\tau(0) = 0$.

Heaps are thus in one-to-one correspondence with finite paths in the graph $(\mathfrak{C}, \to)$ of *non-empty* cliques. By convention, let us extend any such finite path by infinitely many occurrences of the *empty* clique 0. Observe that 0 is an absorbing vertex of the graph of cliques $(\mathscr{C}, \to)$, since $0 \to \gamma \iff \gamma = 0$, and $\gamma \to 0$ holds for every $\gamma \in \mathscr{C}$. With this convention, heaps are now in one-to-one correspondence with infinite paths in $(\mathscr{C}, \to)$ that reach the 0 node—and then stay in it.

3.4 Infinite Heaps and Boundary

We define an *infinite heap* as any infinite admissible sequence of cliques in the graph $(\mathscr{C}, \to)$, that does not reach the empty clique. The set of infinite heaps is called the *boundary at infinity of* $\mathcal{M}$, or simply the *boundary* of $\mathcal{M}$, and we denote it by $\partial\mathcal{M}$ [2]. By contrast, elements of $\mathcal{M}$ might be called *finite heaps*. Extending the previous terminology, we still refer to the cliques γ_n such that $\xi = (\gamma_n)_{n\geq 1}$ as to the CF decomposition of an infinite heap ξ.

It is customary to introduce the following notation:

$$\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}.$$

Elements of $\overline{\mathcal{M}}$ are thus in one-to-one correspondence with infinite paths in $(\mathscr{C}, \to)$; those that reach 0 correspond to heaps, and those that do not reach 0 correspond to infinite heaps. For $\xi \in \partial\mathcal{M}$, we put $|\xi| = \infty$.

We wish to extend to $\overline{\mathcal{M}}$ the order $\leq$ previously defined on $\mathcal{M}$. For this, we use the representation of heaps, either finite or infinite, as infinite paths in the graph $(\mathscr{C}, \to)$, and we put, for $\xi = (\gamma_1, \gamma_2, \ldots)$ and $\xi' = (\gamma'_1, \gamma'_2, \ldots)$:

$$\xi \leq \xi' \iff \forall n \geq 1 \quad \gamma_1 \cdot \cdots \cdot \gamma_n \leq \gamma'_1 \cdot \cdots \cdot \gamma'_n. \tag{5}$$

Proposition 3.1 ([2]) *The relation defined in* (5) *makes* $(\overline{\mathcal{M}}, \leq)$ *a partial order, which extends* $(\mathcal{M}, \leq)$ *and has the following properties:*

1. $(\overline{\mathcal{M}}, \leq)$ *is complete with respect to:*

(a) *Least upper bounds* (lub) *of non-decreasing sequences: for every sequence* $(x_n)_{n\geq 1}$ *such that* $x_n \in \overline{\mathcal{M}}$ *and* $x_n \leq x_{n+1}$ *for all integers* $n \geq 1$, *the* lub $\bigvee_{n\geq 1} x_n$ *exists in* $\overline{\mathcal{M}}$.

(b) *Greatest lower bounds of arbitrary subsets.*

2. *For every heap* $\xi \in \overline{\mathcal{M}}$*, either finite or infinite, the following subset:*

$$L(\xi) = \{x \in \overline{\mathcal{M}} \ : \ x \le \xi\}, \tag{6}$$

is a complete lattice with 0 *and* ξ *as minimal and maximal elements.*
3. (Finiteness property of elements of $\mathcal{M}$ in the sense of [11]) *For every finite heap* $x \in \mathcal{M}$*, and for every non-decreasing sequence* $(x_n)_{n\ge 0}$ *of heaps, holds:*

$$\bigvee_{n\ge 0} x_n \ge x \implies \exists n \ge 0 \quad x_n \ge x.$$

4. *The elements of* $\mathcal{M}$ *form a basis of* $\overline{\mathcal{M}}$ *in the sense of* [11]*: for all* $\xi, \xi' \in \overline{\mathcal{M}}$ *holds:*

$$\xi \ge \xi' \iff (\forall x \in \mathcal{M} \quad \xi' \ge x \implies \xi \ge x).$$

3.5 Visual Cylinders and Bernoulli Measures

For $x \in \mathcal{M}$ a heap, the *visual cylinder of base* x is the following non-empty subset of $\partial\mathcal{M}$:

$$\uparrow x = \{\xi \in \partial\mathcal{M} \ : \ x \le \xi\}.$$

We equip the boundary $\partial\mathcal{M}$ with the σ-algebra:

$$\mathfrak{F} = \sigma\langle \uparrow x, \ x \in \mathcal{M}\rangle,$$

generated by the countable collection of visual cylinders. From now on, when referring to the *boundary*, we shall always mean the measurable space $(\partial\mathcal{M}, \mathfrak{F})$.

We say that a probability measure $\mathbb{P}$ on the boundary is a *Bernoulli measure* whenever it satisfies:

$$\forall x, y \in \mathcal{M} \quad \mathbb{P}\big(\uparrow (x \cdot y)\big) = \mathbb{P}(\uparrow x) \cdot \mathbb{P}(\uparrow y). \tag{7}$$

We shall furthermore impose the following condition:

$$\forall x \in \mathcal{M} \quad \mathbb{P}(\uparrow x) > 0. \tag{8}$$

If $\mathbb{P}$ is a Bernoulli measure on the boundary, the positive function $f : \mathcal{M} \to \mathbb{R}$ defined by:

$$\forall x \in \mathcal{M} \quad f(x) = \mathbb{P}(\uparrow x),$$

is called the *valuation associated to* $\mathbb{P}$. By definition of Bernoulli measures, f is multiplicative: $f(x \cdot y) = f(x) \cdot f(y)$. In particular, the values of f on $\mathcal{M}$ are entirely determined by the finite collection $(p_a)_{a\in\Sigma}$ of *characteristic numbers of* $\mathbb{P}$ defined by the value of f on single pieces:

$$\forall a \in \Sigma \quad p_a = f(a). \tag{9}$$

The condition (8) is equivalent to impose $p_a > 0$ for all $a \in \Sigma$.

For any heap $x \in \mathcal{M}$, $\mathbb{P}(\uparrow x)$ corresponds to the probability of seeing x at bottom of a random infinite heap with law $\mathbb{P}$. By definition of Bernoulli measures, this probability is equal to the product $p_{a_1} \times \cdots \times p_{a_n}$, where the word $a_1 \ldots a_n$ is any representative word of the heap x.

3.6 Interpretation of the Introductory Probabilistic Protocol

The definition of Bernoulli measures already allows us to interpret the probabilistic protocol introduced in § 2 by means of a Bernoulli measure $\mathbb{P}$ on the boundary of a heap monoid. Obviously, the heap monoid to consider coincides with our running example $\mathcal{T} = \langle a, b, c \mid ab = ba \rangle$ on three generators. Let us check that the measure $\mathbb{P}$ defined by the law of infinite heaps generated by the described protocol is indeed Bernoulli.

Any heap $x \in \mathcal{T}$ can be uniquely described under the form:

$$x = (a^{r_1} \cdot b^{s_1}) \cdot c \cdot \cdots \cdot (a^{r_k} \cdot b^{s_k}) \cdot c \cdot a^{r_{k+1}} \cdot b^{s_{k+1}},$$

for some integers $k, r_1, s_1, \ldots, r_{k+1}, s_{k+1} \geq 0$. For such a heap x, referring to the description of the probabilistic protocol, the associated visual cylinder $\uparrow x$ is described by:

$$\uparrow x = \begin{cases} \text{Round 1:} & N_a = r_1,\ N_b = s_1 \\ \quad\vdots & \\ \text{Round k:} & N_a = r_k,\ N_b = s_k \\ \text{Round k+1:} & N_a \geq r_{k+1},\ N_b \geq s_{k+1} \end{cases}$$

and is given probability:

$$\mathbb{P}(\uparrow x) = (\lambda\lambda')^k (1-\lambda)^{r_1+\cdots+r_k} (1-\lambda')^{s_1+\cdots+s_k} (1-\lambda)^{r_{k+1}} (1-\lambda')^{s_{k+1}}.$$

If $f : \mathcal{M} \to \mathbb{R}$ is the multiplicative function defined by:

$$f(a) = 1 - \lambda, \qquad f(b) = 1 - \lambda', \qquad f(c) = \lambda\lambda', \tag{10}$$

it is thus apparent that $\mathbb{P}(\uparrow x) = f(x)$ holds for all $x \in \mathcal{M}$. Since $\mathbb{P}(\uparrow x)$ is multiplicative, the measure $\mathbb{P}$ is Bernoulli.

In passing, we notice that, whatever the choices of $\lambda, \lambda' \in (0, 1)$, the equation:

$$1 - f(a) - f(b) - f(c) + f(a)f(b) = 0$$

is satisfied, as one shall expect from (14)–(a) below.

3.7 Compatible Heaps

In the sequel, we shall often use the following facts. We say that two heaps $x, y \in \mathcal{M}$ are *compatible* if there exists $z \in \mathcal{M}$ such that $x \leq z$ and $y \leq z$. It follows in particular from Proposition 3.1 that the following propositions are equivalent:

(i) $x, y \in \mathcal{M}$ are compatible;
(ii) $\uparrow x \cap \uparrow y \neq \emptyset$;
(iii) the *lub* $x \vee y$ exists in $\mathcal{M}$.

In this case, we also have:

$$\uparrow x \cap \uparrow y = \uparrow (x \vee y).$$

If $\mathbb{P}$ is a Bernoulli probability measure on $\partial\mathcal{M}$, and if x and y are two compatible heaps, then it is immediate to obtain, by multiplicativity of $\mathbb{P}$:

$$\mathbb{P}\big(\uparrow x \mid \uparrow y\big) = \mathbb{P}\big(\uparrow ((x \vee y) - y)\big) = \mathbb{P}\big(\uparrow (x - (x \wedge y))\big). \tag{11}$$

3.8 Möbius Transform and Markov Chain of Cliques

Call *valuation* any positive and multiplicative function $f : \mathcal{M} \to \mathbb{R}$; the valuations induced by Bernoulli measures are particular examples of valuations. Any valuation is characterized by its values on single pieces, as in (9).

If $f : \mathcal{M} \to \mathbb{R}$ is any valuation, then the *Möbius transform* of f is the function $h : \mathscr{C} \to \mathbb{R}$ defined by [17,20]:

$$\forall \gamma \in \mathscr{C} \quad h(\gamma) = \sum_{\gamma' \in \mathscr{C}\,:\,\gamma' \geq \gamma} (-1)^{|\gamma'|-|\gamma|} f(\gamma'). \tag{12}$$

It is proved in [2] that, for a given valuation $f : \mathcal{M} \to \mathbb{R}$, there exists a Bernoulli measure on the boundary that induces f if and only if its Möbius transform h satisfies the following two conditions:

$$(a)\ h(0) = 0\,; \qquad\qquad (b)\ \forall \gamma \in \mathfrak{C} \quad h(\gamma) > 0. \tag{13}$$

Note that Condition (a) is a polynomial condition in the characteristic numbers, and Condition (b) corresponds to a finite series of polynomial inequalities. For instance, for the heap monoid $\mathcal{T}$ on three generators $\mathcal{T} = \langle a, b, c \ :\ ab = ba \rangle$, we obtain:

$$(a)\ 1 - p_a - p_b - p_c + p_a p_b = 0, \quad (b) \begin{cases} h(a) > 0 \iff p_a(1 - p_b) > 0 \\ h(b) > 0 \iff p_b(1 - p_a) > 0 \\ h(c) > 0 \iff p_c > 0 \\ h(ab) > 0 \iff p_a p_b > 0 \end{cases} \tag{14}$$

Returning to the study of a general heap monoid, and when considering the case where all coefficients p_a are equal, say to p, then both conditions in (13) reduce to the following: p is the (known to be unique [8,12,14]) root of smallest modulus of the *Möbius polynomial of the heap monoid* $\mathcal{M}$, defined by:

$$\mu_{\mathcal{M}}(X) = \sum_{c \in \mathscr{C}} (-1)^{|c|} X^{|c|}.$$

The associated Bernoulli measure is then called the *uniform measure on the boundary*.

For the running example $\mathcal{T}$, one has $\mu_{\mathcal{T}}(X) = 1 - 3X + X^2$, and the uniform measure is given by $\mathbb{P}(\uparrow x) = p^{|x|}$ with $p = (3 - \sqrt{5})/2$.

In the remaining of this subsection, we characterize the process of cliques that compose a random infinite heap under a Bernoulli measure.

For each integer $n \geq 1$, the mapping which associates to an infinite heap ξ the n^{th} clique γ_n such that $\xi = (\gamma_1, \gamma_2, \ldots)$ is measurable, and defines thus a random variable $C_n : \partial\mathcal{M} \to \mathfrak{C}$. Furthermore, the sequence $(C_n)_{n \geq 1}$ is a *time homogeneous and ergodic Markov chain* [2], of which:

1. The initial distribution is the restriction $h\big|_{\mathfrak{C}}$, where h is the Möbius transform defined in (12).
2. For each non-empty clique $\gamma \in \mathfrak{C}$, put:

$$g(\gamma) = \sum_{\gamma' \in \mathfrak{C} \,:\, \gamma \to \gamma'} h(\gamma'). \tag{15}$$

Then the transition matrix $P = (P_{\gamma,\gamma'})_{(\gamma,\gamma') \in \mathfrak{C}}$ of the chain is given by:

$$P_{\gamma,\gamma'} = \begin{cases} 0, & \text{if } (\gamma \to \gamma') \ \text{ does } \textit{not} \text{ hold,} \\ h(\gamma')/g(\gamma), & \text{if } (\gamma \to \gamma') \ \text{ holds.} \end{cases} \tag{16}$$

In general, the initial measure $h\big|_{\mathfrak{C}}$ does *not* coincide with the stationary measure of the chain of cliques.

4 Asynchronous Stopping Times and the Strong Bernoulli Property

In this section, we introduce asynchronous stopping times and their associated shift operators. They will be our basic tools to formulate and prove the law of large numbers in the subsequent sections.

4.1 Definition and Examples

Intuitively, an asynchronous stopping time is a way to select sub-heaps from infinite heaps, such that for each infinite heap one can decide at each "time instant" whether the sub-heap in question has already been reached or not. The formal definition follows.

Definition 4.1 An *asynchronous stopping time*, or *AST* for short, is a mapping $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$, which we sometimes denote by $\xi \mapsto \xi_V$, such that:

1. $\xi_V \leq \xi$ for all $\xi \in \partial\mathcal{M}$;
2. $\forall \xi, \xi' \in \partial\mathcal{M} \quad (|\xi_V| < \infty \wedge \xi_V \leq \xi') \implies \xi'_V = \xi_V$.

We say that V is $\mathbb{P}$-a.s. *finite* whenever $\xi_V \in \mathcal{M}$ for $\mathbb{P}$-a.s. every $\xi \in \partial\mathcal{M}$.

In the above definition, we think of ξ_V as "ξ cut at V". $V = 0$ is a first, trivial example of *AST*. Actually, the property 2 of the above definition implies that if $\xi_V = 0$ for some $\xi \in \partial\mathcal{M}$, then $V = 0$ on $\partial\mathcal{M}$. Another simple example is the following. Let $x \in \mathcal{M}$ be a fixed heap. Define $V_x : \partial\mathcal{M} \to \overline{\mathcal{M}}$ by:

$$V_x(\xi) = \begin{cases} x, & \text{if } x \leq \xi, \\ \xi, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that V_x is an *AST*. We recover the previous example by setting $x = 0$.

Consider the sequence of random cliques $(C_k)_{k\geq 1}$ associated with infinite heaps ξ, and define for each integer $k \geq 1$ the random heap $Y_k = C_1 \cdot \dots \cdot C_k$. Then, by construction, $Y_k \leq \xi$ holds; however, the mapping Y_k is *not* an *AST*, except if $\mathcal{M}$ is the free monoid, which corresponds to the empty independence relation I on Σ.

We will frequently use the following remark, which is a direct consequence of the definition: let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an *AST*, and let $\mathcal{V}$ be the set of finite values assumed by V. Then holds:

$$\forall x \in \mathcal{V} \qquad \{V = x\} = \uparrow x.$$

The following proposition provides less trivial examples of *AST* that we will use throughout the rest of the paper. Let us first introduce a new notation. If $a \in \Sigma$ is a piece, and $x \in \mathcal{M}$ is a heap, the number of occurrences of a in a representative word of x does not depend on the representative word and is thus attached to the heap x. We denote it by $|x|_a$.

Proposition 4.2 *The two mappings $\partial\mathcal{M} \to \overline{\mathcal{M}}$, $\xi \mapsto \xi_V$ described below define asynchronous stopping times:*

1. *For $\xi = (\gamma_1, \gamma_2, \dots)$, let $R_\xi = \{k \geq 1 \ : \ \gamma_k$ is maximal in $\mathscr{C}\}$, and put:*

$$\xi_V = \begin{cases} \gamma_1 \cdot \dots \cdot \gamma_n \text{ with } n = \min R_\xi, & \text{if } R_\xi \neq \emptyset, \\ \xi, & \text{otherwise.} \end{cases} \tag{17}$$

2. *Let $a \in \Sigma$ be some fixed piece. For $\xi \in \partial\mathcal{M}$, let $L(\xi)$ be the complete lattice defined in* (6). *Then put:*

$$\xi_V = \bigwedge H_a(\xi), \qquad H_a(\xi) = \{x \in L(\xi) \cap \mathcal{M} \ : \ |x|_a > 0\}, \tag{18}$$

where the greatest lower bound defining ξ_V is taken in $L(\xi)$. It is called the first hitting time of a.

It $H_a(\xi) \neq \emptyset$, then $\xi_V \in H_a(\xi)$.

For the first hitting time of a, since the greatest lower bound in (18) is taken in the complete lattice $L(\xi)$, if $H_a(\xi) = \emptyset$ then $\xi_V = \max L(\xi) = \xi$.

Proof 1. The condition $\xi_V \leq \xi$ is obvious on (17). Hence, let $\xi, \xi' \in \partial\mathcal{M}$ such that $\xi_V \in \mathcal{M}$ and $\xi_V \leq \xi'$. We need to show that $\xi'_V = \xi_V$. For this, let $\xi = (\gamma_1, \gamma_2, \ldots)$ and $\xi' = (\gamma'_1, \gamma'_2, \ldots)$, and let $n = \min R_\xi$ and $u = \gamma_1 \cdot \cdots \cdot \gamma_n$. By hypothesis, we have $u \leq \xi'$ and thus $u \leq \gamma'_1 \cdot \cdots \cdot \gamma'_n$, by (5).
It follows from [2, Lemma 8.1] that the sequences $(\gamma_i)_{1 \leq i \leq n}$ and $(\gamma'_i)_{1 \leq i \leq n}$ are related as follows: for each integer $i \in \{1, \ldots, n\}$, there exists a clique δ_i such that $\delta_i \wedge \gamma_i = 0, \ldots, \delta_i \wedge \gamma_n = 0$, and $\gamma'_i = \gamma_i \cdot \delta_i$. But γ_n is maximal, therefore, $\delta_i = 0$ for all $i \in \{1, \ldots, n\}$, and thus $\gamma'_i = \gamma_i$. From $\gamma'_n = \gamma_n$ follows at once that $\min R_{\xi'} \leq n$. And since $\gamma'_i = \gamma_i$ for all $i < n$, no clique γ'_i is maximal in $\mathscr{C}$, otherwise it would contradict the definition of ξ_V. Hence, finally $\min R_{\xi'} = n$, from which follows $\xi'_V = \gamma'_1 \cdot \cdots \cdot \gamma'_n = \gamma_1 \cdot \cdots \cdot \gamma_n = \xi_V$.
2. Again, it is obvious on (18) that $\xi_V \leq \xi$. Let $\xi, \xi' \in \partial\mathcal{M}$ be such that $\xi_V \in \mathcal{M}$ and $\xi_V \leq \xi'$. Then $H_a(\xi) \neq \emptyset$, and we observe that $H_a(\xi)$ actually has a minimum, $\xi_V = \min H_a(\xi)$. This is best seen with the resource interpretation of heap monoids introduced in [7].

It follows that $\xi_V \in H_a(\xi')$ and thus $H_a(\xi') \neq \emptyset$. Henceforth, as above, we deduce $\xi'_V = \min H_a(\xi')$, and $\xi'_V \leq \xi_V$. It follows that $\xi'_V \in H_a(\xi)$ and thus $\xi_V \leq \xi'_V$ and finally $\xi_V = \xi'_V$. □

4.2 Action of the Monoid on its Boundary and Shift Operators

In order to define the shift operator associated with an AST, we first describe the natural left action of a heap monoid on its boundary.

For $x \in \mathcal{M}$ and $\xi \in \partial\mathcal{M}$ an infinite heap, the visually intuitive operation of piling up ξ upon x should yield an infinite heap. However, some pieces in the first layers of ξ might fall off and fill up empty slots in x. Hence, the CF decomposition of $x \cdot \xi$ cannot be defined as the mere concatenation of the CF decompositions of x and ξ.

The proper definition of the concatenation $x \cdot \xi$ is as follows. Let $\xi = (\gamma_1, \gamma_2, \ldots)$. The sequence of heaps $(x \cdot \gamma_1 \cdot \cdots \cdot \gamma_n)_{n \geq 1}$ is obviously non-decreasing. According to point a1 of Proposition 3.1, we may thus consider:

$$x \cdot \xi = \bigvee_{n \geq 1} (x \cdot \gamma_1 \cdot \cdots \cdot \gamma_n), \quad \text{which exists in } \partial\mathcal{M},$$

and then we have:

$$\forall x, y \in \mathcal{M} \quad \forall \xi \in \partial\mathcal{M} \quad x \cdot (y \cdot \xi) = (x \cdot y) \cdot \xi.$$

It is then routine to check that, for each $x \in \mathcal{M}$, the mapping:

$$\Phi_x : \partial\mathcal{M} \to \uparrow x, \quad \xi \mapsto x \cdot \xi,$$

is a bijection. Therefore, we extend the notation $y - x$, licit for $x, y \in \mathcal{M}$ with $x \leq y$, by allowing y to range over $\partial\mathcal{M}$, as follows:

$$\forall x \in \mathcal{M} \quad \forall \xi \in \uparrow x \quad \xi - x = \Phi_x^{-1}(\xi).$$

Hence, $\zeta = \xi - x$ denotes the tail of ξ "after" x, for $\xi \geq x$. It is characterized by the property $x \cdot \zeta = \xi$, and this allows us to introduce the following definition.

Definition 4.3 Let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an AST. The *shift operator associated to* V is the mapping $\theta_V : \partial\mathcal{M} \to \partial\mathcal{M}$, which is partially defined by:

$$\forall \xi \in \partial\mathcal{M} \quad \xi_V \in \mathcal{M} \implies \theta_V(\xi) = \xi - \xi_V.$$

The domain of definition of θ_V is $\{|\xi_V| < \infty\}$.

4.3 The Strong Bernoulli Property

The strong Bernoulli property has with respect to the definition of Bernoulli measures, the same relationship than the strong Markov property with respect to the mere definition of Markov chains. Its formulation is also similar (see for instance [16]). In particular, it involves a σ-algebra associated with an AST, defined as follows.

Definition 4.4 Let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an AST, and let $\mathcal{V}$ be the collection of finite values assumed by V. We define the σ-algebra $\mathfrak{F}_V$ as

$$\mathfrak{F}_V = \sigma\langle \uparrow x \ : \ x \in \mathcal{V}\rangle.$$

With the above definition, we have the following result.

Theorem 4.5 (Strong Bernoulli Property) *Let $\mathbb{P}$ be a Bernoulli measure on $\partial\mathcal{M}$, let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an AST and let $\psi : \partial\mathcal{M} \to \mathbb{R}$ be a $\mathfrak{F}$-measurable function, either non-negative or $\mathbb{P}$-integrable. Extend $\psi \circ \theta_V$, which is only defined on $\{|\xi_V| < \infty\}$, by $\psi \circ \theta_V = 0$ on $\{|\xi_V| = \infty\}$. Then:*

$$\mathbb{E}(\psi \circ \theta_V | \mathfrak{F}_V) = \mathbb{E}(\psi), \quad \mathbb{P}\text{-a.s. on } |\xi_V| < \infty, \tag{19}$$

denoting by $\mathbb{E}(\cdot)$ the expectation with respect to $\mathbb{P}$, and by $\mathbb{E}(\cdot|\mathfrak{F}_V)$ the conditional expectation with respect to $\mathbb{P}$ and to the σ-algebra $\mathfrak{F}_V$.

If $V : \partial\mathcal{M} \to \mathcal{M}$ is $\mathbb{P}$-a.s. finite, then the strong Bernoulli property writes as:

$$\mathbb{P}\text{-a.s.} \quad \mathbb{E}(\psi \circ \theta_V | \mathfrak{F}_V) = \mathbb{E}(\psi).$$

In the general case, we may still have an equality valid $\mathbb{P}$-almost surely by multiplying both members of (19) by the characteristic function $\mathbf{1}_{\{V\in\mathcal{M}\}} = \mathbf{1}_{\{|\xi_V|<\infty\}}$, which is $\mathfrak{F}_V$-measurable, as follows:

$$\mathbb{P}\text{-a.s.}\quad \mathbb{E}(\mathbf{1}_{\{V\in\mathcal{M}\}}\psi\circ\theta_V|\mathfrak{F}_V) = \mathbf{1}_{\{V\in\mathcal{M}\}}\mathbb{E}(\psi). \tag{20}$$

Proof By the standard approximation technique for measurable functions, it is enough to show the result for ψ of the form $\psi = \mathbf{1}_{\uparrow y}$ for some heap $y\in\mathcal{M}$. For such a function ψ, let $Z = \mathbb{E}(\psi\circ\theta_V|\mathfrak{F}_V)$.

Let $\mathcal{V}$ be the set of finite values assumed by V. We note that the visual cylinders $\uparrow x$, for x ranging over $\mathcal{V}$, are pairwise disjoint since $V(\xi) = x$ on $\uparrow x$ for $x\in\mathcal{V}$. Hence $\mathfrak{F}_V$ is atomic. Therefore, if $\hat{Z}:\mathcal{V}\to\mathbb{R}$ denotes the function defined by:

$$\forall x\in\mathcal{V}\qquad \hat{Z}(x) = \mathbb{E}(\psi\circ\theta_V|V=x),$$

then a version of Z is given by:

$$Z(\xi) = \begin{cases} 0, & \text{if}\quad V(\xi)=\xi,\\ \hat{Z}(x), & \text{if}\quad V(\xi)=x \text{ with } x\in\mathcal{M}.\end{cases}$$

For $x\in\mathcal{V}$ and for $\xi\geq x$, one has

$$\psi\circ\theta_V(\xi) = \psi(\xi - x) = \mathbf{1}_{\uparrow y}(\xi - x) = \mathbf{1}_{\uparrow(x\cdot y)}(\xi).$$

And since $\{V=x\} = \uparrow x$ for $x\in\mathcal{V}$, this yields:

$$\hat{Z}(x) = \mathbb{E}(\psi\circ\theta_V|\uparrow x) = \frac{1}{\mathbb{P}(\uparrow x)}\mathbb{P}(\uparrow(x\cdot y)) = \mathbb{P}(\uparrow y) = \mathbb{E}\psi,$$

by the multiplicativity property of $\mathbb{P}$. The proof is complete. □

5 Iterating Asynchronous Stopping Times

This section studies the iteration of asynchronous stopping times, defined in a very similar way as the iteration of classical stopping times for standard probabilistic processes; see for instance [16]. Properly dealing with iterated AST is a typical example of use of the strong Bernoulli property, as in Proposition 5.3 below.

5.1 Iterated Stopping Times

Proposition 5.1 *Let $V:\partial\mathcal{M}\to\overline{\mathcal{M}}$ be an AST. Let $V_0=0$, and define the mappings $V_n:\partial\mathcal{M}\to\overline{\mathcal{M}}$ by induction as follows:*

$$\forall\xi\in\partial\mathcal{M}\qquad V_{n+1}(\xi) = \begin{cases}\xi, & \text{if } V_n(\xi)\in\partial\mathcal{M}\\ V_n(\xi)\cdot V\big(\xi - V_n(\xi)\big), & \text{if } V_n(\xi)\in\mathcal{M}\end{cases}$$

Then $(V_n)_{n\geq 0}$ is a sequence of AST.

Proof The proof is by induction on the integer $n \geq 0$.

The case $n = 0$ is trivial. Hence, for $n \geq 1$, and assuming that V_{n-1} is an AST, let $\xi, \xi' \in \partial\mathcal{M}$ be such that:

$$V_n(\xi) \in \mathcal{M}, \quad V_n(\xi) \leq \xi'.$$

It implies in particular that $V_{n-1}(\xi) \in \mathcal{M}$ and $V_{n-1}(\xi) \leq \xi'$, from which follows by the induction hypothesis that $V_{n-1}(\xi') = V_{n-1}(\xi)$. Putting $x = V_{n-1}(\xi) = V_{n-1}(\xi')$ on the one hand, there are thus two infinite heaps ζ and ζ' such that $\xi = x \cdot \zeta$ and $\xi' = x \cdot \zeta'$. Putting $y = V(\xi - V_{n-1}(\xi))$ on the other hand, the assumption $V_n(\xi) \leq \xi'$ writes as: $x \cdot y \leq x \cdot \zeta'$, which implies $y \leq \zeta'$ by cancellativity of the monoid. But since V is an AST, this implies in turn $V(\zeta') = y$, and finally, by definition of V_n:

$$V_n(\xi') = V_{n-1}(\xi') \cdot V(\xi' - V_{n-1}(\xi')) = x \cdot V(\zeta') = x \cdot y = V_n(\xi).$$

This shows that V_n is an AST, completing the induction. $\square$

Definition 5.2 Let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an AST. The sequence $(V_n)_{n\geq 0}$ of AST defined as in Proposition 5.3 is called the *iterated sequence of stopping times* associated with V.

Proposition 5.3 *Let $\mathbb{P}$ be a Bernoulli measure equipping the boundary $\partial\mathcal{M}$. Let $(V_n)_{n\geq 0}$ be the iterated sequence of stopping times associated with an* AST $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ *which we assume to be $\mathbb{P}$-a.s. finite. Let also $(\Delta_n)_{n\geq 1}$ be the sequence of increments:*

$$\forall n \geq 0 \qquad \Delta_{n+1} = V \circ \theta_{V_n}, \qquad\qquad V_{n+1} = V_n \cdot \Delta_{n+1}.$$

Then, $(\Delta_n)_{n\geq 1}$ is an i.i.d. *sequence of random variables with values in $\mathcal{M}$, with the same distribution as V.*

Proof We first show that $V_n \in \mathcal{M}$ for all integers $n \geq 1$ and $\mathbb{P}$-almost surely. For this, we apply the strong Bernoulli property (Theorem 4.5) with AST V_{n-1} and with the function $\psi = \mathbf{1}_{\{V\in\mathcal{M}\}}$ to get:

$$\mathbb{P}\text{-a.s.} \quad \mathbb{E}\big(\mathbf{1}_{\{V_{n-1}\in\mathcal{M}\}}\psi \circ \theta_{V_{n-1}} | \mathfrak{F}_{V_{n-1}}\big) = \mathbf{1}_{\{V_{n-1}\in\mathcal{M}\}}\mathbb{E}\psi.$$

But $\mathbf{1}_{\{V_n\in\mathcal{M}\}} = \mathbf{1}_{\{V_{n-1}\in\mathcal{M}\}}\psi \circ \theta_{V_{n-1}}$, and $\mathbb{E}\psi = \mathbb{P}(V \in \mathcal{M}) = 1$ by hypothesis. Hence, the equation above writes as:

$$\mathbb{P}\text{-a.s.} \quad \mathbb{E}(\mathbf{1}_{\{V_n\in\mathcal{M}\}} | \mathfrak{F}_{V_{n-1}}) = \mathbf{1}_{\{V_{n-1}\in\mathcal{M}\}}.$$

Taking the expectations of both members yields: $\mathbb{P}(V_n \in \mathcal{M}) = \mathbb{P}(V_{n-1} \in \mathcal{M})$. Hence by induction, since $\mathbb{P}(V_0 \in \mathcal{M}) = 1$, we deduce that $\mathbb{P}(V_n \in \mathcal{M}) = 1$ for all integers $n \geq 1$.

To complete the proof of the proposition, we show that for any non-negative functions $\varphi_1, \ldots, \varphi_n : \mathcal{M} \to \mathbb{R}$, holds:

$$\mathbb{E}\big(\varphi_1(\Delta_1) \cdot \cdots \cdot \varphi_n(\Delta_n)\big) = \mathbb{E}\varphi_1(V) \cdot \cdots \cdot \mathbb{E}\varphi_n(V). \tag{21}$$

The case $n = 0$ is trivial. Assume the hypothesis true at rank $n-1 \geq 0$. Applying the strong Bernoulli property (Theorem 4.5) with the *AST* V_{n-1} yields, since $V_{n-1} \in \mathcal{M}$ $\mathbb{P}$-almost surely:

$$\mathbb{P}\text{-a.s.} \quad \mathbb{E}\big(\varphi_n(\Delta_n)|\mathfrak{F}_{V_{n-1}}\big) = \mathbb{E}\varphi_n(V). \tag{22}$$

Let A be the left-hand member of (21). Since $\Delta_1, \ldots, \Delta_{n-1}$ are $\mathfrak{F}_{V_n}$-measurable, we compute as follows, using the standard properties of conditional expectation [3]:

$$\begin{aligned}
A &= \mathbb{E}\big(\mathbb{E}(\varphi_1(\Delta_1) \cdot \cdots \cdot \varphi_n(\Delta_n)|\mathfrak{F}_{V_{n-1}})\big) \\
&= \mathbb{E}\big(\varphi_1(\Delta_1) \cdot \cdots \cdot \varphi_{n-1}(\Delta_{n-1}) \cdot \mathbb{E}(\varphi_n(\Delta_n)|\mathfrak{F}_{V_{n-1}})\big) \\
&= \mathbb{E}(\varphi_1(\Delta_1) \cdot \cdots \cdot \varphi_{n-1}(\Delta_{n-1})\big) \cdot \mathbb{E}\varphi_n(V) && \text{by} \quad (22) \\
&= \mathbb{E}\varphi_1(V) \cdot \cdots \cdot \mathbb{E}\varphi_n(V),
\end{aligned}$$

the later equality by the induction hypothesis. This proves (21). □

5.2 Exhaustive Asynchronous Stopping Times

Lemma 5.4 *Let V be an AST, that we assume to be $\mathbb{P}$-a.s. finite. Let $(V_n)_{n\geq 0}$ be the associated sequence of iterated stopping times. Then the two following properties are equivalent:*

(i) $\mathfrak{F} = \bigvee_{n\geq 0} \mathfrak{F}_{V_n}$.
(ii) $\xi = \bigvee_{n\geq 0} V_n(\xi)$ *for $\mathbb{P}$-a.s. every $\xi \in \partial\mathcal{M}$.*

Proof (i) implies (ii). By the Martingale convergence theorem [3, Th. 35.6], we have for every $x \in \mathcal{M}$:

$$\mathbb{P}\text{-a.s.} \quad \lim_{n\to\infty} \mathbb{E}(\mathbf{1}_{\uparrow x}|\mathfrak{F}_{V_n}) = \mathbf{1}_{\uparrow x}. \tag{23}$$

Let $n \geq 0$ be an integer, and let $\mathcal{V}_n$ denote the set of finite heaps assumed by V_n. Then, by the *AST* property of V_n, we have $\{V_n = y\} = \uparrow y$ for every $y \in \mathcal{V}_n$, and thus, for every $x \in \mathcal{M}$ compatible with y:

$$\mathbb{P}(\uparrow x\,|V_n = y) = \mathbb{P}(\uparrow x \,|\, \uparrow y) = \mathbb{P}\big(\uparrow (x - (x \wedge y))\big) \qquad \text{by (11)}$$

Therefore, by (23), we obtain for every $x \in \mathcal{M}$ and for $\mathbb{P}$-a.s. every $\xi \in \uparrow x$:

$$\lim_{n\to\infty} \mathbb{P}(\uparrow (x - (x \wedge \xi_{V_n})) = 1.$$

It implies that the sequence $x \wedge \xi_{V_n}$, which is eventually constant since it is non-decreasing and bounded by x, eventually reaches x, since the only heap $y \in \mathcal{M}$ satisfying $\mathbb{P}(\uparrow y) = 1$ is $y = 0$. In other words, $x \leq \xi_{V_n}$ for n large enough. Hence, $\bigvee_{n\geq 0} \xi_{V_n} \geq x$ for every $x \in \mathcal{M}$ and for $\mathbb{P}$-a.s. every $\xi \in \uparrow x$. In view of the basis property of $\mathcal{M}$ (point 4 of Proposition 3.1), it follows that $\bigvee_{n\geq 0} \xi_{V_n} = \xi$ holds $\mathbb{P}$-almost surely.

(ii) implies (i). Let $\mathfrak{F}'$ be the σ-algebra:

$$\mathfrak{F}' = \bigvee_{n\geq 1} \mathfrak{F}_{V_n}.$$

To show that $\mathfrak{F} = \mathfrak{F}'$, it is enough to show that $\uparrow x \in \mathfrak{F}'$ for every $x \in \mathcal{M}$. But for $x \in \mathcal{M}$, by assumption $\bigvee_{n\geq 1} V_n(\xi) \geq x$ for $\mathbb{P}$-a.s. every $\xi \in \uparrow x$. By the finiteness property of elements of $\mathcal{M}$ (point 3 of Proposition 3.1), it implies, for $\mathbb{P}$-a.s. every $\xi \in \uparrow x$, the existence of an integer $n \geq 0$ such that $V_n(\xi) \geq x$. Letting $\mathcal{V}$ denote the set of finite values assumed by any of the V_n, we have thus:

$$\uparrow x = \bigcup_{v\in\mathcal{V}\,:\,v\geq x} \uparrow v.$$

Since $\mathcal{V}$ is at most countable, it implies $\uparrow x \in \mathfrak{F}'$, which was to be shown. □

Definition 5.5 A $\mathbb{P}$-a.s. finite *AST* that satisfies any of the properties (i)–(ii) of Lemma 5.4 is said to be *exhaustive*.

5.3 Examples of Exhaustive Asynchronous Stopping Times

Proposition 5.6 *Both examples V of AST defined in Proposition 4.2 are exhaustive and satisfy furthermore $\mathbb{E}|V| < \infty$.*

Proof For both examples, that $|V| < \infty$ $\mathbb{P}$-a.s. and also that $\mathbb{E}|V| < \infty$, follow from the two following facts:

1. The Markov chain of cliques $(C_k)_{k\geq 1}$ such that $\xi = (C_1, C_2, \ldots)$ is irreducible with a finite number of states (see § 3.8), and thus is positive recurrent.
2. If α denotes the maximal size of a clique, then $|C_1 \cdot \cdots \cdot C_k| \leq \alpha k$.

We now show that both examples are exhaustive. Let $(V_n)_{n\geq 0}$ be the associated sequence of iterated stopping times.

For V defined in point 1 of Proposition 4.2. Since $V < \infty$ $\mathbb{P}$-a.s., it follows from Proposition 5.3 that $V_n < \infty$ $\mathbb{P}$-a.s. and for all integers $n \geq 0$. Let $\xi = (\gamma_n)_{n\geq 1}$ be an infinite heap. Let $n \geq 0$ be an integer, and let $c_1 \to \ldots \to c_{k_n}$ be the CF decomposition of $V_n(\xi)$. Then, on the one hand, $k_n \geq n$, and on the other hand, since c_{k_n} is maximal and since $\xi \geq V_n(\xi)$, it must hold:

$$\gamma_1 = c_1, \qquad \gamma_2 = c_2, \qquad \ldots \qquad \gamma_{k_n} = c_{k_n}.$$

Hence, if $\xi' = (\gamma'_n)_{n\geq 1}$ denotes:

$$\xi' = \bigvee_{n\geq 0} V_n(\xi),$$

one has $\gamma'_i = \gamma_i$ for all $i \leq k_n$ and for all $n \geq 0$. And since $k_n \geq n$, it implies $\gamma_i = \gamma'_i$ for all integers $i \geq 1$, and thus $\xi' = \xi$. This proves that V is exhaustive.

For V defined in point 2 of Proposition 4.2. Let V be the first hitting time of $a \in \Sigma$. With the same notations as above, let us first show the following claim:

($\Diamond$) For every $b \in \Sigma$: $\mathbb{P}(V \geq b) > 0$.
($\Diamond\Diamond$) For every $b \in \Sigma$, and $\mathbb{P}$-almost surely: $\xi \geq b \implies \xi' \geq b$.

Proof of ($\Diamond$). Since the dependence relation $D = (\Sigma \times \Sigma)\backslash I$ is assumed to make the graph (Σ, D) connected, we pick a sequence $a_1, \ldots, a_j$ of pairwise distinct pieces such that $a_1 = b$, $a_j = a$ and $(a_i, a_{i+1}) \in D$ for all $i \in \{1, \ldots, j-1\}$. Put $x = a_1 \cdot \cdots \cdot a_j$. Then, it is clear that $V(\xi) = x$ for every $\xi \geq x$. Hence $\mathbb{P}(V = x) = \mathbb{P}(\uparrow x) > 0$. Since $b \leq x$, it follows that $\mathbb{P}(V \geq b) \geq \mathbb{P}(\uparrow x) > 0$.

Proof of ($\Diamond\Diamond$). Let $(\Delta_n)_{n\geq 1}$ be the sequence of increments, such that $V_{n+1} = V_n \cdot \Delta_{n+1}$. Then $(\Delta_n)_{n\geq 1}$ being *i.i.d.* with the same law as V according to Proposition 5.3, and since $\mathbb{P}(V \geq b) > 0$, it follows that there exists at least an integer $n \geq 1$ such that $\Delta_n \geq b$, for $\mathbb{P}$-a.s. every $\xi \in \partial\mathcal{M}$. For $\mathbb{P}$-a.s. every $\xi \geq b$, let n be the smallest such integer. Then, the heap $\Delta_1 \cdot \cdots \cdot \Delta_{n-1}$ does not contain any occurrence of b on the one hand and is compatible with b on the other hand. That implies that b commutes with all pieces of $\Delta_1 \cdot \cdots \cdot \Delta_{n-1}$. Therefore, it follows that $b \leq \Delta_1 \cdot \cdots \cdot \Delta_n \leq \xi'$. The claim ($\Diamond\Diamond$) is proved.

Now, to prove that V is exhaustive, let $x \in \mathcal{M}$ be a heap. We show that, $\mathbb{P}$-a.s., $\xi \geq x \implies \xi' \geq x$, which will complete the proof *via* the basis property of $\mathcal{M}$ (point 4 of Proposition 3.1). Putting $y = \xi' \wedge x$, and assuming $\xi \geq x$, we prove that $y = x$ holds $\mathbb{P}$-almost surely. Assume $y \neq x$. Since $y \leq x$, there is thus a piece $b \in \Sigma$ such that $y \cdot b \leq x$ holds and $(\xi' - y) \geq b$ does not hold. Let N be the smallest integer such that $V_N(\xi) \wedge x = y$; such an integer exists, by the finiteness property of $\partial\mathcal{M}$ (point 3 of Proposition 3.1). Let $z = V_N(\xi)$. Then, it follows from the definition of the sequence $(V_n)_{n\geq 0}$ that holds:

$$\forall n \geq N \qquad V_n(\xi) = z \cdot V_{N-n}(\xi - z).$$

According to the property ($\Diamond\Diamond$), for $\mathbb{P}$-a.s. every ξ such that $\xi - z \geq b$, there exists an integer $k \geq 0$ such that $V_k(\xi - z) \geq b$. But then, $V_{N+k}(\xi) \geq y \cdot b$, and thus $\xi' \wedge x \geq y \cdot b$, contradicting the definition of y. It follows that $y \neq x$ can only occur with probability 0, which was to be proved. $\square$

6 The Cut-Invariant Law of Large Numbers

6.1 Statement of the Law of Large Numbers

We first define ergodic sums and ergodic means associated with an AST and with a cost function. The setting of the section is that of a heap monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$ together with a Bernoulli measure $\mathbb{P}$ on $(\partial\mathcal{M}, \mathfrak{F})$.

If $\varphi : \Sigma \to \mathbb{R}$ is a function, seen as a cost function, it is clear that φ has a unique extension on $\mathcal{M}$ which is additive; we denote this extension by $\langle\varphi, \cdot\rangle$. Hence, if the Σ-word $x_1 \ldots x_n$ is a representative of a heap x, then:

$$\langle\varphi, x\rangle = \varphi(x_1) + \cdots + \varphi(x_n).$$

In particular, if 1 denotes the constant function, equal to 1 on Σ, one has: $\langle 1, x\rangle = |x|$ for every $x \in \mathcal{M}$.

Definition 6.1 Let $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ be an AST, which we assume to be $\mathbb{P}$-a.s. finite, and let $\varphi : \Sigma \to \mathbb{R}$ be a cost function. Consider the sequence of iterated stopping times $(V_n)_{n\geq 0}$ associated with V. The *ergodic sums* associated with V are the random variables in the sequence $(S_{V,n}\varphi)_{n\geq 0}$ defined $\mathbb{P}$-a.s. by:

$$\forall n \geq 0 \qquad S_{V,n}\varphi = \langle\varphi, \xi_{V_n}\rangle.$$

The *ergodic means* associated with V are the random variables in the sequence $(M_{V,n})_{n\geq 1}$ defined $\mathbb{P}$-a.s. by:

$$\forall n \geq 0 \qquad M_{V,n}\varphi = \frac{S_{V,n}\varphi}{S_{V,n}1} = \frac{\langle\varphi, \xi_{V_n}\rangle}{|\xi_{V_n}|}.$$

Theorem 6.2 *Let $\mathcal{M}(\Sigma, I)$ be a heap monoid, equipped with a Bernoulli measure $\mathbb{P}$ on $(\partial\mathcal{M}, \mathfrak{F})$ and with a cost function $\varphi : \Sigma \to \mathbb{R}$.*

Then, for every exhaustive asynchronous stopping time $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ such that $\mathbb{E}|V| < \infty$ holds, the ergodic means $(M_{V,n}\varphi)_{n\geq 1}$ converge $\mathbb{P}$-a.s. toward a constant. Furthermore, this constant does not depend on the choice of the exhaustive AST V such that $\mathbb{E}|V| < \infty$.

Before we proceed with the proof of Theorem 6.2, we state a corollary which provides a practical way of computing the limit of ergodic means.

Corollary 6.3 *Let $\varphi : \Sigma \to \mathbb{R}$ be a cost function. Let π be the invariant measure of the Markov chain of cliques associated with a Bernoulli measure $\mathbb{P}$ on $\partial\mathcal{M}$. Then the limit $M\varphi$ of the ergodic means $M_{V,n}\varphi$, for any exhaustive AST V such that $\mathbb{E}|V| < \infty$ holds, is given by:*

$$M\varphi = \Big(\sum_{\gamma\in\mathfrak{C}} \pi(\gamma)|\gamma|\Big)^{-1} \sum_{\gamma\in\mathfrak{C}} \pi(\gamma)\langle\varphi, \gamma\rangle. \tag{24}$$

Proof According to Theorem 6.2, to compute the value $M\varphi$, we may choose any *AST* of finite length in average. According to Proposition 4.2, the *AST* $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ defined in point 1 of Proposition 4.2 is eligible. Let $(V_n)_{n\geq 1}$ be the associated sequence of iterated stopping times. Then $V_n = C_1 \cdot \dots \cdot C_{K_n}$ for some integer K_n by construction of V, where $(C_k)_{k\geq 1}$ is the Markov chain of cliques associated with infinite heaps. Furthermore, $\lim_{n\to\infty} K_n = \infty$ holds since V is exhaustive. But then the ergodic means are given by:

$$M_{V,n}\varphi = \frac{K_n}{|C_1| + \cdots + |C_{K_n}|} \cdot \frac{\varphi(C_1) + \cdots + \varphi(C_{K_n})}{K_n}.$$

The equality (24) follows then from the law of large numbers [6] for the ergodic Markov chain $(C_k)_{k\geq 1}$. □

6.2 Direct Computation for the Introductory Probabilistic Protocol

We have computed in Sect. 2 the asymptotic density of pieces for the heap monoid $\mathcal{T} = \langle a, b, c \mid ab = ba\rangle$ by computing ergodic means associated either with the first hitting time of c or with the first hitting time of a. The fact that the results coincide can be seen as an instance of Theorem 6.2. Corollary 6.3 provides a direct way of computing the limit density vector, without having to describe an infinite set of heaps as we did in Sect. 2, which would become much less tractable for a general heap monoid. Let us check that we recover the same values for the density vector $\gamma = \big(\gamma_a\ \gamma_b\ \gamma_c\big)$.

We have already obtained in (10) the values of the characteristic numbers of the associated Bernoulli measure: $f(a) = 1 - \lambda$, $f(b) = 1 - \lambda'$, $f(c) = \lambda\lambda'$. Let us use the short notations a, b, c for $f(a), f(b), f(c)$. The Möbius transform is then the following vector, indexed by cliques a, b, c, ab in this order:

$$h = \big(a(1-b)\ b(1-a)\ c\ ab\big)$$

Using the equality $1 - a - b - c + ab = 0$, and according to the results recalled in Sect. 3.8, the transition matrix of the chain of cliques is given by:

$$\begin{aligned} P &= \begin{pmatrix} a & 0 & \frac{c}{1-b} & 0 \\ 0 & b & \frac{c}{1-a} & 0 \\ a(1-b) & b(1-a) & c & ab \\ a(1-b) & b(1-a) & c & ab \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 0 & \lambda & 0 \\ 0 & 1-\lambda' & \lambda' & 0 \\ \lambda'(1-\lambda) & \lambda(1-\lambda') & \lambda\lambda' & (1-\lambda)(1-\lambda') \\ \lambda'(1-\lambda) & \lambda(1-\lambda') & \lambda\lambda' & (1-\lambda)(1-\lambda') \end{pmatrix} \end{aligned}$$

Direct computations give the left invariant probability vector π of P:

$$\begin{pmatrix}\pi_a\\ \pi_b\\ \pi_c\\ \pi_{ab}\end{pmatrix} = \frac{1}{\lambda^2+\lambda'^2-\lambda\lambda'(\lambda+\lambda'-1)}\begin{pmatrix}(1-\lambda)\lambda'^2\\ (1-\lambda')\lambda^2\\ \lambda\lambda'(\lambda+\lambda'-\lambda\lambda')\\ \lambda\lambda'(1-\lambda)(1-\lambda')\end{pmatrix}$$

Using the notion of limit for ergodic means, the density vector defined in Sect. 2 is $\gamma = \begin{pmatrix}\gamma_a & \gamma_b & \gamma_c\end{pmatrix} = \begin{pmatrix}M\mathbf{1}_{\{a\}} & M\mathbf{1}_{\{b\}} & M\mathbf{1}_{\{c\}}\end{pmatrix}$, which yields, according to the result of Corollary 6.3:

$$\begin{pmatrix}\gamma_a\\ \gamma_b\\ \gamma_c\end{pmatrix} = \frac{1}{\pi_a+\pi_b+\pi_c+2\pi_{ab}}\begin{pmatrix}\pi_a+\pi_{ab}\\ \pi_b+\pi_{ab}\\ \pi_c\end{pmatrix} = \frac{1}{\lambda+\lambda'-\lambda\lambda'}\begin{pmatrix}\lambda'(1-\lambda)\\ \lambda(1-\lambda')\\ \lambda\lambda'\end{pmatrix}$$

As expected, we recover the values found in Sect. 2.

6.3 Proof of Theorem 6.2

The proof is divided into two parts, each one gathered in a subsection: first, the proof of convergence of the ergodic means (Sect. 6.3.1); and second, the proof that the limit does not depend on the choice of the *AST* $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$, provided that $\mathbb{E}|V| < \infty$ holds (Sect. 6.3.2).

6.3.1 Convergence of Ergodic Means

Using the notations introduced in Theorem 6.2, let $(\Delta_n)_{n\geq 1}$ be the sequence of increments associated with the sequence $(V_n)_{n\geq 1}$. The increments are defined as in Proposition 5.3. Then we have:

$$M_{V,n}\varphi = \frac{\langle\varphi, \Delta_1\cdot\cdots\cdot\Delta_n\rangle}{\langle 1, \Delta_1\cdot\cdots\cdot\Delta_n\rangle} = \frac{n}{\langle 1, \Delta_1\rangle+\cdots+\langle 1, \Delta_n\rangle}\cdot\frac{\langle\varphi, \Delta_1\rangle+\cdots+\langle\varphi, \Delta_n\rangle}{n}$$

Let $M = \max|\varphi|$. Then the assumption $\mathbb{E}|\xi_V| < \infty$ implies:

$$\mathbb{E}|\langle\varphi, \xi_V\rangle| \leq M\mathbb{E}|\xi_V| < \infty.$$

Since $(\Delta_n)_{n\geq 1}$ is *i.i.d.* according to Proposition 5.3, each Δ_i being distributed according to ξ_V, the strong law of large numbers for *i.i.d.* sequences implies the $\mathbb{P}$-a.s. convergence:

$$\lim_{n\to\infty}\frac{\langle\varphi, \Delta_1\rangle+\cdots+\langle\varphi, \Delta_n\rangle}{n} = \mathbb{E}\langle\varphi, \xi_V\rangle, \quad \lim_{n\to\infty}\frac{\langle 1, \Delta_1\rangle+\cdots+\langle 1, \Delta_n\rangle}{n} = \mathbb{E}|\xi_V|. \tag{25}$$

It follows in particular from V being exhaustive that $\mathbb{E}|\xi_V| > 0$; otherwise, we would have $\xi_V = 0$, $\mathbb{P}$-a.s., and thus $\xi_{V_n} = 0$, $\mathbb{P}$-a.s. and for all $n \geq 0$, contradicting the $\mathbb{P}$-a.s. equality $\bigvee_{n\geq 0} \xi_{V_n} = \xi$ stated in Lemma 5.4. Hence, from (25), we deduce the $\mathbb{P}$-a.s. convergence:

$$\lim_{n\to\infty} M_{V,n}\varphi = \frac{\mathbb{E}\langle \varphi, \xi_V\rangle}{\mathbb{E}|\xi_V|}.$$

6.3.2 Uniqueness of the Limit

We start with a couple of lemmas.

Lemma 6.4 *Let $f : \mathcal{M} \to \mathbb{R}$ be the valuation defined by $f(x) = \mathbb{P}(\uparrow x)$ for all $x \in \mathcal{M}$, and let $B = (B_{\gamma,\gamma'})_{(\gamma,\gamma')\in\mathfrak{C}\times\mathfrak{C}}$ be the non-negative matrix defined by:*

$$\forall(\gamma,\gamma') \in \mathfrak{C}\times\mathfrak{C} \qquad B_{\gamma,\gamma'} = \begin{cases} 0, & \textit{if } \ \neg(\gamma \to \gamma'),\\ f(\gamma'), & \textit{if } \ \gamma \to \gamma'. \end{cases}$$

Then B has spectral radius 1.

Proof This lemma is shown in [1], we reproduce the argument below. First, we observe that the non-negative matrix B is primitive. Indeed, it is irreducible since the graph of non-empty cliques $(\mathfrak{C}, \to)$ is strongly connected according to [14, Lemma 3.2]. And it is aperiodic since $c \to c$ holds for any clique $c \in \mathfrak{C}$.

Let $h : \mathscr{C} \to \mathbb{R}$ be the Möbius transform defined in (12), and let $g = (g(\gamma))_{\gamma\in\mathfrak{C}}$ be the normalization vector defined in (15). The following identity is proved in [2, Prop. 10.3] to hold for all $\gamma \in \mathfrak{C}$: $h(\gamma) = f(\gamma)g(\gamma)$. It implies:

$$(Bg)_\gamma = \sum_{\gamma'\in\mathfrak{C}\,:\,\gamma\to\gamma'} f(\gamma')g(\gamma') = \sum_{\gamma'\in\mathfrak{C}\,:\,\gamma\to\gamma'} h(\gamma') = g(\gamma).$$

Hence, g is B-invariant on the right. Since $h > 0$ on $\mathfrak{C}$, the vector g is positive. Therefore, g is a right Perron eigenvector of B which implies the statement. □

Lemma 6.5 *Let $a \in \Sigma$ be a piece, and let $\mathcal{M}'_a$ be the sub-monoid of $\mathcal{M}$ consisting of heaps with no occurrence of a. Then:*

$$\sum_{x\in\mathcal{M}'_a} \mathbb{P}(\uparrow x) < \infty, \qquad \sum_{x\in\mathcal{M}'_a} |x|\,\mathbb{P}(\uparrow x) < \infty, \qquad \sum_{x\in\mathcal{M}'_a} |x|^2\,\mathbb{P}(\uparrow x) < \infty.$$

Proof Of course, it is enough to prove the last one of the three inequalities. Let $\|\cdot\|$ denote the spectral radius of a non-negative matrix. Let B the matrix defined as in Lemma 6.4, and let B_a be the matrix obtained by replacing in B all entries (γ,γ') by 0 as long as γ or γ' contains an occurrence of a. Then, the non-negative matrices B and B_a satisfy $B_a \leq B$ and $B_a \neq B$. Since B is primitive, and since $\|B\| = 1$ by Lemma 6.4, it follows from Perron–Frobenius theory [19, Chapter 1] that $\|B_a\| < 1$.

The result follows now by decomposing heaps $x \in \mathcal{M}'_a$ according to their height $\tau(x)$ on the one hand, and by observing the obvious estimation $|x| \leq \alpha\tau(x)$ for all $x \in \mathcal{M}$ on the other hand, where α is the maximal size of cliques. Considering the vector $I = (I_\gamma)_{\gamma\in\mathfrak{C}}$ with $I_\gamma = 1$ for all $\gamma \in \mathfrak{C}$, and its transpose I^t, this yields:

$$\sum_{x\in\mathcal{M}'_a} |x|^2\mathbb{P}(\uparrow x) \leq \alpha^2 I^t \Big(\sum_{k\geq 0} k^2 B_a^k\Big) I < \infty$$

since $\|B_a\| < 1$. This completes the proof. □

Lemma 6.6 *Let $a \in \Sigma$ be a piece. Let V be the first hitting time of a, and let $(V_k)_{k\geq 0}$ be the associated sequence of iterated stopping times. Fix $x \neq 0$ a heap, and let $J_x : \partial\mathcal{M} \to \mathbb{N} \cup \{\infty\}$ be the random variable defined by:*

$$J_x(\xi) = \inf\{k \geq 0 \ : \ V_k(\xi) \geq x\}.$$

Then, the random variable $U_x : \partial\mathcal{M} \to \overline{\mathcal{M}}$ defined by:

$$U_x(\xi) = \begin{cases} V_{J_x(\xi)}(\xi), & \text{if } J_x(\xi) < \infty, \\ \xi, & \text{if } J_x(\xi) = \infty, \end{cases}$$

is an AST*, and there exists a constant $C \geq 0$, independent of x, such that:*

$$\mathbb{E}\big((|U_x| - |x|)^2 \bigm| \uparrow x\big) \leq C. \tag{26}$$

Proof The fact that U_x is an AST is an easy consequence of the V_k's being AST (Proposition 5.1). Since V is exhaustive by Proposition 5.6, in particular $J_x(\xi) < \infty$ for $\mathbb{P}$-a.s. every $\xi \in \uparrow x$. Henceforth, the conditional expectation in (26) is computed as the following sum:

$$\mathbb{E}\big((|U_x| - |x|)^2 \bigm| \uparrow x\big) = \frac{1}{\mathbb{P}}(\uparrow x) \sum_{y\in\mathcal{U}_x} (|y| - |x|)^2\mathbb{P}\big(\{U_x = y\} \cap \uparrow x\big),$$

where $\mathcal{U}_x$ denotes the set of finite values assumed by U_x. Since U_x is an AST, we have for all $y \in \mathcal{U}_x$:

$$\begin{aligned} &\{U_x = y\} = \uparrow y, \\ &\frac{1}{\mathbb{P}(\uparrow x)}\mathbb{P}\big(\{U_x = y\} \cap \uparrow x\big) = \frac{\mathbb{P}\big(\uparrow (y \vee x)\big)}{\mathbb{P}(\uparrow x)} = \mathbb{P}\big(\uparrow (y - x)\big), \end{aligned}$$

the later equality since $x \leq y$ and by the multiplicativity property of $\mathbb{P}$. Therefore:

$$\mathbb{E}\big((|U_x| - |x|)^2 \bigm| \uparrow x\big) = \sum_{y\in\mathcal{U}_x} |y - x|^2\mathbb{P}\big(\uparrow (y - x)\big) \tag{27}$$

Recall that each heap can be seen itself as a partially ordered labeled set [23], where elements are labeled by Σ. Assume first that x contains a unique maximal piece, say $b \in \Sigma$. Such a heap is called *pyramidal*. Then for each $y \in \mathcal{U}_x$, the heap $z = y - x$ has the following shape, for some integer $k \geq 0$: $z = \delta_1 \cdot \dots \cdot \delta_{k-1} \cdot \delta_k$, where the δ_i's for $i \in \{1, \dots, k-1\}$ result from the action of the hitting time V prior to $V_k \geq x$. In particular, the $k-1$ first heaps δ_i do not have any occurrence of b; whereas δ_k writes as $\delta_k = u \cdot a$ for some heap u with no occurrence of a. Denoting by $\mathcal{M}'_a$ and $\mathcal{M}'_b$, respectively, the sub-monoids of $\mathcal{M}$ of heaps with no occurrence of a and of b, we have thus $z = v \cdot u \cdot a$, for some $v \in \mathcal{M}'_b$ and $u \in \mathcal{M}'_a$. Hence, from (27), we deduce:

$$\mathbb{E}\big((|U_x| - |x|)^2 \bigm| \uparrow x\big) \leq \sum_{\substack{u \in \mathcal{M}'_a \\ v \in \mathcal{M}'_b}} (|u| + |v| + 1)^2 \mathbb{P}(\uparrow u) \cdot \mathbb{P}(\uparrow v) \cdot \mathbb{P}(\uparrow a).$$

Since a and b range over a finite set, it follows from Lemma 6.5 that the sum above in the right member is bounded by a constant. The result (26) follows.

We have proved the result if x is pyramidal. The general case follows since every heap x writes as an upper bound $x = x_1 \vee \dots \vee x_n$ of at most α pyramidal heaps, with α the maximal size of cliques. $\square$

Lemma 6.7 *Let W be an* AST *such that $\mathbb{E}|W| < \infty$. Let $a \in \Sigma$ be a piece. Let V be the first hitting time of a, and let $(V_k)_{k \geq 0}$ be the associated sequence of iterated stopping times. Let $K : \partial\mathcal{M} \to \mathbb{N} \cup \{\infty\}$ be the random integer defined by:*

$$K(\xi) = \inf\{k \geq 0 \ : \ V_k(\xi) \geq W(\xi)\}.$$

Then the mapping $U : \partial\mathcal{M} \to \overline{\mathcal{M}}$ defined by:

$$U(\xi) = \begin{cases} V_{K(\xi)}(\xi), & \text{if } K(\xi) < \infty, \\ \xi, & \text{if } K(\xi) = \infty \end{cases}$$

is an AST, *and there is a constant $C \geq 0$ such that:*

$$\mathbb{E}\big((|U| - |W|)^2\big) \leq C.$$

Proof Let $\mathcal{W}$ denote the set of finite values assumed by W. Since $\mathbb{E}|W| < \infty$, in particular $W < \infty$ $\mathbb{P}$-almost surely, and therefore:

$$\begin{aligned} \mathbb{E}\big((|U| - |W|)^2\big) &= \sum_{w \in \mathcal{W}} \mathbb{P}(\uparrow w)\mathbb{E}\big((|U| - |w|)^2 \bigm| \uparrow w\big) \\ &= \sum_{w \in \mathcal{W}} \mathbb{P}(\uparrow w)\mathbb{E}\big((|U_w| - |w|)^2 \bigm| \uparrow w\big) \qquad \text{with the notation } U_x \text{ of Lemma 6.6} \\ &\leq \sum_{w \in \mathcal{W}} \mathbb{P}(\uparrow w) C \qquad \text{with the constant } C \text{ from Lemma 6.6} \\ &\leq C \end{aligned}$$

The latter inequality follows from the fact that the visual cylinders $\uparrow w$, for w ranging over $\mathcal{W}$, are pairwise disjoint since W takes different values on different such visual cylinders. The proof of Lemma 6.7 is complete. □

Finally, we will use the following elementary analytic result.

Lemma 6.8 *Let $(X_k)_{k\geq 1}$ be a sequence of real random variables defined on some common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and such that $\mathbb{E}|X_k|^2 \leq C < \infty$ for some constant C. Then $\lim_{k\to\infty} X_k/k = 0$ holds $\mathbb{P}$-almost surely.*

Proof Let $Y_k = X_k/k$. To prove the $\mathbb{P}$-a.s. limit $Y_k \to 0$, we use the following well-known sufficient criterion:

$$\forall \epsilon > 0 \qquad \sum_{k\geq 1} \mathbb{P}(|Y_k| > \epsilon) < \infty.$$

Applying Markov inequality yields:

$$\sum_{k\geq 1} \mathbb{P}(|Y_k| > \epsilon) = \sum_{k\geq 1} \mathbb{P}(|X_k|^2 > k^2\epsilon^2) \leq \frac{1}{\epsilon^2} \sum_{k\geq 1} \frac{C}{k^2} < \infty,$$

which shows the result. □

We now proceed with the proof of uniqueness of the limit in Theorem 6.2. The setting is the following. Let W be an exhaustive *AST* such that $\mathbb{E}|W| < \infty$, let $(W_n)_{n\geq 0}$ be the associated sequence of iterated stopping times. By the first part of the proof (Sect. 6.3.1), we know that the ergodic means $M_{W,n}\varphi$ converge $\mathbb{P}$-a.s. toward a constant, say $M(W, \varphi)$.

Pick $a \in \Sigma$ a piece, and let V be the first hitting time of a. Let $(V_n)_{n\geq 0}$ be the associated sequence of iterated stopping times, and let $M(V, \varphi)$ be the limit of the associated ergodic means $M_{V,n}\varphi$. We shall prove that $M(W, \varphi) = M(V, \varphi)$. This will conclude the proof of Theorem 6.2.

We consider for each integer $j \geq 0$ the following random integer $K_j : \partial\mathcal{M} \to \mathbb{N} \cup \{\infty\}$:

$$K_j(\xi) = \inf\big\{k \geq 0 \ : \ V_k(\xi) \geq W_j(\xi)\big\},$$

and the *AST* $V_j' : \partial\mathcal{M} \to \overline{\mathcal{M}}$ defined by:

$$V_j'(\xi) = \begin{cases} V_{K_j(\xi)}(\xi), & \text{if} \quad K_j < \infty, \\ \xi, & \text{if} \quad K_j = \infty. \end{cases}$$

Since $W_j \leq V_j'$ by construction, we put $\Delta_j = V_j' - W_j$, so that $V_j' = W_j \cdot \Delta_j$. Then, by Lemma 6.7, there is a constant $C \geq 0$ such that:

$$\forall j \geq 0 \qquad \mathbb{E}|\Delta_j|^2 \leq C.$$

Hence, applying Lemma 6.8 with $X_j = |\Delta_j|$, and since $|\langle\varphi, \Delta_j\rangle| \leq M|\Delta_j|$ if $M = \max\{|\varphi(x)| \ : \ x \in \Sigma\}$, we have:

$$\mathbb{P}\text{-a.s.} \quad \lim_{j\to\infty} \frac{|\Delta_j|}{j} = 0, \qquad\qquad \mathbb{P}\text{-a.s.} \quad \lim_{j\to\infty} \frac{\langle\varphi, \Delta_j\rangle}{j} = 0. \tag{28}$$

We also have, according to the result of Sect. 6.3.1:

$$\mathbb{P}\text{-a.s.} \quad \lim_{j\to\infty} \frac{|W_j|}{j} = \mathbb{E}|W| > 0, \qquad \mathbb{P}\text{-a.s.} \quad \lim_{j\to\infty} \frac{\langle\varphi, W_j\rangle}{|W_j|} = M(W, \varphi). \tag{29}$$

The ergodic means can be compared as follows:

$$\begin{aligned} M_{V',j}\varphi - M_{W,j}\varphi &= \frac{\langle\varphi, W_j\rangle + \langle\varphi, \Delta_j\rangle}{|W_j| + |\Delta_j|} - \frac{\langle\varphi, W_j\rangle}{|W_j|} \\ &= \frac{1}{|W_j| + |\Delta_j|}\langle\varphi, \Delta_j\rangle - \frac{|\Delta_j|}{|W_j| + |\Delta_j|} \cdot \frac{\langle\varphi, W_j\rangle}{|W_j|} \end{aligned}$$

Using (28) (29), both terms in the right member above go to 0, and therefore, $M(V', \varphi) = M(W, \varphi)$.

But, since $\lim_{j\to\infty} K_j = \infty$, we clearly have $M(V', \varphi) = M(V, \varphi)$, and thus finally: $M(W, \varphi) = M(V, \varphi)$, which was to be shown. The proof of Theorem 6.2 is complete.

7 A Cut-Invariant Law of Large Numbers for Sub-Additive Functions

In Sect 6, we have obtained a strong law of large numbers relative to functions of the kind $\langle\varphi, \cdot\rangle : \mathcal{M} \to \mathbb{R}$, which are additive by construction—and any additive function on $\mathcal{M}$ is of this form.

Interesting asymptotic quantities, however, are not always of this form. For instance, the ratio between the *length* and the *height* of heaps, $|x|/\tau(x)$, has been introduced in [14, 18] as a measure of the *speedup* in the execution of asynchronous processes.

The height function is sub-additive on $\mathcal{M}$: $\tau(x \cdot y) \leq \tau(x) + \tau(y)$. This constitutes a motivation for extending the strong law of large numbers to sub-additive functions. We shall return to the computation of the speedup in Sect. 7.2, after having established a convergence result for ergodic ratios with respect to sub-additive functions (Theorem 7.1).

7.1 Statement of the Law of Large Numbers for Sub-additive Functions

As for additive functions, we face the following issues: (1) define proper ergodic ratios with respect to a given AST; (2) prove the almost sure convergence of these ratios; (3) study the uniqueness of the limit when the AST varies.

We restrict the proof of uniqueness to *first hitting times* only.

Theorem 7.1 *Let a heap monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be equipped with a Bernoulli measure $\mathbb{P}$, and let $\varphi : \mathcal{M} \to \mathbb{R}$ be a sub-additive function, that is to say, φ satisfies $\varphi(x \cdot y) \le \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{M}$. We assume furthermore that φ is non-negative on $\mathcal{M}$.*

Let $a \in \Sigma$ be a piece of the monoid, and let $(V_n)_{n\ge 0}$ be the sequence of iterated stopping times associated with the first hitting time of a. Then the ratios $\varphi(V_n)/|V_n|$ converge $\mathbb{P}$-a.s. as $n \to \infty$, toward a constant which is independent of the chosen piece a.

We gather into two separate subsections the proof of convergence (Sect. 7.1.1), and the proof that the limit is independent of the chosen piece (Sect. 7.1.2).

7.1.1 Proof of Convergence

The proof is based on Kingman sub-additive Ergodic Theorem, of which we shall use the following formulation [21]: let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, let $T : \Omega \to \Omega$ be a measure preserving and ergodic transformation, and let $(g_n)_{n\ge 1}$ be a sequence of integrable real-valued functions satisfying $g_{n+m} \le g_n + g_m \circ T^n$ for all integers $n, m \ge 1$. Then, g_n/n converge $\mathbb{P}$-a.s. toward a constant $g \ge -\infty$.

Lemma 7.2 *If $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$ is an exhaustive AST, then the shift operator $\theta_V : \partial\mathcal{M} \to \partial\mathcal{M}$ which is $\mathbb{P}$-a.s. defined on $\partial\mathcal{M}$, is measure preserving and ergodic.*

Proof To prove that θ_V is $\mathbb{P}$-invariant, it is enough to show $\mathbb{P}(\theta_V^{-1}(\uparrow x)) = \mathbb{P}(\uparrow x)$ for all heaps $x \in \mathcal{M}$. Let $x \in \mathcal{M}$. The equality $\xi = \xi_V \cdot \theta_V(\xi)$ holds $\mathbb{P}$-a.s. since $V < \infty$ $\mathbb{P}$-almost surely. Therefore, denoting by $\mathcal{V}$ the set of finite values assumed by V, one has:

$$\mathbb{P}\text{-a.s.}\quad \theta_V^{-1}(\uparrow x) = \bigcup_{v\in\mathcal{V}} \uparrow (v \cdot x).$$

The visual cylinders $\uparrow v$, for v ranging over $\mathcal{V}$, are pairwise disjoint, since V assumes distinct values on each of them. Hence, passing to the probabilities and using the Bernoulli property:

$$\mathbb{P}\big(\theta_V^{-1}(\uparrow x)\big) = \mathbb{P}(\uparrow x) \sum_{v\in\mathcal{V}} \mathbb{P}(\uparrow v) = \mathbb{P}(\uparrow x)\mathbb{P}(|V| < \infty) = \mathbb{P}(\uparrow x).$$

This proves that θ_V is $\mathbb{P}$-invariant.

We now show the ergodicity of θ_V. Let $f : \partial\mathcal{M} \to \mathbb{R}$ be a bounded measurable and θ_V-invariant function. Since V is exhaustive, $\mathfrak{F} = \bigvee_{n\ge 1} \mathfrak{F}_{V_n}$ by Lemma 5.4. Hence, by the Martingale convergence theorem [3, Th. 35.6]:

$$f = \lim_{n\to\infty} \mathbb{E}(f|\mathfrak{F}_{V_n}) \quad \mathbb{P}\text{-a.s.} \tag{30}$$

Since $V_n \in \mathcal{M}$ with probability 1, the strong Bernoulli property (Theorem 4.5) implies:

$$\mathbb{E}(f \circ \theta_{V_n} | \mathfrak{F}_{V_n}) = \mathbb{E}(f) \quad \mathbb{P}\text{-a.s.}$$

But, since f is assumed to be θ_V-invariant, and noting that $\theta_{V_n} = (\theta_V)^n$ by construction, the above writes as: $\mathbb{E}(f|\mathfrak{F}_{V_n}) = \mathbb{E}(f)$, which yields $f = \mathbb{E}(f)$ by (30), proving the ergodicity of θ_V. □

We now prove the following result, which is slightly strongest than the convergence part in the statement of Theorem 7.1:

(†) *For every exhaustive* AST $V : \partial\mathcal{M} \to \overline{\mathcal{M}}$, *if* $(V_n)_{n\geq 1}$ *is the sequence of iterated stopping times associated with* V, *the sequence* $\varphi(V_n)/|V_n|$ *is* $\mathbb{P}$*-a.s. convergent, toward a constant.*

Since, by Proposition 5.6, first hitting times are exhaustive, this statement implies indeed the convergence statement in Theorem 7.1.

For the proof of (†), let $g_n = \varphi(V_n)$ for $n \geq 0$. An easy induction shows that for any integers $n, m \geq 0$, one has:

$$V_{n+m} = V_n \cdot (V_m \circ \theta_{V_n}), \qquad \theta_{V_n} = (\theta_V)^n,$$

and thus by sub-additivity of φ: $g_{n+m} \leq g_n + g_m \circ (\theta_V)^n$.

The application of Kingman sub-additive Ergodic Theorem recalled above is permitted by the measure-preserving property and the ergodicity of θ_V proved in Lemma 7.2. It implies the $\mathbb{P}$-a.s. convergence of $g_n/n = \varphi(V_n)/n$ toward a constant. Since $\lim_{n\to\infty} |V_n|/n = \mathbb{E}|V|$ with probability 1 by Theorem 6.2, we deduce the $\mathbb{P}$-a.s. convergence of the ratios $\varphi(V_n)/|V_n|$ as $n \to \infty$ toward a constant, which proves (†).

7.1.2 Proof of Uniqueness

To complete the proof of Theorem 7.1, it remains only to show that the limit of the ratios $\varphi(V_n)/|V_n|$ is independent of the AST V, that is to say, of the piece for which V is the first hitting time. For this, we first show the following result:

(‡) *Let* $\varphi : \mathcal{M} \to \mathbb{R}$ *be a sub-additive and non-negative function. Let* $W : \partial\mathcal{M} \to \overline{\mathcal{M}}$ *be an* AST *such that* $\mathbb{E}|W| < \infty$, *let* $(W_n)_{n\geq 0}$ *be the associated sequence of iterated stopping times, and let* MW *be the* $\mathbb{P}$*-a.s. limit of* $\varphi(W_n)/|W_n|$. *Let also* V *be the first hitting time of some piece* a, *let* $(V_n)_{n\geq 0}$ *be the associated sequence of iterated stopping times, and let* MV *be the* $\mathbb{P}$*-a.s. limit of* $\varphi(V_n)/|V_n|$. *Then* $MV \leq MW$.

For the proof of (‡), we follow the same line of proof as for the uniqueness in the proof of Theorem 6.2 (Sect. 6.3.2). Using the very same notations for V'_n and Δ_n, we have $V'_n = W_n \cdot \Delta_n$, and thus:

$$\frac{\varphi(V_n')}{|V_n'|} - \frac{\varphi(W_n)}{|W_n|} = \underbrace{\frac{\varphi(W_n \cdot \Delta_n) - \varphi(W_n)}{|W_n| + |\Delta_n|}}_{A_n} - \underbrace{\frac{|\Delta_n|\varphi(W_n)}{|W_n|(|W_n| + |\Delta_n|)}}_{B_n}$$

The sub-additivity of φ and the existence of the CF decomposition of heaps shows that $\varphi(x) \leq C_1 x$ for all $x \in \mathcal{M}$, and for some real constant C_1. Therefore, using again the sub-additivity of φ, we obtain:

$$A_n \leq C_1 \frac{|\Delta_n|}{|W_n| + |\Delta_n|}, \qquad \text{and thus :} \quad \limsup_{n\to\infty} A_n \leq 0.$$

The ratios $\varphi(W_n)/|W_n|$ being bounded since they have a finite limit, it is clear that the terms B_n converge to 0. We deduce:

$$\limsup_{n\to\infty} \left(\frac{\varphi(V_n')}{|V_n'|} - \frac{\varphi(W_n)}{|W_n|} \right) \leq 0.$$

But the ratios $\varphi(V_n)/|V_n|$ also have a limit, and clearly $\lim \varphi(V_n')/|V_n'| = \lim \varphi(V_n)/|V_n|$. Hence we obtain:

$$\lim_{n\to\infty} \frac{\varphi(V_n)}{|V_n|} \leq \lim_{n\to\infty} \frac{\varphi(W_n)}{|W_n|},$$

which proves (‡).

It is now clear that if both V and W are first hitting times, then $MV = MW$ since $MV \leq MW$ and $MW \leq MV$ by applying (‡) twice. This completes the proof of Theorem 7.1.

7.2 Computing the Speedup

Let us define the *speedup* of the pair $(\mathcal{M}, \mathbb{P})$, where $\mathbb{P}$ is a Bernoulli measure on the boundary $\partial\mathcal{M}$ of a heap monoid $\mathcal{M}$, as the $\mathbb{P}$-a.s. limit of the inverse of the ergodic ratios:

$$\mathbb{P}\text{-a.s.} \quad \rho = \lim_{n\to\infty} \frac{|V_n|}{\tau(V_n)},$$

where V is the first hitting time associated with some piece of the monoid. The greater the speedup, the more the parallelism is exploited.

Based on generating series techniques, the authors of [14] obtain an expression for a similar quantity for the particular case of uniform measures. With Bernoulli measure, we obtain a more intuitive formula, easier to manipulate for algorithmic approximation purposes.

Proposition 7.3 *The speedup is given by:*

$$\rho = \sum_{c \in \mathfrak{C}} \pi(\gamma)|\gamma|, \tag{31}$$

where π is the invariant measure of the Markov chain of cliques under the probability measure $\mathbb{P}$.

Proof Let W be the *AST* defined in point 1 of Proposition 4.2. Then W is exhaustive and satisfies $\mathbb{E}|W| < \infty$ according to Proposition 5.6. Let $(W_n)_{n\geq 0}$ be the associated sequence of iterated stopping times. Then, since the height $\tau(\cdot)$ is sub-additive, it follows from (†) in Sect. 7.1.1 that the ratios $\tau(W_n)/|W_n|$ converge $\mathbb{P}$-a.s. toward a constant, say MW. Furthermore, according to (‡) in Sect. 7.1.2, $\rho^{-1} \leq MW$. Hence, to complete the proof of the proposition, it is enough to show the following two points:

1. $MW = \left(\sum_{\gamma\in\mathfrak{C}} \pi(\gamma)|\gamma|\right)^{-1}$.
2. $MW \leq \rho^{-1}$.

Proof of point 1. For $\xi \in \partial\mathcal{M}$ an infinite heap given by $\xi = (\gamma_i)_{i\geq 1}$, let $Y_n \in \mathcal{M}$ be defined for each integer $n \geq 0$ by $Y_n = \gamma_1 \cdot \cdots \cdot \gamma_n$. For each integer $n \geq 0$, there is an integer K_n such that $W_n = Y_{K_n}$, and $\lim_{n\to\infty} K_n = \infty$. Therefore:

$$MW = \lim_{n\to\infty} \frac{\tau(Y_{K_n})}{|Y_{K_n}|}. \tag{32}$$

But we have $\tau(Y_j) = j$ for each integer $j \geq 1$. Therefore, applying the strong law of large numbers [6] to the ergodic Markov chain $(C_n)_{n\geq 1}$, we get:

$$\frac{\tau(Y_j)}{|Y_j|} = \frac{j}{|Y_j|} = \frac{j}{|C_1| + \cdots + |C_j|} \to_{j\to\infty} \Big(\sum_{\gamma\in\mathfrak{C}} \pi(\gamma)|\gamma|\Big)^{-1}. \tag{33}$$

Point 1 results from (32) and (33).

Proof of point 2. For each integer $n \geq 0$, let $\tau_n = \tau(V_n)$. Then, the heap Y_{τ_n} has same height as V_n and has no lesser length. Therefore, the ratios satisfy:

$$\frac{\tau(Y_{\tau_n})}{|Y_{\tau_n}|} = \frac{\tau(V_n)}{|Y_{\tau_n}|} \leq \frac{\tau(V_n)}{|V_n|}.$$

Passing to the limit, we obtain $MW \leq \rho^{-1}$, completing the proof. □

For the example monoid $\mathcal{T} = \langle a, b, c \mid ab = ba\rangle$ equipped with the uniform measure $\mathbb{P}$ given by $\mathbb{P}(\uparrow x) = p^{|x|}$ with $p = (3 - \sqrt{5})/2$, the computation goes as follows. Referring to the computations already performed in Sect. 6.2, the invariant measure π is:

$$\pi = \frac{1}{2p+1} \begin{pmatrix} p \\ p \\ -3p+2 \\ 3p-1 \end{pmatrix} \begin{matrix} a \\ b \\ c \\ ab \end{matrix}$$

According to Proposition 7.3, the speedup is:

$$\rho = \pi_a + \pi_b + \pi_c + 2\pi_{ab} = \frac{5p}{2p+1} = \frac{5(7-\sqrt{5})}{22} \approx 1.0827\cdots$$

Our method allows for robust algorithmic approximation of the speedup, through the following steps: 1. Approximating the root of the Möbius polynomial; 2. Determining the invariant measure of the matrix (16); 3. Computing the speedup through formula (31).

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