

Normalisation-by-evaluation for λ -calculi

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Idea of NbE

- ▶ computational behaviour of λ -calculi traditionally studied through reduction seen as a rewrite system
- ▶ alternative: *evaluate* a λ -term in a constructive meta-language and *reify*-back the result into the object-language

$$M \longmapsto \|M\| \longmapsto M'$$

where $M =_{\beta\eta} M'$

Advantages of NbE

- ▶ proof of Church-Rosser not necessary
- ▶ easier to deal with η -equality when not considering it as reduction

NbE for Simply Typed λ -Calculus

(Sketch on whiteboard)

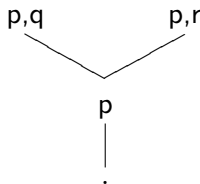
Connections to Concepts from Intuitionistic Logic

Making the previous construction more precise:

- ▶ **Evaluation is Soundness:** given $\Gamma \vdash t : A$, in every world w of every Kripke model, $w \Vdash \Gamma$ implies $w \Vdash A$
- ▶ **Reification is Completeness:** if in every world w of every Kripke model, $w \Vdash \Gamma$ implies $w \Vdash A$, then there exists a derivation $\Gamma \vdash t : A$

NbE = Completeness \circ Soundness

Kripke Semantics (Possible Worlds Semantics)



- ▶ a possible world is determined by atomic formulas known to hold so far
- ▶ at any later world we enrich our knowledge

Kripke Semantics: Formal Definition

A *Kripke model* is given by:

- ▶ a partial order (K, \leq) of **worlds**
- ▶ a domain of constants $D : K \rightarrow Set$, monotone
- ▶ a relation “ \Vdash ” between worlds and *atomic* formulas, called **forcing**

Forcing is extended to composite formulas inductively:

$w \Vdash$

$A \wedge B$ $w \Vdash A$ and $w \Vdash B$

$A \vee B$ $w \Vdash A$ or $w \Vdash B$

$A \rightarrow B$ for any $w' \geq w$, if $w' \Vdash A$ then $w' \Vdash B$

$\forall x P(x)$ for any $w' \geq w$ and any $a \in D(w')$, $w' \Vdash P(a)$

$\exists x P(x)$ there is $a \in D(w)$ such that $w \Vdash P(a)$

\perp is never forced

Almost Tarski's truth definition, \rightarrow (and \forall) standing out.

Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. $p \vee \neg p$
2. $\neg\neg p \rightarrow p$
3. 2 implies 1
4. $(p \rightarrow q) \rightarrow (\neg p \vee q)$
5. $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$
6. $\neg\forall x\neg P(x) \rightarrow \exists x P(x)$

Instances of NbE

- ▶ Simply typed lambda calculus (Berger-Schwichtenberg 1991)
- ▶ Untyped lambda calculus (Filinski-Rohde, 2004)
- ▶ Martin-Löf type theory (Abel-Coquand-Dybjer, 2007)
- ▶ ...

Part II: $\bar{\lambda}\mu\tilde{\mu}$ and its Kripke Semantics

joint work with Hugo Herbelin and Gyesik Lee (ROSAEC, Korea)

Proof Systems For Classical Logic

- ▶ call/cc and Pierce's Law (Griffin, 1990)
- ▶ $\lambda\mu$ -calculus (Parigot, 1992)
- ▶ $\bar{\lambda}\mu$ -calculus (Herbelin, 1995)
- ▶ $\bar{\lambda}\mu\tilde{\mu}$ – *calculus* (Curien-Herbelin, 2000)

$\bar{\lambda}\mu\tilde{\mu}$: Syntax and Reduction Rules

Syntax 3 categories: **commands**, **terms** and **evaluation contexts**

$$c ::= \langle t \| e \rangle$$

$$t ::= x \mid \mu\alpha.c \mid \lambda x.t$$

$$e ::= \alpha \mid \tilde{\mu}x.c \mid t \cdot e$$

Reduction (μ) $\langle \mu\alpha.c \| e \rangle \rightarrow c[e/\alpha]$

$$(\tilde{\mu}) \langle t \| \tilde{\mu}x.c \rangle \rightarrow c[t/x]$$

$$(\beta) \langle \lambda x.t \| t' \cdot e \rangle \rightarrow \langle t' \| \tilde{\mu}x.\langle t \| e \rangle \rangle$$

Critical pair $\langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle$: CBN, CBV strategies

The Sequent Calculus $LK_{\mu\tilde{\mu}}$

$$\frac{\Gamma \vdash v : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle v \parallel e \rangle : (\Gamma \vdash \Delta)} \text{ (Cut)}$$

$$\overline{(x : A), \Gamma \vdash x : A \mid \Delta} \text{ (Ax}_R\text{)}$$

$$\overline{\Gamma \mid \alpha : A \vdash (\alpha : A), \Delta} \text{ (Ax}_L\text{)}$$

$$\frac{c : (\Gamma \vdash (\alpha : A), \Delta)}{\Gamma \vdash \mu\alpha.c : A \mid \Delta} (\mu)$$

$$\frac{c : ((x : A), \Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c : A \vdash \Delta} (\tilde{\mu})$$

$$\frac{(x : A), \Gamma \vdash (t : B) \mid \Delta}{\Gamma \vdash \lambda x.t : A \rightarrow B \mid \Delta} (\rightarrow_R)$$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid t \cdot e : A \rightarrow B \vdash \Delta} (\rightarrow_L)$$

Kripke Semantics for $LK_{\mu\tilde{\mu}}$: Motivation

2 “new” ingredients for the model cake:

exploding nodes

- ▶ Intuitionistic completeness for Kripke semantics has seen only a **classical** proof
- ▶ Actually, the proof **cannot be** intuitionistic (Gödel, Kreisel, 1962)
- ▶ Yet, an **intuitionistic** proof by Veldman (1976), introducing exploding nodes
- ▶ The same relaxation happens for the intuitionistic proof for classical logic (Krivine, 1996) and for Beth models (Friedman, 60's or 70's)

“refinement” of \Vdash

- ▶ identify “strongly refutes” as primitive, define “forcing” and “refutation” by orthogonality

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$\exists x P(x)$ there is $a \in D(w)$ such that $w \Vdash P(a)$

\perp is never forced

Almost Tarski's truth definition, \rightarrow (and \forall) standing out.

Kripke Semantics for Classical Logic

A *Kripke model* is given by:

- ▶ a partial order (K, \leq) of **worlds**
- ▶ a domain of constants $D : K \rightarrow \text{Set}$, monotone
- ▶ a relation “ $(\cdot) : (\cdot) \Vdash_s$ ” between worlds and *atomic* formulas, called **forcing strong refutation**
- ▶ a marker of **exploding** worlds \Vdash_\perp

Then we define by orthogonality:

forcing $w : \Vdash A$ iff for all $w' \geq w$, $w' : A \Vdash_s$ implies $w' : \Vdash_\perp$

refutation $w : A \Vdash$ iff for all $w' \geq w$, $w' : \Vdash A$ implies $w' : \Vdash_\perp$

mutually with the extension of *strong* refutation to composite formulas: ...

Kripke Semantics for Classical Logic

... *mutually* with the extension of *strong* refutation to composite formulas:

$w : \Box \Vdash$

$A \vee B$ w refutes A and w refutes B

$A \wedge B$ w refutes A or w refutes B

$A \rightarrow B$ w forces A and w refutes B

$\forall x P(x)$ there is $a \in D(w)$ such that w refutes $P(a)$

$\exists x P(x)$ for all $w' \geq w$ and all $a \in D(w')$, w refutes $P(a)$

\perp always

\top never

Resemblance to Intuitionistic Forcing

$$w : \Vdash A \rightarrow B \iff \text{for all } w' \geq w, w' : \Vdash A \Rightarrow w' : \Vdash B \quad (1)$$

$$w : \Vdash \forall x. A(x) \iff \text{for all } w' \geq w \text{ and } t \in D(w'), w' : \Vdash A(t) \quad (2)$$

$$w : \Vdash \perp \iff w : \Vdash \perp \quad (3)$$

$$w : \Vdash \top \iff \text{true} \quad (4)$$

$$w : \Vdash A \wedge B \iff w : \Vdash A \text{ and } w : \Vdash B \quad (5)$$

$$w : \Vdash A \vee B \iff w : \Vdash A \text{ or } w : \Vdash B \quad (6)$$

$$w : \Vdash \exists x. A(x) \iff \text{for some } t \in D(w), w : \Vdash A(t) \quad (7)$$

Soundness of $LK_{\mu\tilde{\mu}}$ for Kripke Semantics

$c : (\Gamma \vdash \Delta) \implies$ for any $w, w : \Vdash \Gamma$ and $w : \Delta \Vdash$ implies $w : \Vdash \perp$

$\Gamma \vdash t : A \mid \Delta \implies$ for any $w, w : \Vdash \Gamma$ and $w : \Delta \Vdash$ implies $w : \Vdash A$

$\Gamma \mid e : A \vdash \Delta \implies$ for any $w, w : \Vdash \Gamma$ and $w : \Delta \Vdash$ implies $w : A \Vdash$

Proof.

By mutual induction on the derivations.



The Universal Kripke Model

Context Semantics

We prove Completeness for a **universal** Kripke model. From that, Completeness for any Kripke model follows.

The Universal Kripke model **U** is obtained by putting

possible worlds K is $\{(\Gamma, \Delta) \mid \Gamma : \text{tvar} \rightarrow \text{typ}, \Delta : \text{evar} \rightarrow \text{typ}\}$

partial order $(\Gamma, \Delta) \leq (\Gamma', \Delta')$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$

exploding nodes $(\Gamma, \Delta) : \vdash_{\perp}$ iff $\Gamma \vdash_{cf} \Delta$

Strong Completeness of Kripke Semantics for $LK_{\mu\tilde{\mu}}$

$(\Gamma, \Delta) : \Vdash A \implies$ there is a term t such that $\Gamma \vdash_{cf} t : A \mid \Delta$
 $(\Gamma, \Delta) : A \Vdash \implies$ there is an ev. context e such that $\Gamma \mid e : A \vdash_{cf} \Delta$

Proof.

By induction on the type A . □

Remarks:

- ▶ Only case for disjunction not straightforward
- ▶ Richer semantics – simpler completeness proof

Strong Completeness of Kripke Semantics for $LK_{\mu\tilde{\mu}}$

- ▶ Normalisation-by-evaluation
- ▶ Experiments show: All reduction rules preserved
- ▶ Experiments show: Call-by-name discipline

Future Work

- ▶ Compare with the *call-by-value* variant of Kripke semantics
- ▶ Compare to a constructive Henkin-style proof (Krivine, Berardi-Valentini)
- ▶ Formalise in Coq the quantifier part