Normalisation-by-evaluation for λ-calculi

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Idea of NbE

- computational behaviour of λ-calculi traditionally studied through reduction seen as a rewrite system
- alternative: *evaluate* a λ-term in a constructive meta-language and *reify*-back the result into the object-language

$$M \longmapsto \|M\| \longmapsto M'$$

where $M =_{\beta\eta} M'$

Advantages of NbE

- proof of Church-Rosser not necessary
- easier to deal with η-equality when not considering it as reduction

NbE for Simply Typed λ-Calculus

(Sketch on whiteboard)

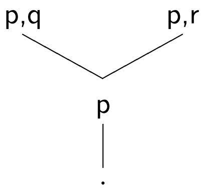
Connections to Concepts from Intuitionistic Logic

Making the previous construction more precise:

- **Evaluation** is **Soundness**: given $\Gamma \vdash t : A$, in every world w of every Kripke model, $w \Vdash \Gamma$ implies $w \Vdash A$
- **Reification** is **Completeness**: if in every world w of every Kripke model, $w \Vdash \Gamma$ implies $w \Vdash A$, then there exists a derivation $\Gamma \vdash t : A$

NbE $=$ Completeness $\circ$ Soundness

Kripke Semantics (Possible Worlds Semantics)



- a possible world is determined by atomic formulas known to hold so far
- at any later world we enrich our knowledge

Kripke Semantics: Formal Definition

A *Kripke model* is given by:

- a partial order (K,$\leq$) of **worlds**
- a domain of constants $D : K \to Set$, monotone
- a relation "$\Vdash$" between worlds and *atomic* formulas, called **forcing**

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Almost Tarski's truth definition, $\to$ (and $\forall$) standing out.

Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

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2. $\neg\neg p \to p$
3. 2 implies 1
4. $(p \to q) \to (\neg p \vee q)$

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4. $(p \rightarrow q) \rightarrow (\neg p \vee q)$
5. $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$

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4. $(p \rightarrow q) \rightarrow (\neg p \vee q)$
5. $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$
6. $\neg \forall x \neg P(x) \rightarrow \exists x P(x)$

Instances of NbE

- Simply typed lambda calculus (Berger-Schwichtenberg 1991)
- Untyped lambda calculus (Filinski-Rohde, 2004)
- Martin-Löf type theory (Abel-Coquand-Dybjer, 2007)
- ...

Part II: $\bar{\lambda}\mu\tilde{\mu}$ and its Kripke Semantics

joint work with Hugo Herbelin and Gyesik Lee (ROSAEC, Korea)

Proof Systems For Classical Logic

- call/cc and Pierce's Law (Griffin, 1990)

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- $\bar{\lambda}\mu\tilde{\mu} - calculus$ (Curien-Herbelin, 2000)

$\bar{\lambda}\mu\tilde{\mu}$: Syntax and Reduction Rules

Syntax 3 categories: **commands**, **terms** and **evaluation contexts**

$$
\begin{aligned}
c &::= \langle t \| e \rangle \\
t &::= x \mid \mu\alpha.c \mid \lambda x.t \\
e &::= \alpha \mid \tilde{\mu}x.c \mid t \cdot e
\end{aligned}
$$

Reduction

$$
\begin{aligned}
(\mu)\quad & \langle \mu\alpha.c \| e \rangle \to c[e/\alpha] \\
(\tilde{\mu})\quad & \langle t \| \tilde{\mu}x.c \rangle \to c[t/x] \\
(\beta)\quad & \langle \lambda x.t \| t' \cdot e \rangle \to \langle t' \| \tilde{\mu}x.\langle t \| e \rangle \rangle
\end{aligned}
$$

Critical pair $\langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle$: CBN, CBV strategies

The Sequent Calculus $\mathrm{LK}_{\mu\tilde{\mu}}$

$$\frac{\Gamma \vdash v : A|\Delta \qquad \Gamma|e : A \vdash \Delta}{\langle v\|e\rangle : (\Gamma \vdash \Delta)}\ (\mathrm{Cut})$$

$$\frac{}{(x : A), \Gamma \vdash x : A|\Delta}\ (\mathrm{Ax}_R) \qquad \frac{}{\Gamma|\alpha : A \vdash (\alpha : A), \Delta}\ (\mathrm{Ax}_L)$$

$$\frac{c : (\Gamma \vdash (\alpha : A), \Delta)}{\Gamma \vdash \mu\alpha.c : A|\Delta}\ (\mu) \qquad \frac{c : ((x : A), \Gamma \vdash \Delta)}{\Gamma|\tilde{\mu}x.c : A \vdash \Delta}\ (\tilde{\mu})$$

$$\frac{(x : A), \Gamma \vdash (t : B)|\Delta}{\Gamma \vdash \lambda x.t : A \to B|\Delta}\ (\to_R) \qquad \frac{\Gamma \vdash t : A|\Delta \qquad \Gamma|e : B \vdash \Delta}{\Gamma|t \cdot e : A \to B \vdash \Delta}\ (\to_L)$$

Kripke Semantics for $LK_{\mu\tilde{\mu}}$: Motivation

2 "new" ingredients for the model cake:

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"refinement" of $\Vdash$

- identify "strongly refutes" as primitive, define "forcing" and "refutation" by orthogonality

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Than we define by orthogonality:

forcing $w :\Vdash A$ iff for all $w' \geq w$, $w' : A \Vdash_s$ implies $w' :\Vdash_\perp$

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$\top$	never

Resemblance to Intuitionistic Forcing

$$w :\Vdash A \to B \iff \text{ for all } w' \geq w, w' :\Vdash A \Rightarrow w' :\Vdash B \tag{1}$$

$$w :\Vdash \forall x.A(x) \iff \text{ for all } w' \geq w \text{ and } t \in D(w'), w' :\Vdash A(t) \tag{2}$$

$$w :\Vdash \bot \iff w :\Vdash_{\bot} \tag{3}$$

$$w :\Vdash \top \iff \mathit{true} \tag{4}$$

$$w :\Vdash A \wedge B \iff w :\Vdash A \text{ and } w :\Vdash B \tag{5}$$

$$w :\Vdash A \vee B \Longleftarrow w :\Vdash A \text{ or } w :\Vdash B \tag{6}$$

$$w :\Vdash \exists x.A(x) \Longleftarrow \text{ for some } t \in D(w), w :\Vdash A(t) \tag{7}$$

Soundness of $\mathrm{LK}_{\mu\tilde{\mu}}$ for Kripke Semantics

$$
\begin{aligned}
c : (\Gamma \vdash \Delta) &\implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w :\Vdash_{\perp} \\
\Gamma \vdash t : A|\Delta &\implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w :\Vdash A \\
\Gamma|e : A \vdash \Delta &\implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w : A \Vdash
\end{aligned}
$$

Proof.

By mutual induction on the derivations. □

The Universal Kripke Model

Context Semantics

We prove Completeness for a **universal** Kripke model. From that, Completeness for any Kripke model follows.

The Universal Kripke model **U** is obtained by putting

possible worlds K is $\{(\Gamma, \Delta) | \Gamma : \mathrm{tvar} \to \mathrm{typ}, \Delta : \mathrm{evar} \to \mathrm{typ}\}$

partial order $(\Gamma, \Delta) \leq (\Gamma', \Delta')$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$

exploding nodes $(\Gamma, \Delta) :\Vdash_{\perp}$ iff $\Gamma \vdash_{cf} \Delta$

Strong Completeness of Kripke Semantics for $\mathrm{LK}_{\mu\tilde{\mu}}$

$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \text{there is a term } t \text{ such that } \Gamma \vdash_{cf} t : A|\Delta$$
$$(\Gamma, \Delta) : A \Vdash \Longrightarrow \text{there is an ev. context } e \text{ such that } \Gamma|e : A \vdash_{cf} \Delta$$

Proof.
By induction on the type A. □

Remarks:

- Only case for disjunction not straightforward
- Richer semantics – simpler completeness proof

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- Experiments show: Call-by-name discipline

Future Work

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- Formalise in Coq the quantifier part